V.M. Starzhinskii

An Advanced Course of THEORETICAL MECHANICS for Engineering Students

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ИЗДАТЕЛЬСТВО • МИР •

В. М. СТАРЖИНСКИЙ

ТЕОРЕТИЧЕСКАЯ МЕХАНИКА

Краткий курс по полной программе втузов

ИЗДАТЕЛЬСТВО «НАУКА»
МОСКВА

V.M. Starzhinskii

An Advanced Course of Theoretical Mechanics

FOR ENGINEERING STUDENTS

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2 Kinemasics

3 Dynamics

Preface to the English Edition

The present textbook comprises a course in theoretical mechanics together with some elements of analytical mechanics.

The first part of the book is devoted to statics of a rigid body. The study of the equilibrium of a body under the action of an arbitrary system of forces is preceded by a separate presentation of the case of a plane system of forces. For the latter system the equilibrium involving friction forces and the determination of internal forces in the bars of trusses are considered in detail (Chap. 4).

In the second part of the book (Kinematics) the major role is played by Chap. 9 (General Case of Motion of a Free Rigid Body). This chapter is based on Euler's theorem on instantaneous motion characterized by the velocity field of the particles of a body at each given instant. However, since this general approach may seem too complicated, in Chap. 10 where the plane motion is studied, besides references to Euler's theorem, the proofs independent of that theorem are also given.

In the third part of the book (Dynamics) Chap. 19 (General Principles of Dynamics of a System of Particles) is the central one. In this chapter we employ the teaching methods of the university courses by N. E. Joukowski, S. A. Chaplygin and N. G. Chetayev. Their approach based on the D'Alembert-Lagrange general equation of dynamics makes it possible not to introduce the reactions of ideal constraints and provides a general method of solving problems: the determination of virtual displacements and the first integrals corresponding to them. However, in the Appendix a more traditional derivation of the general principles is also given.

The book includes some optional material: stability of equilibrium and small vibrations (Chap. 20), advanced topics in dynamics of a rigid body (Chap. 22), the theory of impact and the theory of motion of a body with variable mass (Chap. 23), the motion of a particle in a field of central force and the trajectories of the Earth's artificial satellites (Chap. 24), and a detailed representation of mechanics of a flexible inextensible thread (Chap. 25) concluding the book.

The book contains more than 200 problems; for 120 of them detailed solutions and for the rest hints and answers are given. This enables the reader to use the book both for studying the theory and for problem-solving practice.

The textbook is intended for engineering students and can be used for studying under the guidance of a teacher as well as for self-instruction.

V. M. Starzhinskii

Introduction

1. Subject of Theoretical Mechanics. *Theoretical mechanics* is a science treating of the simplest form of motion of substance, namely the general laws of mechanical motion and of equilibrium of material bodies or their parts.

In the broad sense motion of substance is understood as any change of the state of a material body or bodies in thermal, chemical, electromagnetic, intratomic and other processes. Theoretical mechanics is confined to the simplest form of motion—mechanical motion.

By *mechanical motion* is meant variation with time of the position of material bodies relative to one another. Since the state of equilibrium is a special case of mechanical motion, theoretical mechanics also includes the study of equilibrium of bodies.

The observation of various natural phenomena shows that not all properties of the bodies involved in the phenomenon in question equally affect the course of the phenomenon or its final result. For instance, experiments show that the forces with which a beam having two supports acts on them are essentially dependent on the position of the supports and are practically independent of the deflection of the beam (provided that the deflection is small). Therefore when determining these forces we may conditionally replace the real beam by a nondeformable (perfectly rigid) one. In the investigation of other phenomena analogous arguments lead us to the notions of models of bodies such as a material point (a particle), a point charge, etc. An attempt to solve even the simplest problems without introducing such simplified models would fail. However, one should bear in mind that in nature there are no perfectly rigid bodies, particles, point charges, etc. and that all these are abstractions which enable us to consider theoretically the phenomena in question and to solve the required problems.

The present course is devoted to the study of classical mechanics based on the laws which were first stated exactly by G. Galileo (1564-1642) and I. Newton (1642-1727). At the end of the 19th century and the beginning of the 20th century it was found that the

laws of classical mechanics do not apply to motion of microparticles and to bodies moving with velocities close to that of light. At the beginning of the 20th century relativistic mechanics was created; relativistic mechanics is based on the relativity theory of A. Einstein (1879-1955). The relativity theory establishes the general laws concerning the relationship between space, time, mass and energy. Relativistic mechanics is applicable to the study of motion with velocity close to the speed of light and indicates the limits within which the laws of classical mechanics remain valid. This however does not diminish the role of classical mechanics as a practical method for studying motion of macroscopic bodies whose velocities are small compared with that of light, that is the motion usually dealt with in engineering.

2. Methods of Theoretical Mechanics. Theoretical mechanics, like other natural sciences, widely uses the method of abstraction. The application of this method and the generalization of the results of human experience, technological practice and experiment made it possible to establish some general laws playing the role of axioms. All the further propositions of classical mechanics can be derived from these axioms using logical argument and mathematical calculations. Since theoretical mechanics mostly deals with quantitative relationships it is clear that mathematical analysis must play a very important role in it. However, although the course of theoretical mechanics is saturated with mathematics and contains few references to experimental studies, this does not at all mean that theoretical mechanics can do without experimental verification of its laws and conjectures. On the contrary, as in all other branches of knowledge, the final proof of the propositions of theoretical mechanics lies in experiment and practice. The history of the development of science and, particularly, of theoretical mechanics confirms that only experiment and practice can decide whether or not a hypothesis or a theory is correct.

3. Historical Notes. Theoretical mechanics is closely connected with practice and is one of the most ancient sciences. Although the earliest manuscripts on mechanics known to us belong to the 4th century B.C. the remnants of ancient structures show that even much earlier the ancients were familiar with some elements of mechanics.

The beginning of the development of mechanics was primarily connected with statics—the science treating of the equilibrium of material bodies. As early as the 3rd century B.C. the scientific basis of statics was founded, mainly in the works of the great Greek scientist Archimedes (*circa* 287-212 B.C.). He elaborated the exact solution of the problem of equilibrium of the lever, introduced the concept of the centre of gravity, discovered the well-known law of hydrostatics named after him, etc.

Although scientists were interested in questions concerning the motion of bodies as early as the ancient times, the creation of kinematics and, particularly, of dynamics (the branch of theoretical mechanics studying the motion of bodies in connection with their interaction) was started only at the end of the 16th century and at the beginning of the 17th century. As was mentioned, in the foundation of mechanics the major role was played by G. Galileo and I. Newton. The period of almost 2000 years separating the times of Archimedes and Newton can be characterized, in relation to the development of mechanics, as the time of accumulation of much experimental material concerning various kinds of mechanical motion (in particular, the motion of celestial bodies) and of systematic, although slow, development of mathematical methods. This experimental material, the development of mathematics, the great discoveries made by N. Copernicus (1473-1543) and J. Kepler (1571-1630) and, particularly, G. Galileo, immediate predecessors of I. Newton, enabled the latter to discover the general laws of mechanics (named after him) and to create adequate mathematical methods (differential and integral calculus) making it possible to apply these general laws and their consequences to the solution of practical problems.*

In the 18th and 19th centuries the development of theoretical mechanics was mainly connected with the creation and application of analytical methods (L. Euler (1707-1783), J. D'Alembert (1717-1783), J. L. Lagrange (1736-1813), K. G. J. Jacobi (1804-1851), W. R. Hamilton (1805-1865), J. H. Poincaré (1854-1912) and others) and geometrical methods (L. Poinsot (1777-1859) and others).

A valuable contribution to the development of theoretical mechanics was made by Russian scientists: M. V. Ostrogradsky (1801-1862), P. L. Chebyshev (1821-1894), S. V. Kovalevskaya (1850-1891), A. M. Lyapunov (1857-1918), N. E. Joukowski (1847-1921), I. V. Meshchersky (1859-1935), S. A. Chaplygin (1869-1942), A. N. Krylov (1863-1945), N. G. Chetayev (1902-1959) and others.

The growth of modern technology led to the development of some special branches of theoretical mechanics such as hydrodynamics, aerodynamics, gas dynamics, theory of elasticity, theory of plasticity, strength of materials, etc. which at present are independent branches of science. However, the problem-solving methods of these sciences are based on those of theoretical mechanics. That is why theoretical mechanics is one of the basic topics studied in engineering colleges.

4. **Branches of Theoretical Mechanics.** The course of theoretical mechanics divides into three parts: statics, kinematics and dynamics.

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* Simultaneously with and independently of I. Newton differential and integral calculus was discovered by G. W. Leibniz (1646-1716).

Statics deals with the laws of equilibrium of material bodies and the rules of reduction of systems of forces to the simplest form. In *kinematics* the motion of the bodies is considered purely geometrically irrespective of the factors causing the motion. Finally, *dynamics* studies the motion of the bodies in connection with the forces acting on them and causing their motion.

STATICS
OF A PERFECTLY
RIGID BODY

Part 1

Chapter 1 System of Concurrent Forces

§ 1. Elements of Vector Algebra *

1.1. Scalar and Vector Quantities. In all branches of theoretical mechanics we encounter two types of quantities: scalar quantities and vector quantities.

A *scalar quantity* or, simply, a *scalar* is a quantity which is specified only by its numerical value relative to the chosen system of units and is not connected with any direction in space. For instance, mass and volume of a body, temperature and energy are scalars.

A *vector quantity* or, simply, a *vector* is specified not only by its numerical value but also by a definite direction in space. For instance, force and velocity are vectors.

A vector is a directed line segment. A vector is completely determined by two points taken in a definite order: the first point is called the origin (or tail or point of application) of the vector, and the second point, the end of the vector, is called the terminus (or tip or terminal point) of the vector. In the figures vectors are represented by line segments (Fig. 1.1) at whose ends arrows are shown. The direction of the arrow indicates that of the vector. The length of the line segment (relative to the chosen scale) represents the numerical value characterizing the vector; the numerical value of the vector is called the *modulus* (or *absolute value* or *magnitude*) *of the vector*. The straight line kl along which the vector is directed is referred to as the *line of action of the vector.*

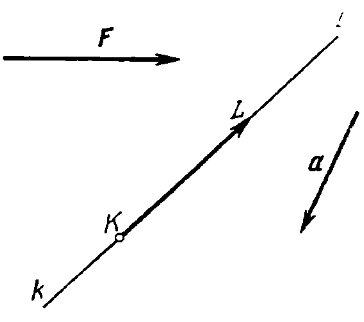

Fig. 1.1

In the text and in the figures the vectors are denoted either by one ($\mathbf{a}$, $\mathbf{F}$) or by two ($\mathbf{KL}$) boldface-type letters. In the latter case the

* In this section we give only a brief review of vector algebra; for greater detail see [7].

first letter (for instance, the letter K in the notation of the vector KL) denotes the origin of the vector and the second letter denotes its terminus. The modulus (magnitude or absolute value) of the vector is denoted by the same standard-type letters; for instance, a, F, KL.

In theoretical mechanics we deal with quantities characterized by free vectors, sliding vectors and bound vectors. By a *free vector* is meant a vector whose point of application can be transferred to any point in space. A *sliding vector* is a vector whose point of application can be transferred along its line of action. Finally, a *bound* (or *localized*) *vector* is a vector whose point of application is fixed. Mechanical quantities to which the enumerated three types of vectors correspond will be indicated in the corresponding parts of the course.

A bound (localized) vector is specified by six independent scalar quantities; for instance, these can be the three coordinates of the origin of the vector and the three coordinates of its terminus.

A free vector $\mathbf{a}$ is specified by three independent scalar quantities; these can be the projections a_x, a_y and a_z of the vector on the axes Ox, Oy and Oz of the Cartesian coordinates.

A sliding vector is specified by five independent scalars. As is known, a straight line in space is determined by four constants, for instance, by the coefficients a, b, p and q of the equations

$$x = az + p, \quad y = bz + q$$

specifying that line. As parameters specifying the sliding vector we can take the quantities a, b, p and q, and the quantity whose absolute value is equal to the modulus of the vector and whose sign (plus or minus) is chosen depending on whether a certain coordinate (for instance, z) increases or decreases along the direction of that sliding vector. The algebra of sliding vectors differs from that of free vectors.

However, the study of sliding and bound vectors can be in many cases reduced to the study of free vectors; therefore we shall limit ourselves to algebra of free vectors.

1.2. Basic Definitions and Rules of Operations on Free Vectors. 1. Two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal if their lengths (moduli) are equal, their directions coincide and their lines of action are parallel (or coincide).

The equality of vectors is written in the same way as the equality of scalar quantities*

$$\mathbf{a} = \mathbf{b} \tag{1.1}$$

* The formulas, figures, examples and problems are numbered thus: the first number indicates the chapter and the second is the number of that formula (or figure or example or problem) within that chapter.

2. To add together several vectors we must construct the first vector, then place the point of application of the second vector at the terminus of the first, then construct the third vector placing its point of application at the terminus of the second, etc. After that we must join the origin of the first vector with the terminus of the last, which results in the vector representing the *vector sum of the given vectors* (Fig. 1.2). When speaking of a sum of vectors we shall always mean their vector sum; the sum of n given vectors $\boldsymbol{a}_\nu$ $(\nu = 1, \ldots, n)$ will be written in the form

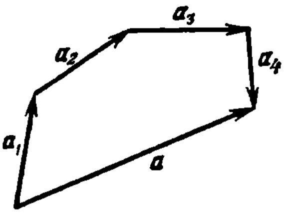

Fig. 1.2

$$\boldsymbol{a} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \ldots + \boldsymbol{a}_n, \text{ that is } \boldsymbol{a} = \sum_{\nu=1}^{n} \boldsymbol{a}_\nu \tag{1.2}$$

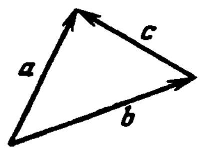

Fig. 1.3

3. Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, to subtract the second vector ($\boldsymbol{b}$) from the first ($\boldsymbol{a}$) we must apply both vectors at one point and then draw a vector joining the terminus of the subtrahend vector with the terminus of the minuend vector (Fig. 1.3). The vector $\boldsymbol{c}$ thus constructed is the difference between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$$

4. When a vector is multiplied by a number λ the modulus of the vector is multiplied by the absolute value of the number while the direction of the vector is retained if $\lambda > 0$ and is changed to the opposite if $\lambda < 0$.

5. A vector whose modulus is equal to unity is spoken of as a *unit vector*. Every vector $\boldsymbol{a}$ can be represented in the form

$$\boldsymbol{a} = a\boldsymbol{a}^0 \tag{1.3}$$

where a is the modulus (magnitude) of the vector $\boldsymbol{a}$, and $\boldsymbol{a}^0$ is a unit vector whose direction coincides with that of the vector $\boldsymbol{a}$.

6. The *projection of a vector on an axis* is a scalar quantity equal to the product of the modulus of the vector by the cosine of the angle between the positive direction of the axis and the vector (Fig. 1.4):

$$a_x \equiv \operatorname{proj}_x \boldsymbol{a} = a \cos \alpha \tag{1.4}$$

Here a_x is the projection of the vector $\boldsymbol{a}$ on the axis Ox. It should be noted that $a_x > 0$ if $0 \leqslant \alpha < \pi/2$, $a_x = 0$ if $\alpha = \pi/2$ and $a_x < 0$ if $\pi/2 < \alpha \leqslant \pi$.

7. The *projection of a vector sum of several vectors on an axis* is equal to the algebraic sum of the projections of the given vectors on that

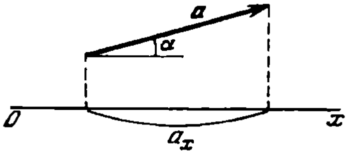

Fig. 1.4

axis (Fig. 1.5.):

$$a_x = a_{1x} + a_{2x} + a_{3x}$$

or, in the case of n vectors $\boldsymbol{a}_\nu$ ($\nu = 1, \ldots, n$),

$$a_x = a_{1x} + a_{2x} + \ldots + a_{nx} = \sum_{\nu=1}^{n} a_{\nu x} \tag{1.5}$$

8. Every vector $\boldsymbol{a}$ can be represented in the form

$$\boldsymbol{a} = a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k} \tag{1.6}$$

where a_x, a_y and a_z are the projections of the vector $\boldsymbol{a}$ on the Cartesian coordinate axes, and $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors along the axes Ox, Oy and Oz respectively.

The modulus of a vector $\boldsymbol{a}$ is expressed in terms of its projections according to the formula

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{1.7}$$

and the direction of the vector is specified by the angles which it forms with the coordinate axes:

$$\cos(\widehat{\boldsymbol{a}, x}) = \frac{a_x}{a}, \quad \cos(\widehat{\boldsymbol{a}, y}) = \frac{a_y}{a}, \quad \cos(\widehat{\boldsymbol{a}, z}) = \frac{a_z}{a} \tag{1.8}$$

9. A *scalar product* of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the product of their moduli by the cosine of the angle between them (Fig. 1.6):

$$(\boldsymbol{a}, \boldsymbol{b}) = ab \cos \varphi \tag{1.9}$$

The scalar product does not change when the factors are interchanged:

$$(\boldsymbol{b}, \boldsymbol{a}) = (\boldsymbol{a}, \boldsymbol{b})$$

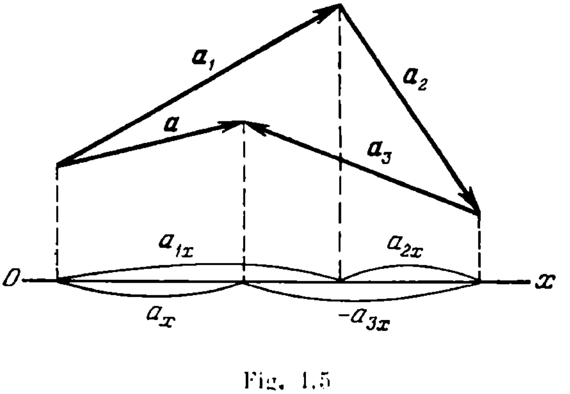

Fig. 1.5

The expression of the scalar product in terms of the projections of the vector factors on the coordinate axes has the form

$$(\boldsymbol{a}, \boldsymbol{b}) = a_x b_x + a_y b_y + a_z b_z \qquad (1.10)$$

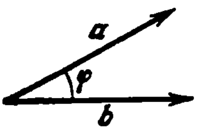

Fig. 1.6

The condition of orthogonality of two vectors different from zero is that their scalar product is equal to zero.

From (1.9) it follows that if $\boldsymbol{a}$ and $\boldsymbol{b}$ are nonzero vectors, that is if $a \neq 0$ and $b \neq 0$, then

$$\cos\varphi = \frac{(\boldsymbol{a}, \boldsymbol{b})}{ab} \qquad (1.11)$$

The expression of $\cos\varphi$ can also be written thus:

$$\cos\varphi = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}} \qquad (1.11a)$$

The last formula is obtained if we express the scalar product $(\boldsymbol{a}, \boldsymbol{b})$ and the moduli a and b of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in terms of their projections on the coordinate axes using formulas (1.10) and (1.7).

10. By the ***vector product*** of two given vectors is meant a third vector whose modulus is equal to the product of the moduli of the given vectors by the sine of the angle between them and which is perpendicular to the plane containing these vectors and is directed according to the right-hand screw rule. This rule indicates that when contemplating from the terminus of the vector equal to the vector product $\boldsymbol{c}$ we see the rotation through the smallest angle from the first factor $\boldsymbol{a}$ to the second factor $\boldsymbol{b}$ in the counterclockwise direction (Fig. 1.7). The vector product is denoted using square brackets. Thus, if

$$\boldsymbol{c} = [\boldsymbol{a}, \boldsymbol{b}] \qquad (1.12)$$

then

$$c = ab \sin\varphi \qquad (1.13)$$

When the factors $\boldsymbol{a}$ and $\boldsymbol{b}$ are interchanged the vector product changes its sign:

$$[\boldsymbol{b}, \boldsymbol{a}] = -[\boldsymbol{a}, \boldsymbol{b}] \qquad (1.14)$$

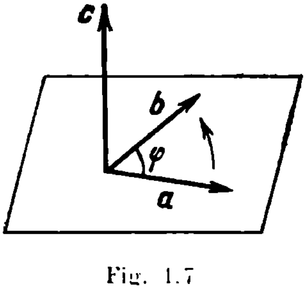

Fig. 1.7

The vector product of two nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to zero if and only if the vectors are parallel or lie in one straight line (that is if $\varphi = 0$ or $\varphi = \pi$). Consequently, the condition of parallelism of two vectors is that their vector product is equal to zero.

The vector product $\boldsymbol{c} = [\boldsymbol{a}, \boldsymbol{b}]$ is expressed in terms of the projections of the

vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ on the coordinate axes thus:

$$\boldsymbol{c} = [\boldsymbol{a}, \boldsymbol{b}] = (a_y b_z - a_z b_y)\,\boldsymbol{i} + (a_z b_x - a_x b_z)\,\boldsymbol{j} + (a_x b_y - a_y b_x)\,\boldsymbol{k} \qquad (1.15)$$

Formula (1.15) can be rewritten in the form of the determinant:

$$\boldsymbol{c} = [\boldsymbol{a}, \boldsymbol{b}] = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad (1.16)$$

The projections of the vector product on the coordinate axes are

$$\begin{aligned} c_x &= [\boldsymbol{a}, \boldsymbol{b}]_x = a_y b_z - a_z b_y \\ c_y &= [\boldsymbol{a}, \boldsymbol{b}]_y = a_z b_x - a_x b_z \\ c_z &= [\boldsymbol{a}, \boldsymbol{b}]_z = a_x b_y - a_y b_x \end{aligned} \qquad (1.17)$$

11. The *triple scalar product* of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is a number equal to the scalar product of the vector product of two of the vectors by the third:

$$([\boldsymbol{a}, \boldsymbol{b}], \boldsymbol{c}) \qquad (1.18)$$

The triple scalar product is a number whose absolute value is equal to the volume of the parallelepiped constructed on the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

The equality of the triple scalar product to zero expresses the condition that the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are coplanar, that is these three vectors are parallel to one plane.

12. The *triple vector product* of three given vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is a vector $\boldsymbol{d}$ constructed according to the following rule:

$$\boldsymbol{d} = [\boldsymbol{a}, [\boldsymbol{b}, \boldsymbol{c}]] \qquad (1.19)$$

The computation of the triple vector product can be carried out using the formula

$$\boldsymbol{d} = (\boldsymbol{a}, \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{b}, \boldsymbol{a})\,\boldsymbol{c} \qquad (1.20)$$

where $(\boldsymbol{a}, \boldsymbol{c})$ and $(\boldsymbol{b}, \boldsymbol{a})$ are the scalar products of the corresponding vectors. Projecting (1.20) on the coordinate axes we obtain

$$\begin{aligned} d_x &= (\boldsymbol{a}, \boldsymbol{c})\, b_x - (\boldsymbol{b}, \boldsymbol{a})\, c_x \\ d_y &= (\boldsymbol{a}, \boldsymbol{c})\, b_y - (\boldsymbol{b}, \boldsymbol{a})\, c_y \\ d_z &= (\boldsymbol{a}, \boldsymbol{c})\, b_z - (\boldsymbol{b}, \boldsymbol{a})\, c_z \end{aligned} \qquad (1.21)$$

Substituting into (1.21) the expressions of the scalar products in terms of the projections of the corresponding vectors (see (1.10)) we can obtain the expressions of the projections of the triple vector product on the coordinate axes in terms of the projections of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ on those axes.

§ 2. Basic Notions of Statics

2.1. Perfectly Rigid Body. In science when studying a natural phenomenon some abstractions are used in order to concentrate attention on most significant features of the phenomenon and to discard less significant features. In theoretical mechanics abstractions of this kind are the notions of a material point (a particle) and of a (perfectly) rigid body.

A *material point* (a *particle*) is a physical body whose dimensions can be neglected in the conditions of the problem in question. A particle differs from a geometrical point in that the former is supposed to carry a (concentrated) mass. Because of that a particle possesses the property of inertness (see Sec. 1.1 of Chap. 13) and can interact with other particles.

In mechanics every physical body is understood as a system of particles. A *system of particles* is a definite collection of particles interacting with one another according to the law of action and reaction (see Sec. 2.5, Axiom 3). By a *perfectly rigid body* is meant a physical body possessing the property that the distances between any two of its points remain invariable under all conditions. In other words, a perfectly rigid body and its any part retain an invariable geometrical form, that is they are nondeformable.

2.2. Force. Now we pass to the definition of force. In mechanics by *force* is meant a quantitative measure of mechanical interaction between material bodies. The interaction may result in a change of kinematical state of the material bodies, that is not only in a change in their position in space but also in a change of the velocities of the particles of the body. This definition of the accelerating property of a force will be discussed in full in dynamics (see Sec. 1.1 of Chap. 13). In relation to the problems of statics, we shall understand a force as an action of one body on another resulting in the appearance of pressure, attraction or repulsion.

The simplest example of a force is the *force of gravity*. This is the force with which every body is attracted to the Earth; as a result, every constrained body presses its support (this is the static action of the force); a free body falls on the Earth with acceleration g under the action of that force (this is the dynamic action of the force).

In the present course we shall use only the *International System of Units* (SI) (see Sec. 2.3). The unit of force in the International System of Units is called the *newton* (N) and is defined as a force which imparts to a mass* equal to the mass of an adopted platinum standard (the *kilogram-mass*) an acceleration of 1 m/s^2. On the relationship between the newton and the *kilogram-force* (used in the technical system of units) see Sec. 2.3.

* On the definition of mass see Sec. 1.1 of Chap. 13.

The action of a force on a body is specified by the *point of application* of the force*, by its direction and its numerical value (magnitude). The straight line along which the given force is directed is called the *line of action* of the force. The *magnitude* (the *modulus*) of the force is found by the comparison with unit force (for instance, with the aid of a dynamometer).

A *force is a vector quantity* and therefore graphically it is represented by a vector. The length of that vector expresses the modulus (the numerical value) of the force relative to the chosen scale, and the straight line on which the vector lies and the direction of the vector indicate the line of action and the direction of the force respectively. We shall specify the position of force vectors in space with the aid of a rectangular Cartesian coordinate system connected with the Earth. Coordinate systems (frames of reference) will be discussed in greater detail in kinematics and dynamics (see the corresponding parts of the course).

2.3. Units of Mechanical Quantities. There are three metric systems of units that have been in use:

(a) the absolute physical system of units cgs in which the basic units are the centimetre, the gram (mass) and the second;

(b) the absolute practical system of units in which the basic units are the metre, the kilogram (mass) and the second;

(c) the technical system of units in which the basic units are the metre, the kilogram-force and the second.

In all the systems of units the unit of a plane angle is the *radian* and the unit of a solid angle is the *steradian.*

In 1960 the International Conference of Weights and Measures recommended as preferable for all branches of science and engineering the International System of Units (SI). For the mechanical units the SI system coincides with the absolute practical system.

Definition of Basic Units in the SI. The *unit of length*, the *metre*, is defined as the distance between two marks on a standard platinum-iridium bar. The *unit of mass*, the *kilogram-mass*, is the mass of an adopted platinum standard. The *unit of time*, the *second*, is equal to 1/86 400 of the mean solar day.**

Derived Units. The derived units are those expressed in terms of the basic units according to the physical laws. The *dimensional formula*, or, simply, the *dimension* of a mechanical quantity is a formula indicating what operations involving multiplication and division must be performed on the basic units in order to obtain the given quantity. The symbols L, M and T in these formulas denote the units of length, mass and time respectively. For instance, the

* In Sec. 2.5 we shall show that the force vector is a sliding vector provided that the force is applied to a perfectly rigid body.

** On systems of units see [6].

unit of force in the International System of Units (SI), the newton (N), is a derived unit. The newton is defined as a force imparting to the unit mass (1 kg) the unit acceleration (1 m/s^2). The dimensional formula for the force in the International System of Units (SI) is LMT^{-2}. The kilogram-force (kgf) (which is one of the basic units in the technical system and a derived unit in the absolute practical system) is defined as a force which imparts an acceleration of 9.80665 m/s^2 to a mass equal to that of the adopted platinum standard of 1 kg. Replacing the standard acceleration 9.80665 m/s^2 by its approximate value 9.81 m/s^2 we conclude that the kilogram-force is 9.81 times as great as the newton:

$$1 \text{ kgf} = 9.81 \text{ N} \quad (1 \text{ N} = 0.102 \text{ kgf})$$

Thus one kilogram-force is equal to 9.81 newtons (one newton is equal to 0.102 kilogram-force).

In order to name the multiples and submultiples of adopted units certain prefixes are used. For instance, one kilonewton (kN) is equal to a thousand newtons (to 102 kgf in the technical system of units); one millinewton (mN) is equal to one thousandth of the newton.

Besides the newton, the derived units (used in statics) in the International System of Units (SI) are:

(a) the square metre, m^2, for measuring area;

(b) the cubic metre, m^3, for measuring volume;

(c) kg/m^3 for measuring density;

(d) for measuring the absolute value of the moment of a force about a point or an axis the derived unit with dimensional formula L^2MT^{-2} is used (its abbreviated notation is $N \cdot m$);

(e) N/m^2, the pascal (newton per square metre), for measuring pressure (stress);

(f) N/m^3 (newton per cubic metre) for measuring specific weight.

In order to pass from the units of the technical system to the units of the International System (SI) the following relations should be used:

for the moment of a force:

$$1 \text{ kgf}\cdot\text{m} = 9.81 \text{ N}\cdot\text{m} \quad (1 \text{ N}\cdot\text{m} = 0.102 \text{ kgf}\cdot\text{m})$$

for specific weight:

$$1 \text{ kgf/m}^3 = 9.81 \text{ N/m}^3 \ (1 \text{ N/m}^3 = 0.102 \text{ kgf/m}^3)$$

The derived units used in kinematics and dynamics will be discussed in the second and the third parts of the course. We once again stress that throughout the present course only the International System of Units (SI) is used.

2.4. System of Forces. A collection of forces applied to a given body is called a *system of forces*. If a rigid body remains at rest or is in inertial motion (for instance, if all the particles of the body are

in rectilinear motion with the same constant velocity) under the action of a force system, such a state of the body is referred to as a *state of equilibrium*. A system of forces under whose action the body is in equilibrium is said to be *balanced*.

Inertial motion of bodies will be considered in dynamics (see Sec. 1.4 of Chap. 19 and Sec. 2.3 of Chap. 22); in statics by the equilibrium of a body we shall mean the state of rest. When a given rigid body is acted upon by a balanced system of forces we usually say that these *forces are in equilibrium*. Every force belonging to a balanced system of forces is called the *balancing* (the *equilibrant*) *force* relative to all the other forces of that system.

Two systems of forces $\{F_1, F_2, \ldots, F_n\}$ and $\{P_1, P_2, \ldots, P_m\}$ applied to a given rigid body are said to be *equivalent* if they can be transformed into one another with the aid of the following *elementary operations*:

(a) the transfer of the points of application of the forces along their lines of action;

(b) composition and resolution of forces applied at one point;

(c) addition and subtraction of a balanced system of forces.

The equivalence of two systems of forces will be written using the symbol $\sim$:

$$\{F_1, F_2, \ldots, F_n\} \sim \{P_1, P_2, \ldots, P_m\}$$

A balanced system of forces $\{Q_1, Q_2, \ldots, Q_l\}$ is said to be *equivalent to zero*:

$$\{Q_1, Q_2, \ldots, Q_l\} \sim 0$$

If a system of forces $\{F_1, F_2, \ldots, F_n\}$ is equivalent to one force R, that is

$$\{F_1, F_2, \ldots, F_n\} \sim R$$

then this force is called the *resultant* of the given system of forces. In this case the forces $F_1, F_2, \ldots, F_n$ are called the *components* of the force R. The operation of replacing a force system $\{F_1, F_2, \ldots$ $\ldots, F_n\}$ by its resultant R is called the *composition* (*addition*) *of forces*. The reverse operation of replacing a force R by its components $F_1, F_2, \ldots, F_n$ is called the *resolution of the given force into its components*.

2.5. Axioms of Statics. The application of the method of abstraction and the generalization of the results of human experience, direct observations and technological practice made it possible to establish some general laws of statics. These laws are called the *axioms of statics*. All further propositions of elementary statics are derived from these axioms by means of mathematical argument. Here the term "elementary" underlines the distinction between statics considered in the first part of the book and analytic statics presented in Chap. 17.

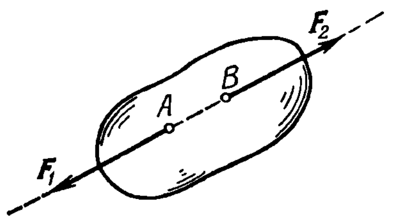

Fig. 1.8

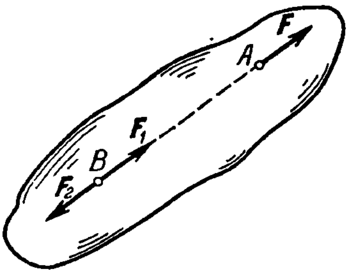

Fig. 1.9

AXIOM 1. *For two forces applied to a perfectly rigid body to be balanced it is necessary and sufficient that these forces should be equal in their moduli and have opposite directions along the straight line joining their point of application* (Fig. 1.8).

Thus, if $\{\boldsymbol{F}_1, \boldsymbol{F}_2\} \sim 0$ then $\boldsymbol{F}_2 = -\boldsymbol{F}_1$ and the vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are collinear, that is they lie in one straight line.

A system of two forces, applied to a rigid body, which have equal moduli and opposite directions along one and the same straight line is the simplest force system equivalent to zero.

COROLLARY. *If a system of forces possesses the resultant then the equilibrant and the resultant forces have equal moduli, lie in one straight line and have opposite dire..tions.*

AXIOM 2. *Given a system of forces applied to a perfectly rigid body, the addition to or the subtraction from that system of a balanced force system does not change the action of the given system on the body.*

COROLLARY. *The point of application of a given force acting on a perfectly rigid body can be transferred along the line of action of that force without changing the action of the force on the body.*

Proof. Let a perfectly rigid body be acted upon by a force $\boldsymbol{F}$ applied at a point A (Fig. 1.9). Let us take an arbitrary point B lying on the line of action of the force $\boldsymbol{F}$ and apply at that point two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ having opposite directions along the line AB and equal moduli coinciding with that of the force $\boldsymbol{F}$: $F_1 = F_2 = F$. Since the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are balanced (Axiom 1), it follows from Axiom 2 that the system of the three forces $\boldsymbol{F}$, $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is equivalent to the force $\boldsymbol{F}$. The forces $\boldsymbol{F}$ and $\boldsymbol{F}_2$ are also balanced and therefore, by Axiom 2, they can be discarded; after that there remains the force $\boldsymbol{F}_1$ applied at the point B. Hence, the force $\boldsymbol{F}$ applied at the point A is equivalent to the force $\boldsymbol{F}_1$ applied at the point B, which is what we had to prove.

The possibility of transferring the point of application of a force along its line of action shows that a force acting on a perfectly rigid

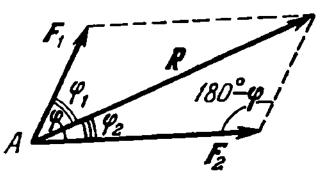

Fig. 1.10

body can be interpreted as a sliding vector. For an elastic body a force vector is not sliding but bound.

AXIOM 3. *The forces with which two bodies act on each other are always equal in their moduli and have opposite directions along one straight line.*

This axiom stated by I. Newton is called the *law of action and reaction.* In this connection it should be noted that although the forces of action and reaction have equal magnitudes and opposite directions along one line, they cannot be called balanced. Indeed, this would be senseless because only forces applied to one body can be called balanced, and the forces of action and reaction are applied to two different bodies.

AXIOM 4. *If a given system of forces applied to a rigid body is balanced then it is also balanced when acting on any other rigid body.*

According to this axiom the dimensions and the shape of the body are of no importance in the study of statics of a perfectly rigid body.

AXIOM 5. *If a deformable body is in equilibrium then its equilibrium is not disturbed when the body solidifies* (that is becomes rigid).

From this law called the *principle of solidification* it follows that when a deformable body is in equilibrium the conditions of the equilibrium of that body regarded as perfectly rigid should also hold (the conditions stated in Axioms 4 and 5 are necessary for the equilibrium of a deformable body but, generally speaking, are not sufficient).

2.6. Parallelogram Law. Theorem on Three Forces. To the axioms enumerated above we should add the following important axiom on composition of two forces (known as the *parallelogram law*):

The resultant of two forces applied at one point of a rigid body is applied at that same point, and its modulus and direction are specified by the diagonal of the parallelogram constructed on these forces (Fig. 1.10).

Denoting by $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ two forces applied at one point of a body and by $\boldsymbol{R}$ their resultant we can write $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2$.

The modulus of the resultant is found using the formula

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\varphi} \tag{1.22}$$

The application of the law of sines results in

$$\frac{F_1}{\sin\varphi_2} = \frac{F_2}{\sin\varphi_1} = \frac{R}{\sin(180^\circ - \varphi)}$$

whence

$$\sin\varphi_1 = \frac{F_2}{R}\sin\varphi, \quad \sin\varphi_2 = \frac{F_1}{R}\sin\varphi \tag{1.23}$$

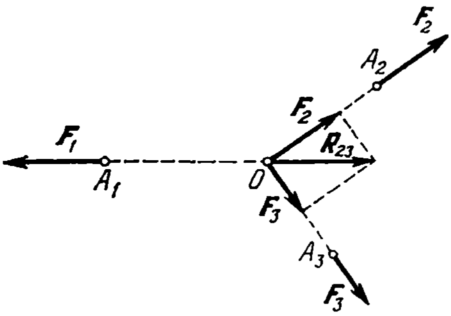

Fig. 1.11

Formulas (1.22) and (1.23) determine the magnitude and the direction of the resultant of two forces F_1 and F_2 applied at one point and forming an angle φ.

The parallelogram law makes it possible to prove the following theorem:

TREOREM ON THREE FORCES. *If a rigid body is in equilibrium under the action of three nonparallel forces lying in one plane then the lines of action of these forces intersect at one point.*

Proof. Let a body be in equilibrium under the action of three forces F_1, F_2 and F_3 lying in one plane and applied at point A_1, A_2 and A_3 of the body respectively (Fig. 1.11). Let us transfer two of the forces, for instance, F_2 and F_3 to the point O of intersection of their lines of action and then add them together using the parallelogram law. This results in the system of two forces F_1 and R_{23} equivalent to the former system of the three forces F_1, F_2 and F_3. According to Axiom 1, the equilibrium of a body acted upon by two forces is possible if and only if these forces are equal in magnitude, lie in one straight line and have opposite directions. Consequently, the line of action of the force F_1 which coincides with that of the force R_{23} passes through the point O. The theorem is proved.

As will be shown in Sec. 3.1 of Chap. 5, if a system of three forces is balanced then these forces lie in one plane. This means that the words "lying in one plane" can be deleted from the conditions of the above theorem.

2.7. External and Internal Forces. Forces acting on a body are divided into two classes, namely external forces and internal forces. By *external forces* are meant those forces with which other bodies act on the particles of the given rigid body. By *internal forces* are meant the forces of interaction between the particles of the body.

LEMMA. *The internal forces acting within a given perfectly rigid body form a balanced force system and do not affect the equilibrium conditions of the body.*

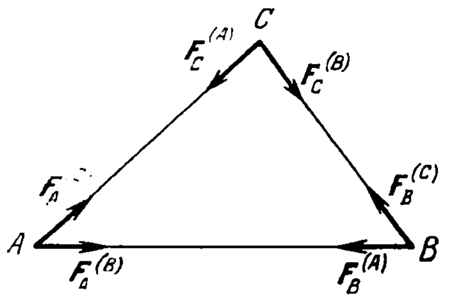

Fig. 1.12

Proof. The internal forces acting within the given rigid body can be represented as being resolved into pairs of mutually balanced forces. For instance, let us consider three particles A, B and C of the body and let us denote the internal forces as is shown in Fig. 1.12. By virtue of Axiom 3, we have

$$F_A^{(B)} = -F_B^{(A)}, \quad F_B^{(C)} = -F_C^{(B)}, \quad F_C^{(A)} = -F_A^{(C)}$$

According to Axiom 1 the system of internal forces of the given body applied at its three points is equivalent to zero:

$$\{F_A^{(B)}, F_B^{(A)}\} + \{F_B^{(C)}, F_C^{(B)}\} + \{F_C^{(A)}, F_A^{(C)}\} \sim 0$$

The same argument remains valid for any number of points (particles), which proves the lemma.

That is why in statics of rigid body only the conditions of equilibrium of external forces are considered.

For the determination of internal forces acting within the body the so-called *method of sections* is used. It is based on the investigation of the equilibrium of parts of the body. If a given body is in equilibrium then its every part is also in equilibrium. The application of the equilibrium conditions to a chosen part of the body makes it possible to determine the corresponding internal forces.

To demonstrate the application of the method of sections we shall consider the following simplest example. Let a thin rectilinear rod AB (Fig. 1.13a) be in equilibrium under the action of two forces F_1 and F_2 with points of application at its ends A and B. Since the rod is in equilibrium, Axiom 1 implies that

$$F_1 = F_2$$

and that the vectors F_1 and F_2 have opposite directions along the straight line AB.

Let us draw a section mn ("make a cut") through the point C of the rod; this section separates the rod into two parts. Next we consider the equilibrium of the left part AC of the rod (Fig. 1.13b). At the point C of the rod a force T is applied; this force is produced by the action of the right part of the rod which has been discarded on the left part. Both the whole rod and its left part are in equilibrium. By Axiom 1 we conclude that

$$T = F_1$$

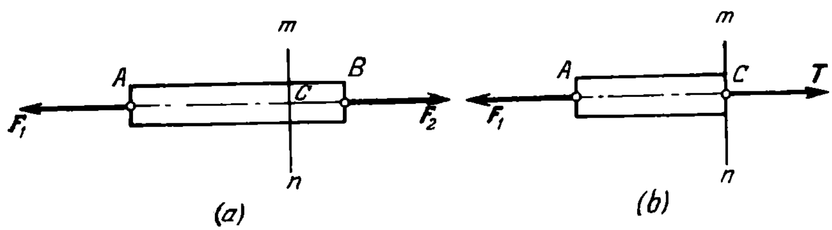

Fig. 1.13

Since the section *mn* was drawn quite arbitrarily we arrive at the following conclusion: in any section of a rod subjected to tension (compression) by external forces there are internal forces acting on the corresponding parts of the rod which are equal to the external forces. The internal force thus determined is referred to as a *tensile* (*compressive*) *stress.*

2.8. Axiom of Constraints. Depending on the conditions of experiment the bodies dealt with in mechanics are classified into free bodies and constrained bodies. A body is said to be *free* if it can move in any direction. For instance, a stone thrown by a person is a free body. A body is said to be *constrained* if it can move only in certain definite directions or cannot move at all. For instance, a railway car is a constrained body because the direction of its motion is determined by the rails. As a rule, when solving problems of statics, we shall deal with constrained rigid bodies whose displacement is restricted by the action of other bodies surrounding the given body.

The material bodies which prevent the motion of constrained bodies (in certain direction) are referred to as *constraints*, and the forces with which these bodies act on the given body are called the *reactions of constraints.* Generally speaking, the direction of the reaction of a constraint is opposite to the direction along which the constraint prevents the given body from motion. This facilitates the determination of the directions of the reactions in the solution of problems.

AXIOM OF CONSTRAINTS. *Any constrained body can be detached from its constraints by replacing the constraints by their reactions, after which the body can be treated as a free body subjected to the action of the given external forces and the reaction forces of the constraints.*

In statics the equilibrium conditions for a free body are investigated. In order to apply these conditions to constrained bodies it is necessary to treat them according to the axiom of constraints. That is why to conclude this section we shall consider the simplest types of constraints and their reactions (at present, without taking into account the friction forces).

2.9. Reactions of Constraints. The constructions of constraints usually have the form of various supports such as hinged joints, pulls, etc. We shall enumerate some types of constraints assuming

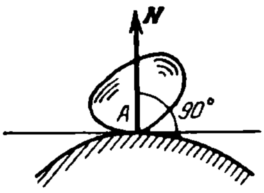

Fig. 1.14

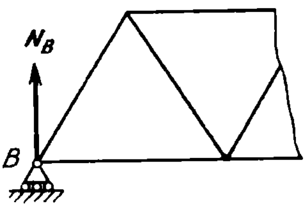

Fig. 1.15

that they are made of perfectly rigid materials and that there is no friction at the points of contact between the bodies in question.

1. *Contact of Bodies.*

(a) Let a body be supported at a point A by a smooth surface (Fig. 1.14). The reaction force of the supporting surface is applied to the body at the point A and its direction is along the normal to the surface. That is why this force is called the *normal reaction* (it is usually denoted by N).

(b) Another example of a constraint of that type is the so-called *movable hinged support* (Fig. 1.15). In the construction of this support a roller is used: in this case a beam or a truss is supported at its end B by a cylindrical roller. The reaction force N_B of the movable hinged support is applied to the beam at the point B and goes along the normal to the supporting surface on which the roller can move.

(c) Let us consider a beam which rests on the floor and on the wall at its end A and on the edge of the dihedral angle at the point B

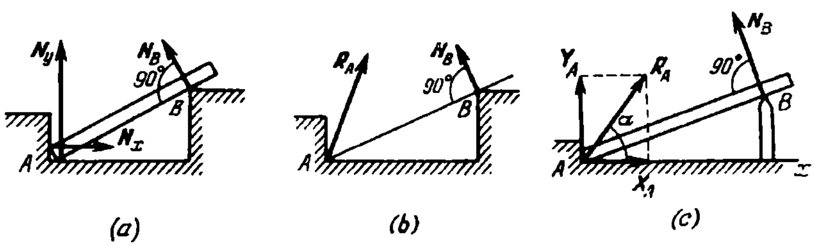

Fig. 1.16

(Fig. 1.16*a*). The directions of the reactions N_x and N_y of the wall and of the floor are along the normals to the corresponding supporting surfaces. *The reaction force N_B of the dihedral angle goes along the normal to the surface of the beam at the point B.* If we neglect the transverse dimension of the beam then the reactions N_x and N_y can be regarded as being applied at A (Fig. 1.16*b*, *c*). The vector sum of the reactions N_x and N_y is represented by one resultant R_A; as to the line of action of the resultant R_A, it is only known that it passes through the point A.

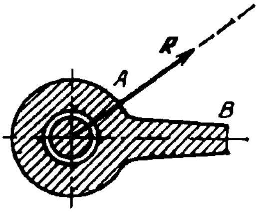

Fig. 1.17

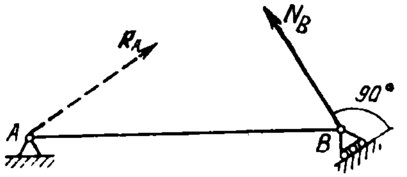

Fig. 1.18

2. *Hinged Joint of Bodies.*

(d) *Cylindrical Bearing.* Such a joint is shown in Fig. 1.17. A fixed cylindrical pin passes through the circular hole in the rod AB, the diameter of the hole being a little greater than that of the pin. The rod AB can only rotate about the axis of the pin. The reaction force R of the cylindrical bearing lies in the plane perpendicular to the axis of the pin and passing through its centre. The modulus and the line of action of R (the line of action lies in that plane) are unknown and are determined depending on the forces acting on the rod AB.

The so-called *immovable hinged support* also belongs to this type of constraint; it is shown in Fig. 1.18 for the beam AB at the point A. The reaction R_A of the immovable hinged support is applied to the beam at the point A and lies in the plane perpendicular to the axis of the pin; however, the direction of the reaction is unknown. In the figure the direction of this reaction is conditionally represented by the dash line.

(e) *Ball-And-Socket Joint.* Consider Fig. 1.19 where the rod AO is shown at one of whose ends there is a spherical surface which can only rotate in its support. In this case the rod AO can only rotate about the centre O of the ball-and-socket joint. Before the problem

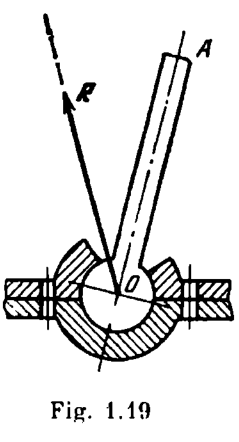

Fig. 1.19

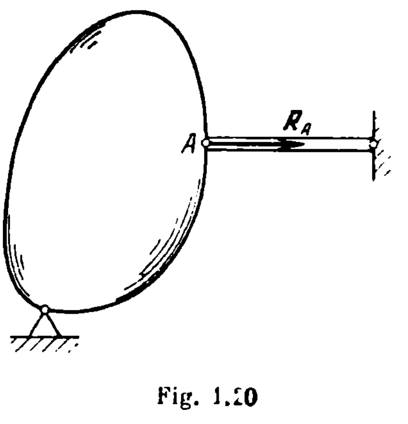

Fig. 1.20

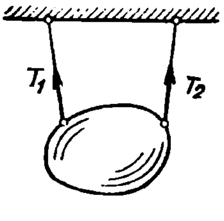

Fig. 1.21

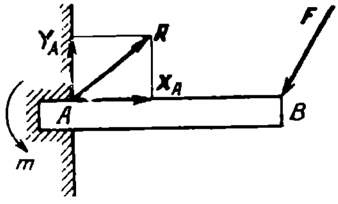

Fig. 1.22

is solved it is only known that the line of action of the reaction R passes through the point O.

(f) *The reaction forces acting on a weightless rod whose ends are hinged* (Fig. 1.20). The reactions acting on such a rod are applied along the axis of the rod. The rod itself is subjected either to tension (in this case the reaction R_A is directed inside the rod along its axis as shown in Fig. 1.20) or to compression (in this case the reaction is directed away from the rod along its axis).

3. *Flexible Constraint* (Thread, Rope, Chain).

(g) The reaction T called the *tension of the thread* is applied at the point at which the thread is attached to the body and is directed along the thread inside it (Fig. 1.21, where T_1 and T_2 are the tensions of the threads).

4. *Rigid Fixing.*

(h) Let us consider the beam AB in Fig. 1.22 whose one end is rigidly fixed (in the wall) while the other end serves as a support for the construction. If the beam is acted upon by some given forces there appear reaction forces in the rigid fixing; in this case the system of reaction forces consists of a force R and a couple with moment m (see Sec. 2.2 of Chap. 2).

In conclusion we stress that, according to Axiom 3, the force with which the given body acts on a constraint and the reaction of the constraint are always equal in their moduli and have opposite directions along one straight line.

§ 3. System of Concurrent Forces

3.1. Composition of Concurrent Forces. Resultant. Statics treats of equilibrium of rigid bodies under the action of forces applied to them; it reduces to the solution of the following two basic problems: (1) the problem of replacing a given system of forces by an equivalent system and (2) the problem of deriving general conditions for equilibrium of rigid bodies. We shall start the study of these problems with the simplest case—the system of concurrent forces.

We stipulate that in the case of a balanced system of forces (equivalent to zero) we shall often speak of the *equilibrium of the system*

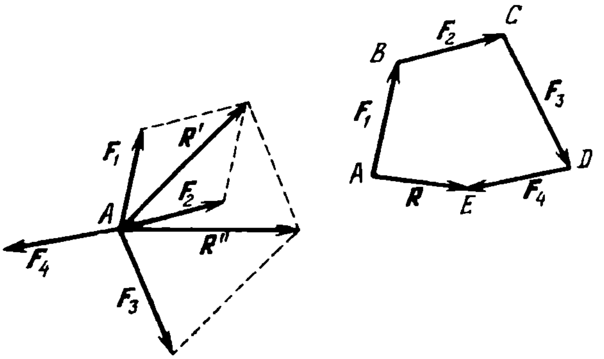

Fig. 1.23

of forces although it would be more correct to speak of the equilibrium of a rigid body under the action of that system of forces.

A system of forces is said to be *concurrent* if the lines of action of all the forces meet at one point. Since every force can be transferred along its line of action, a system of concurrent forces is equivalent to the system of the same forces applied at that point.

The composition (addition) of several forces applied at one point can be carried out by means of the consecutive addition using the parallelogram law (Fig. 1.23): we first add together the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ and find their resultant $\boldsymbol{R}'$; then this resultant is added to the force $\boldsymbol{F}_3$. Constructing the parallelogram on the forces $\boldsymbol{R}'$ and $\boldsymbol{F}_3$ we find their resultant $\boldsymbol{R}''$, etc. The operation of composition of forces can also be performed without constructing each time the force parallelogram; it is sufficient to place the origin of the vector $\boldsymbol{F}_2$ at the terminus B of the vector $\boldsymbol{F}_1$, then apply the vector $\boldsymbol{F}_3$ at the terminus C of the vector $\boldsymbol{F}_2$, etc. Finally, joining the point A of application of the forces with the terminus of the force $\boldsymbol{F}_4$ we obtain the resultant $\boldsymbol{R}$. The last method of finding the resultant is called the *force polygon law*, the broken line $ABCDE$ being the *force polygon* and the line segment AE being its *closing line*.

If we limit ourselves to the parallelogram law then the case of composition of forces applied at one point A and lying in one straight line should be specially stipulated. If we use the polygon law such a stipulation is not needed.

If we are given n forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ concurrent at a point O then their resultant $\boldsymbol{R}$ is applied at the point O and is equal to the vector sum of the given force vectors:

$$\boldsymbol{R}=\boldsymbol{F}_1+\boldsymbol{F}_2+\ldots+\boldsymbol{F}_n=\sum_{\nu=1}^{n}\boldsymbol{F}_\nu \tag{1.24}$$

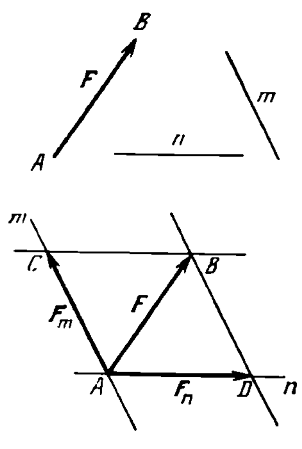

Fig. 1.24

3.2. Resolution of a Given Force into Components. The problem of resolving a force into components reduces to the determination of forces such that their application at the same point produces an action equivalent to that of the given force. Generally speaking, the problem of resolving a given force into two components lying in one plane with the given force is indeterminate. Indeed, the vector addition of the components must result in the given force, that is the given force must be the diagonal of the parallelogram constructed on the sought-for components. It is quite evident that there exists an infinitude of parallelograms whose diagonal coincides with the given force.

The problem of resolving a given force into two nonparallel components lying in one plane with the given force becomes determinate if we set the lines of action m and n of the sought-for components (Fig. 1.24). To find these components it is sufficient to draw straight lines parallel to the given directions m and n through the point A of application of the force $\boldsymbol{F}$ and through its terminus B. The points of intersection of these lines determine the parallelogram $ADBC$ for which the force $\boldsymbol{F}$ serves as diagonal. The vectors $\boldsymbol{AC} = \boldsymbol{F}_m$ and $\boldsymbol{AD} = \boldsymbol{F}_n$ applied at the point A specify the sought-for components:

$$\boldsymbol{F} = \boldsymbol{F}_m + \boldsymbol{F}_n$$

Every force $\boldsymbol{F}$ can be resolved along three arbitrarily chosen directions m, n and p provided that the directions are not parallel to one plane, this resolution being unique (Fig. 1.25). Let us draw through the point of application A of the force $\boldsymbol{F}$ straight lines parallel to the given directions m, n and p. Further, through the terminus B of the force we draw three planes parallel to the faces of the trihedron $Amnp$. These planes intersect the axes of the trihedron at the points

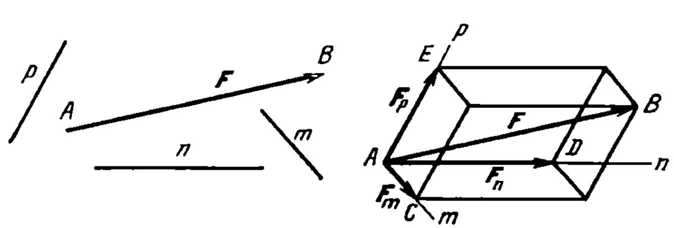

Fig. 1.25

C, D and E. We have thus constructed a parallelepiped whose edges have given directions and whose diagonal is the given force. It is evident that the vectors $AC = \boldsymbol{F}_m$, $AD = \boldsymbol{F}_n$ and $AE = \boldsymbol{F}_p$ are the sought-for three components. Indeed, we have

$$\boldsymbol{F} = \boldsymbol{F}_m + \boldsymbol{F}_n + \boldsymbol{F}_p$$

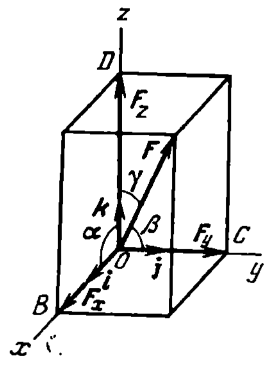

Fig. 1.26

Let a force $\boldsymbol{F}$ form some angles α, β and γ with the axes of a rectangular Cartesian coordinate system $Oxyz$ (Fig. 1.26). In order to resolve the force $\boldsymbol{F}$ along the three coordinate axes let us construct a rectangular parallelepiped for which the force $\boldsymbol{F}$ is a diagonal. The edges of this parallelepiped specify the components which we denote $\boldsymbol{F}_x$, $\boldsymbol{F}_y$ and $\boldsymbol{F}_z$. The moduli of these components taken with plus sign or minus sign (depending on whether the directions of the components coincide with the positive directions of the axes or are opposite to them) are the projections X, Y and Z of the force $\boldsymbol{F}$ on the coordinate axes. Let us denote by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ the unit vectors along the coordinate axes; then the components can be written in the form

$$\boldsymbol{F}_x = X\boldsymbol{i}, \boldsymbol{F}_y = Y\boldsymbol{j}, \boldsymbol{F}_z = Z\boldsymbol{k}; \boldsymbol{F} = \boldsymbol{F}_x + \boldsymbol{F}_y + \boldsymbol{F}_z = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$$

According to formula (1.4) we also have

$$X = F\cos\alpha, \quad Y = F\cos\beta, \quad Z = F\cos\gamma \tag{1.25}$$

Thus, if the modulus F of a force $\boldsymbol{F}$ and the angles α, β and γ between the force and the coordinate axes are known we can find the projections of the force on the coordinate axes using formula (1.25). Conversely, if the projections X, Y and Z of the force $\boldsymbol{F}$ on the coordinate axes are known the modulus of the force and its direction can be found with the aid of the formulas

$$F = \sqrt{X^2 + Y^2 + Z^2}, \quad \cos\alpha = \frac{X}{F}, \quad \cos\beta = \frac{Y}{F}, \quad \cos\gamma = \frac{Z}{F} \tag{1.26}$$

3.3. Geometrical Equilibrium Condition for a System of Concurrent Forces. A system of forces applied at one point is balanced if and only if the resultant of these forces is equal to zero. Since the resultant of a system of concurrent forces is equal to the closing line of the force polygon constructed on these forces (see Sec. 3.1), in the case of equilibrium the force polygon must be closed. Conversely, if the force polygon is closed this means that the resultant of the concurrent forces is equal to zero. We have thus found geometrically the equilibrium condition: *for a system of concurrent forces to be balanced it is necessary and sufficient that the force polygon constructed for this force system should be closed* (Fig. 1.27).

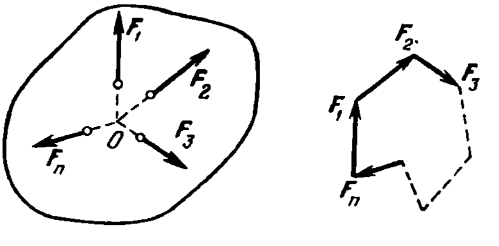

Fig. 1.27

The course of the solution of problems by means of the construction of force polygons is the following:

(a) First of all we must decide for which body the equilibrium is considered.

(b) The action of constraints is replaced by the corresponding reactions (see Secs. 2.8 and 2.9).

(c) According to the condition of the problem we indicate in the figure the points of application and the directions of both the given and the sought-for forces. The forces are applied to the body (which may be a particle, a mechanical unit, etc.) whose equilibrium is considered.

(d) Choosing a definite scale we construct the force polygon beginning with the given forces. Then we find the unknown elements of the force polygon (forces or angles) either using graphical methods or determining the elements of the corresponding triangles.

EXAMPLE 1.1. A body A of weight P is at rest on a smooth inclined plane forming an angle α with the horizon, the body being supported by a thread AB forming an angle β with the vertical (Fig. 1.28*a*). Find the tension of the thread and the pressure the body exerts on the plane.

Solution. The body is in equilibrium under the action of three forces: the force of gravity $\boldsymbol{P}$ directed vertically downward, the reaction $\boldsymbol{T}$ of the thread directed along the thread inside it and the reaction $\boldsymbol{N}$ of the plane perpendicular to that plane (see Sec. 2.9). Let us consider these three forces as applied at one point—the centre of gravity of the body.

We begin the construction of the force triangle starting with the given force $\boldsymbol{P}$. To this end we draw the vector $\boldsymbol{P}$ from an arbitrary point C (Fig. 1.28*b*). Next we place the origin of another force, say $\boldsymbol{N}$, at the terminus D of the vector $\boldsymbol{P}$. Since the modulus of the force $\boldsymbol{N}$ is not yet known we limit ourselves to drawing a straight line through the point D parallel to the vector $\boldsymbol{N}$. Further, since the body is in equilibrium the triangle formed by the forces $\boldsymbol{P}$, $\boldsymbol{N}$ and $\boldsymbol{T}$ must be closed and the terminus of the last of the vectors which we add together (of the vector $\boldsymbol{T}$) must fall at the origin of the first vector (the vector $\boldsymbol{P}$). Consequently, the vector $\boldsymbol{T}$ must lie on the straight line which passes through the point C and is parallel to the tension $\boldsymbol{T}$ of the thread. The point E at which these straight lines intersect determines the terminus of the vector $\boldsymbol{N}$ and the origin of the vector $\boldsymbol{T}$. By the law of sines, we have

$$\frac{P}{\sin[180^\circ-(\alpha+\beta)]}=\frac{T}{\sin\alpha}=\frac{N}{\sin\beta}$$

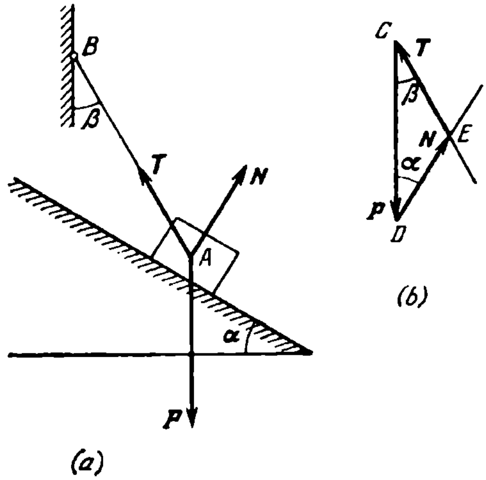

Fig. 1.28

whence it follows that

$$T=\frac{\sin\alpha}{\sin(\alpha+\beta)}P,\quad N=\frac{\sin\beta}{\sin(\alpha+\beta)}P$$

In order to solve the problem graphically we must choose a certain scale and then set off the force P, after which the same geometrical construction should be performed; in this construction the parallelism of the corresponding lines should be strictly observed. Measuring the sides EC and DE with the aid of the scale we find the moduli of the sought-for forces. The tension of the thread (if we consider it as a force applied to the thread) and the pressure exerted by the body on the plane have moduli equal to those of the forces T and N respectively but are of opposite directions because T and N denote the reactions of the thread and of the plane. This problem can also be simply solved by resolving the force P into two components parallel to AB and N.

EXAMPLE 1.2. A homogeneous rod AB is attached to the vertical wall with a hinge A, and its end B is supported by the thread BC, the angle between the rod and the wall being α (Fig. 1.29*a*). It is known that the weight of the rod is equal to P and that $AC = AB$; determine the tension T of the thread and the reaction R_A of the hinge.

Solution. Let us consider the equilibrium of the rod AB. The constraints supporting the rod AB are the thread BC and the hinge A. Replacing the action of the constraints by the reaction forces we conclude that the rod AB is in equilibrium under the action of three nonparallel forces: the force of gravity P acting on the rod AB at its midpoint, the tension T of the thread and the reaction R_A of the hinge A. The only information on the reaction R_A is that the point of application of this force is at A.

The direction of the reaction R_A can be determined on the basis of the theorem on three forces (see Sec. 2.6). Namely, the line of action of the reaction R_A must pass through the point O of intersection of the lines of action of the forces P and T. Let us construct the force triangle beginning with the given force P (Fig. 1.29*b*). Through the terminus of the force vector P we draw a straight line parallel, for instance, to the force T. Since ABC is an isosceles triangle and its vertex angle is α, we have $\angle ACB = 90° - \alpha/2$; therefore

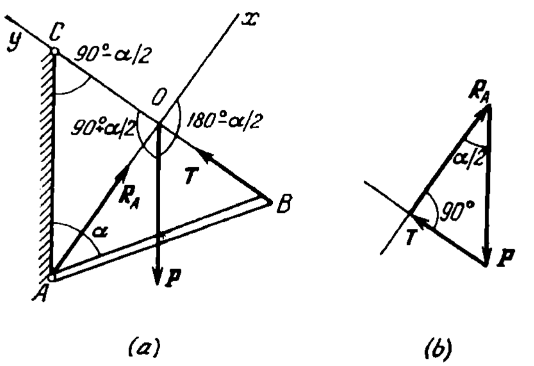

Fig. 1.29

the line of action of the force T in the force triangle must be drawn so that it forms an angle equal to $90° - \alpha/2$ with the vertical. Next we draw through the origin of the vector P a straight line parallel to the force R_A. Here we take into account that $CO = OB$ and therefore $AO \perp CB$, that is the lines of action of the forces R_A and T are mutually perpendicular. The directions of the arrows in the force triangle must be drawn so that the triangle is closed. From the force triangle we obtain

$$R_A = P\cos\frac{\alpha}{2}, \quad T = P\sin\frac{\alpha}{2}$$

3.4. Analytical Equilibrium Conditions for a System of Concurrent Forces. For a system of concurrent forces to be balanced it is necessary and sufficient that the resultant of the system should be equal to zero. Consequently, for the equilibrium it is necessary and sufficient that the projections of the resultant on three noncoplanar axes should be equal to zero. Projecting equality (1.24) on the coordinate axes and using (1.5) we obtain the following expressions for the projections of the resultant:

$$\begin{aligned} R_x &= X_1 + X_2 + \ldots + X_n = \sum_{\nu=1}^{n} X_\nu \\ R_y &= Y_1 + Y_2 + \ldots + Y_n = \sum_{\nu=1}^{n} Y_\nu \\ R_z &= Z_1 + Z_2 + \ldots + Z_n = \sum_{\nu=1}^{n} Z_\nu \end{aligned} \qquad (1.27)$$

Here X_ν, Y_ν and Z_ν ($\nu = 1, 2, \ldots, n$) are the projections of the force F_ν on the coordinate axes Ox, Oy and Oz respectively. Equating

the projections of the resultant to zero we obtain

$$\begin{aligned}
&\sum_{\nu=1}^{n} X_\nu \equiv X_1 + X_2 + \ldots + X_n = 0 \\
&\sum_{\nu=1}^{n} Y_\nu \equiv Y_1 + Y_2 + \ldots + Y_n = 0 \\
&\sum_{\nu=1}^{n} Z_\nu \equiv Z_1 + Z_2 + \ldots + Z_n = 0
\end{aligned} \tag{1.28}$$

It is the system of equalities (1.28) that expresses analytically the equilibrium conditions: *for a system of concurrent forces to be balanced it is necessary and sufficient that the algebraic sum of the projections of all the forces on each of the three coordinate axes should be equal to zero.* The coordinate system $Oxyz$ must not necessarily be rectangular.

In the analytical solution of the problem the unknown forces enter into (1.28); hence system (1.28) must be solved as a system of equilibrium equations. For a plane system of concurrent forces we have not three but only two equations of equilibrium. If the plane in which the forces lie is taken as the coordinate plane Oxy the equations of equilibrium of the plane system of concurrent forces are written in the form

$$\sum_{\nu=1}^{n} X_\nu = 0, \qquad \sum_{\nu=1}^{n} Y_\nu = 0 \tag{1.29}$$

When solving a problem it is advisable to perform consecutively the following operations: we first perform what has been indicated in (a), (b), (c) in Sec. 3.3 and then perform the operations enumerated below:

(d) The origin is taken at the point of intersection of the lines of action of the forces and a certain direction of the coordinate axes is chosen. To simplify the calculations the directions of the axes should be chosen so that they are parallel (perpendicular) to as many forces as possible.

(e) All the forces are projected on each of the coordinate axes; then equilibrium equations (1.28) are written.

(f) The system of equations thus obtained is solved with respect to the unknowns (which are forces, angles, etc.).

EXAMPLE 1.3. Solve the problem stated in Example 1.2 using the analytical method.

Solution. According to indication (d) we draw the coordinate axes Ox and Oy as is shown in Fig. 1.29a. Next we write equilibrium equations (1.29):

$$\sum X = R_A + P\cos\left(180° - \frac{\alpha}{2}\right) = 0, \qquad \sum Y = T + P\cos\left(90° + \frac{\alpha}{2}\right) = 0$$

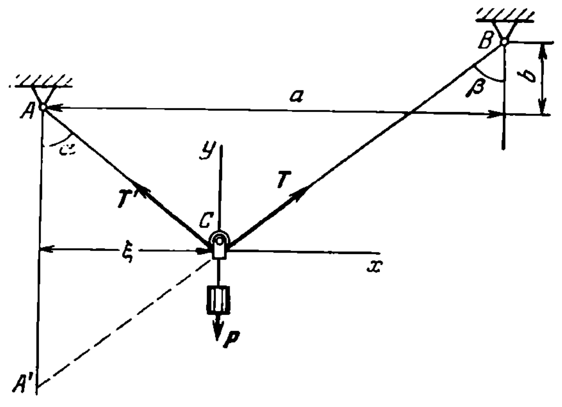

Fig. 1.30

Solving these equations we find

$$R_A = P\cos\frac{\alpha}{2}, \quad T = P\sin\frac{\alpha}{2}$$

which coincides with the result obtained by means of the geometrical method.

EXAMPLE 1.4. Two fixed points A and B (Fig. 1.30) are joined with a taut thread ACB of length l. There is a small pulley C carrying a weight P which can travel along the thread. Find the angles α and β between the parts AC and BC of the thread and the vertical, the tension of both parts of the thread and the distance ξ between the pulley C and the vertical passing through the left support A when the system is in equilibrium. The friction in the axis of the pulley and its weight are neglected.

Solution. Let us consider the equilibrium of the pulley C. Since the ideal (frictionless) pulley does not affect the magnitudes of the tensions of the parts of the thread, the reactions T and T' of the thread directed along CA and CB respectively have equal moduli: $T' = T$. Let us place the origin at the point of intersection of the lines of action of the forces P, T and T' and let the coordinate axes Cx and Cy be drawn horizontally to the right and vertically upward respectively. Now let us set the equations of equilibrium for the forces applied to the pulley C. Projecting the forces P, T and T' on the axes Cx and Cy and equating to zero the sum of the projections on each of the axes we obtain

$$\sum X = T\cos(90° - \beta) + T'\cos(90° + \alpha) = 0$$

$$\sum Y = -P + T\cos\beta + T'\cos\alpha = 0$$

From the first equation the equality $\beta = \alpha$ follows. Therefore from the second equation we find the following expression for the tension T of the thread:

$$T = \frac{P}{2\cos\alpha} = \frac{l}{2\sqrt{l^2 - a^2}}\,P$$

The distance ξ is given by the formula

$$\xi = \frac{AA'}{2}\tan\alpha = \frac{l\cos\alpha - b}{2}\tan\alpha = \frac{a}{2}\left(1 - \frac{b}{\sqrt{l^2 - a^2}}\right)$$

It is readily seen that the greater is b the closer to the vertical passing through the left support is the equilibrium position of the pulley C.

3.5. Double Projection Method. This method can be conveniently applied when the angle between a force $\boldsymbol{F}$ and an axis ol is not known beforehand (Fig. 1.31). Let us draw through the axis ol an arbitrary plane S and project the force $\boldsymbol{F}$ on that plane. The magnitude of this projection is

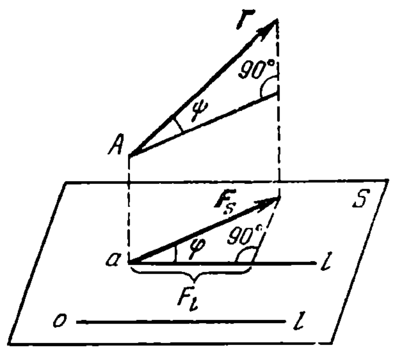

Fig. 1.31

$$F_S = F \cos \psi$$

where ψ is the angle between the force $\boldsymbol{F}$ and the plane S. Now we project the vector $\boldsymbol{F}_S$ on the axis ol; to this end we draw through the origin a of the vector $\boldsymbol{F}_S$ the axis al parallel to the axis ol. Denoting by ψ the angle $(\widehat{\boldsymbol{F}_S, l})$ we obtain

$$F_l = \text{proj}_l \boldsymbol{F}_S = F_S \cos \varphi = F \cos \psi \cos \varphi$$

It should be noted that since ψ is the angle between a force and a plane it must lie within the limits $0 \leqslant \psi \leqslant 90°$ while the angle φ between the vector $\boldsymbol{F}_S$ and the positive direction of the axis al varies between the limits $0 \leqslant \varphi \leqslant 180°$.

EXAMPLE 1.5. A weight Q of 50 kN is suspended from the point D as is shown in Fig. 1.32. The rods shown in the figure have hinged joints at the points A, B and D, and $AD = BD$. Determine the reactions of the rods AD and BD and the tension of the rope CD.

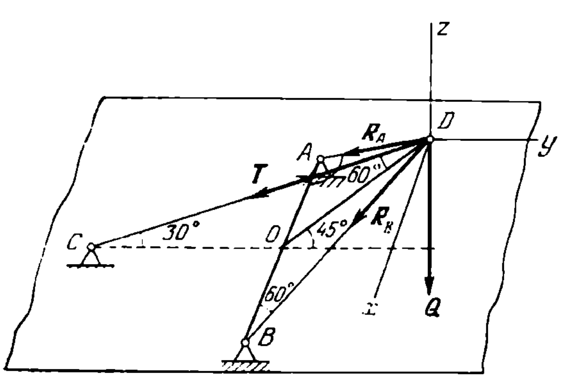

Fig. 1.32

Solution. Let us consider the equilibrium of the joint D. Here a given force is the force $\boldsymbol{Q}$ directed vertically downward. The sought-for forces are the reaction $\boldsymbol{R}_A$ of the rod AD, the reaction $\boldsymbol{R}_B$ of the rod BD and the tension $\boldsymbol{T}$ of the rope CD.

For definiteness, let us assume that the rods AD and BD are subjected to tension; then the reactions of the rods, that is the forces $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ with which the rods act on the joint D, are directed along the rods away from D as is shown in Fig. 1.32. Since all the forces meet at one point D, we can use the analytical equilibrium conditions for a system of concurrent forces to determine the unknown forces.

Let us place the origin at the point D and draw the axis Dz upward perpendicularly to the plane ABC, and let the axes Dx and Dy lie in the plane parallel to ABC as is shown in Fig. 1.32. The force $\boldsymbol{T}$ lies in the vertical plane, and the forces $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ are in the plane ABD.

Now we write the equilibrium equations for the joint D. To this end we project all the forces on the coordinate axes, which results in

$$\sum X = \quad -R_A \cos 60° \quad + R_B \cos 60° \quad = 0 \qquad \text{(a)}$$

$$\sum Y = -T \cos 30° - R_A \cos 30° \cos 45° - R_B \cos 30° \cos 45° = 0 \qquad \text{(b)}$$

$$\sum Z = -T \cos 60° - R_A \cos 30° \cos 45° - R_B \cos 30° \cos 45° - Q = 0 \qquad \text{(c)}$$

If the reader encounters difficulties in projecting forces we recommend him to reread Sec. 3.5. From (a) we find

$$R_B = R_A$$

From (b) we obtain

$$T = -2R_A \cos 45°$$

and, finally, substituting the expression of T into (c), we arrive at

$$2R_A \sin 30° \cos 45° - 2R_A \cos 30° \sin 45° - Q = 0$$

that is

$$-2R_A \sin (45° - 30°) = Q$$

whence

$$R_A = R_B = -\frac{Q}{2 \sin 15°} = -\frac{50}{2 \cdot 0.259} = -96.5 \text{ kN}$$

Here R_A and R_B denote the algebraic values of the reactions. If the solution of the problem yields a negative value of one (or both) of the reactions this means that the direction of the corresponding force should be changed to the opposite. In other words, in this case the real direction of one (or both) of the reactions $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ of the rods AD and BD is opposite to that shown in Fig. 1.32 (and the corresponding rod AD or BD (or both) is subjected not to tension as it was assumed at the beginning of the solution but to compression). The tension of the rope is

$$T = 2 \cdot 96.5 \frac{\sqrt{2}}{2} = 136 \text{ kN}$$

EXAMPLE 1.6. Determine the stresses in the jib AB, the pillar OB and the legs CB and DB of the crane shown in Fig. 1.33 depending on the angle α $(-90° \leqslant \alpha \leqslant 90°)$ of rotation of the jib. All the supports are hinged joints; the angles are shown in Fig. 1.33*a*.

Solution. Let us consider the equilibrium of the joint A. It is acted upon by the given force $\boldsymbol{Q}$ and the reactions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ of the rods AO and AB respectively. For definiteness, at the beginning of the solution of the problem

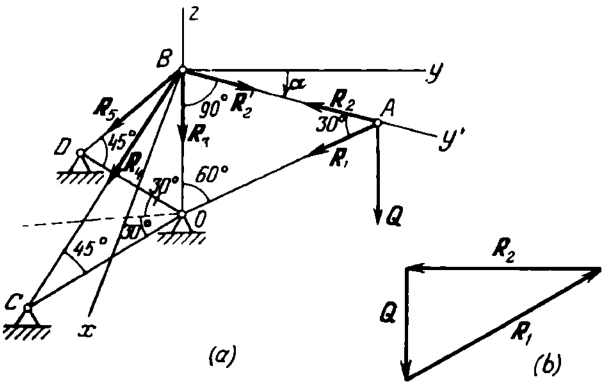

Fig. 1.33

let us suppose that both the rods are subjected to tension; in this case the directions of R_1 and R_2 are as shown in Fig. 1.33*a*. All these concurrent forces lie in one vertical plane *OAB*. Consequently, we can write only two equations of equilibrium for the joint *A*; taking *BA* as the axis *By'* we write

$$\sum Y' = R_1 \cos 150^\circ - R_2 = 0$$

$$\sum Z = -Q + R_1 \cos 120^\circ = 0$$

The solution of these equations yields

$$R_1 = -\frac{Q}{\cos 60^\circ} = -2Q, \quad R_2 = -R_1 \cos 30^\circ = \sqrt{3}\, Q$$

Since R_1 has turned out to be negative ($R_1 < 0$), the rod *AO* is compressed. As to the rod *AB*, it is in tension. This can also be checked by constructing the force triangle for the forces Q, R_1 and R_2 (Fig. 1.33*b*).

Now let us consider the equilibrium of the joint *B*. Since the rod *AB* undergoes tension, its reaction $R_2' = -R_2$ applied to the joint *B* is directed inside the rod. Besides, the joint *B* is acted upon by the reaction R_3 of the pillar *OB* and the reactions R_4 and R_5 of the legs *BC* and *BD*. Let us again assume that all the rods are in tension. Here we have a spatial system of concurrent forces applied to the joint *B*. Let us write three equations of equilibrium for this system taking into account that $R_2' = R_2 = \sqrt{3}Q$:

$$\sum X = \sqrt{3}\, Q \cos (90^\circ - \alpha) + R_4 \cos 45^\circ \cos 60^\circ + R_5 \cos 45^\circ \cos 120^\circ = 0 \qquad (1)$$

$$\sum Y = \sqrt{3}\, Q \cos \alpha + R_4 \cos 45^\circ \cos 150^\circ + R_5 \cos 45^\circ \cos 150^\circ = 0 \qquad (2)$$

$$\sum Z = -R_3 + R_4 \cos 135^\circ + R_5 \cos 135^\circ = 0 \qquad (3)$$

Rewriting equations (1) and (2) in the form

$$\frac{\sqrt{2}}{4}(R_4 - R_5) = -\sqrt{3}\, Q \sin \alpha, \quad -\frac{\sqrt{6}}{4}(R_4 + R_5) = -\sqrt{3}\, Q \cos \alpha$$

we find from them that

$$R_4 = \sqrt{2}\, Q\, (\cos \alpha - \sqrt{3} \sin \alpha), \quad R_5 = \sqrt{2}\, Q\, (\cos \alpha + \sqrt{3} \sin \alpha)$$

Then equation (3) yields

$$R_3 = -\frac{\sqrt{2}}{2}(R_4 + R_5) = -2Q\cos\alpha < 0$$

whence we conclude that the pillar OB is compressed; as to the legs CB and DB, they can undergo tension or compression depending on the value of the angle α. It is not difficult to determine the greatest and the smallest values of R_4 and R_5; however, here we shall not dwell on these calculations.

Problems

PROBLEM 1.1. A body of weight P lies on a smooth inclined plane forming an angle α with the horizon (Fig. 1.34). What force T forming an angle β with the inclined plane should be applied to the weight P for the latter to be in equilibrium ($\alpha + \beta < 90°$)? Solve the problem both graphically and analytically.

Hint. The weight P is under the action of three forces; the third force applied to P is the normal reaction N of the plane.

Answer. $T = \frac{\sin\alpha}{\cos\beta}P$.

PROBLEM 1.2. The beam AB shown in Fig. 1.35 has an immovable hinged support at the end A and a movable hinged support (a roller) at the end B. At the point D of the beam a force F is applied. Determine the reactions of the supports at the points A and B.

Hint. Use the theorem on three forces and compute the angle α, after which the problem can be solved both graphically or analytically.

Answer. $R_A = R_B = \frac{\sqrt{3}}{3}F = 0.577F \quad (\alpha = 30°)$.

PROBLEM 1.3. The beam AB (Fig. 1.36) has a hinged support at the point A and is also supported by the rod CD. At the points C and D there are hinged joints, at the point B a force $F = 50$ kN is applied; $AC = CB$ and $\angle ACD = 90°$. Determine the reactions of the joints A and C neglecting the weight of the beam.

Hint. The reaction of the hinged joint C is directed along the rod CD. To determine the direction of the reaction R_A of the joint A apply the theorem on three forces.

Answer. $R_A = 50$ kN, $R_C = 70.7$ kN.

PROBLEM 1.4. Let us consider the system shown in Fig. 1.37. The rods CA and CB are mutually perpendicular and lie in a horizontal plane. Determine the stresses S_{CA} and S_{CB} in the rods CA and CB and the tension T_{CD} of the chain CD if the lengths of the rods CA and CB and of the chain CD are a, b and d respectively, the construction supporting the weight Q.

Answer. $S_{CA} = -\frac{a}{OD}Q$

$S_{CB} = -\frac{b}{OD}Q$, $T_{CD} = \frac{d}{OD}Q$, where

$OD = \sqrt{d^2 - a^2 - b^2}$.

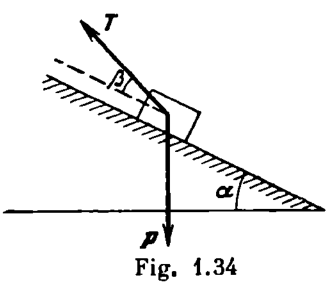

Fig. 1.34

PROBLEM 1.5. The tripod $ABCD$ stands on a smooth horizontal floor (Fig. 1.38). Its legs are of equal length l and are connected at the points B, C and D by a

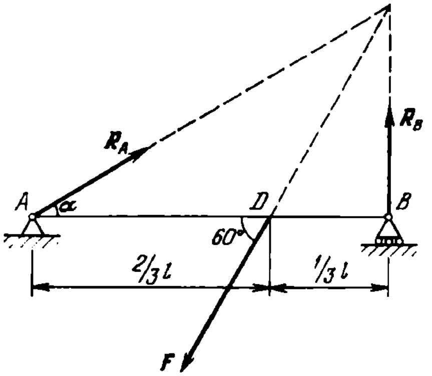

Fig. 1.35

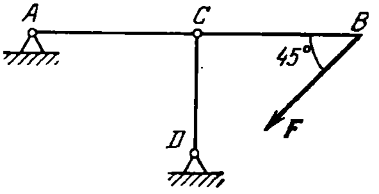

Fig. 1.36

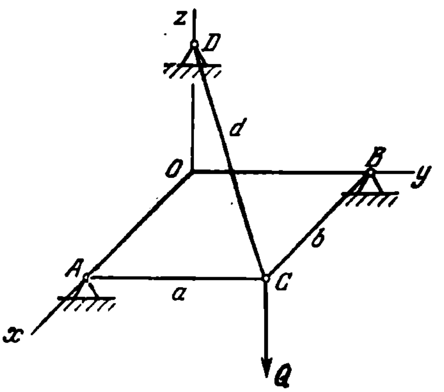

Fig. 1.37

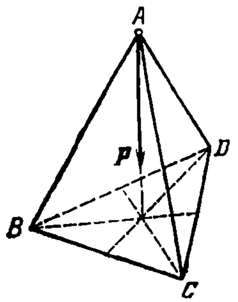

Fig. 1.38

thread BCD forming an equilateral triangle whose sides are equal to l. From the point A a weight P is suspended. Find the stresses S_{AB}, S_{AC} and S_{AD} in the legs of the tripod.

Answer. $S_{AB}=S_{AC}=S_{AD}=-\dfrac{\sqrt{6}}{6}P=-0.408P.$

Chapter 2 System of Two Parallel Forces. Theory of Couples in the Plane

§ 1. System of Two Parallel Forces

A force system is called *plane* if all the forces lie in one plane.

A special case of a plane system of forces is a system of concurrent forces lying in one plane; the rules for composition of concurrent forces and the conditions for their equilibrium were considered in

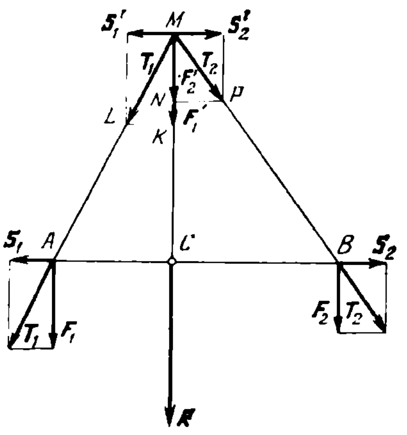

Fig. 2.1

the foregoing section. In Chap. 3 we shall study the general case of a plane system of forces, that is a system of forces placed in an arbitrary manner in one plane. In the present chapter we shall consider a special case of a plane system of forces, namely a system of two parallel forces.

1.1. Composition of Two Parallel Forces. In contrast to the case of two forces intersecting at one point, two parallel forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ applied at points A and B of a rigid body cannot be added together using directly the parallelogram law.

Let us apply at the points A and B of the body two forces S_1 and S_2 equal in their magnitude and having opposite directions (Fig. 2.1). We remind the reader that, according to Axiom 2 (see Sec. 2.5 of Chap. 1), such an operation is allowable. Adding the forces $\boldsymbol{F}_1$ and S_1, and also $\boldsymbol{F}_2$ and S_2 with the aid of the parallelogram law, we arrive at two forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ whose lines of action meet at the point M. Let us transfer the forces $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ to the point M and then resolve them into the components $\boldsymbol{F}_1'$, S_1' and $\boldsymbol{F}_2'$, S_2' as is shown in the figure. Discarding the forces S_1' and S_2' we obtain the two forces $\boldsymbol{F}_1'$ and $\boldsymbol{F}_2'$ lying in one straight line. Hence, the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ applied at the points A and B are equivalent to the forces $\boldsymbol{F}_1'$ and $\boldsymbol{F}_2'$ applied at M; the sum $\boldsymbol{R}$ of these forces is directed along the same line, and $R = F_1' + F_2'$. The point C through which the line of action of the force $\boldsymbol{R}$ passes divides the line segment AB into two parts whose lengths are inversely proportional to the moduli of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$:

$$AC = \frac{F_2}{R} AB, \quad BC = \frac{F_1}{R} AB, \text{ that is } \frac{AC}{BC} = \frac{F_2}{F_1} \tag{2.1}$$

Indeed, the similarity of the triangles ACM and LKM implies that

$$\frac{AC}{S_1} = \frac{CM}{F_1}, \text{ that is } AC \cdot F_1 = CM \cdot S_1 \tag{2.2}$$

From the similarity of the triangles BCM and PNM it follows that

$$\frac{BC}{S_2} = \frac{CM}{F_2}, \text{ that is } BC \cdot F_2 = CM \cdot S_2 \tag{2.3}$$

From (2.3) and (2.2) we obtain $AC \cdot F_1 = BC \cdot F_2$, whence

$$\frac{AC}{F_2} = \frac{BC}{F_1} = \frac{AC + BC}{F_2 + F_1} = \frac{AB}{R}$$

The last relation implies formulas (2.1).

Thus, *two parallel forces* $\boldsymbol{F}_1$ *and* $\boldsymbol{F}_2$ *possess a resultant* $\boldsymbol{R}$ *which is parallel to them and has the same direction, the magnitude of* R *being equal to the sum of the moduli of the given forces. The line of action of the resultant* $\boldsymbol{R}$ *passes through the point* C *dividing the line segment* AB *into parts whose lengths are inversely proportional to the moduli* F_1 *and* F_2 *of the forces* $\boldsymbol{F}_1$ *and* $\boldsymbol{F}_2$.

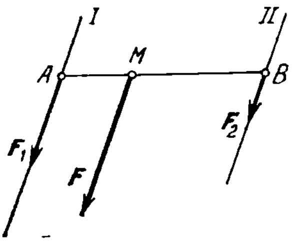

Fig. 2.2

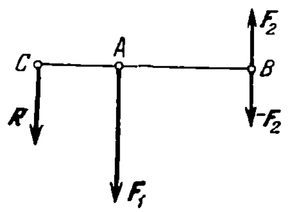

Fig. 2.3

The result we have obtained makes it possible to solve the problem of resolving a given force $\boldsymbol{F}$ applied at a point M into two components $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ along straight lines I and II parallel to the force $\boldsymbol{F}$ and lying on different sides of that force (Fig. 2.2). To this end we draw through the point M a straight line intersecting the straight lines I and II; since a force can be transferred along its line of action, we can assume that the components $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are applied at the points of intersection, that is at A and B. The moduli of the components can be found from relations (2.1) in which R should be replaced by F and C by M:

$$F_1 = \frac{BM}{AB} F, \quad F_2 = \frac{AM}{AB} F \quad (F_1 + F_2 = F) \tag{2.4}$$

1.2. Case of Antiparallel Forces. Let two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ applied at points A and B of a rigid body have parallel lines of action and opposite directions along these lines as is shown in Fig. 2.3. Two forces of this kind are said to be *antiparallel.* Now let us impose an additional condition (which was not introduced in the case of two parallel forces): let the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ have unequal moduli. For definiteness, let us suppose that $F_1 > F_2$.

We shall consider the composition of the two antiparallel forces. Let us resolve the force $\boldsymbol{F}_1$ having the greater modulus into two parallel components; however, in this case the resolution is performed under conditions somewhat different from those stated at the end of Sec. 1.1. Namely, in Sec. 1.1 there were given two straight lines I and II, and in the present case we set one of the components $-\boldsymbol{F}_2$ applied at the point B. Then the modulus of the other component (we denote it by $\boldsymbol{R}$) is equal to $F_1 - F_2$, the point C of its applica-

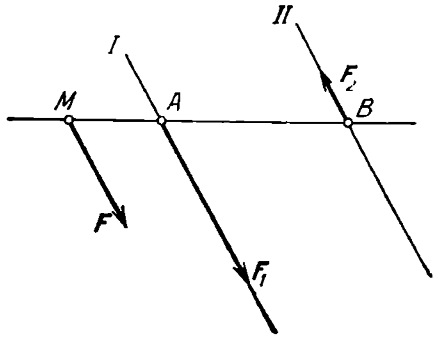

Fig. 4

tion being determined by the proportion

$$\frac{CA}{F_2}=\frac{AB}{F_1-F_2} \tag{2.5}$$

The forces $\boldsymbol{F}_2$ and $-\boldsymbol{F}_2$ with the points of application at B form a balanced force system which, by Axiom 2, can be discarded (see Sec. 2.5 of Chap. 1). The operations we have performed can be written in the form

$$\{\boldsymbol{F}_1,\ \boldsymbol{F}_2\} \sim \{\boldsymbol{R},\ -\boldsymbol{F}_2,\ \boldsymbol{F}_2\} \sim \boldsymbol{R}$$

Thus, the given system of two antiparallel forces is equivalent to the resultant $\boldsymbol{R}$ whose line of action passes through the point C. From (2.5) we obtain the derived proportion:

$$\frac{CA+AB}{F_2+(F_1-F_2)}=\frac{CA}{F_2}, \text{ that is } \frac{CB}{CA}=\frac{F_1}{F_2} \tag{2.6}$$

Hence, *two antiparallel forces* $\boldsymbol{F}_1$ *and* $\boldsymbol{F}_2$ *having different moduli* ($F_1 > F_2$) *possess a resultant* $\boldsymbol{R}$ *whose direction coincides with that of the force having a greater modulus, the modulus of* $\boldsymbol{R}$ *being equal to the difference of the moduli of the given forces:*

$$R = F_1 - F_2 \tag{2.7}$$

The point of intersection of the line of action of the resultant $\boldsymbol{R}$ with the straight line passing through the points of application of the given forces divides the line segment joining these points externally in inverse proportion to the moduli of the given forces (see (2.6)).

Now let us pass to the problem of resolving a given force $\boldsymbol{F}$ into two components $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ along two given straight lines I and II parallel to the line of action of the force $\boldsymbol{F}$ and lying on one side of that line (Fig. 2.4). Let us draw a straight line through the point M of application of the force $\boldsymbol{F}$; it intersects lines I and II at some points A and B. Let it be the component $\boldsymbol{F}_1$ whose line of action is closer to that of $\boldsymbol{F}$; then the modulus of $\boldsymbol{F}_1$ is greater than that of

F_2, and therefore F_1 has the same direction as F; the other component F_2 has the opposite direction. The moduli of the components are determined from proportions (2.5) and (2.6) which can be written in the form

$$\frac{F_1}{BM}=\frac{F_2}{AM}=\frac{F}{AB}$$

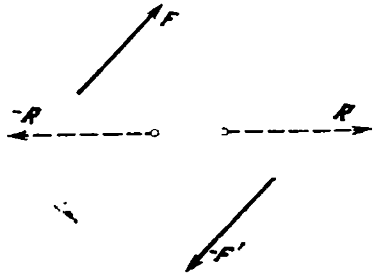

Fig. 2.5

whence

$$F_1=\frac{BM}{AB}F, \quad F_2=\frac{AM}{AB}F \quad (F_1-F_2=F) \qquad (2.8)$$

The problem considered here and the problem solved in Sec. 1.1 (the composition of parallel forces) provide the solution to the problem of balancing two parallel forces by a third force. As was mentioned in Sec. 2.4 of Chap. 1, if a system of forces possesses a resultant then the force having the same modulus as the resultant and the opposite direction along the same straight line is the equilibrant (balancing) force. In the general case the problem of determining the equilibrant force also reduces to the determination of the resultant.

1.3. Couple of Forces. It now remains to consider the case excluded in Sec. 1.2, namely a system of two antiparallel forces (F and $-F'$) having equal moduli: $F = F'$ (Fig. 2.5). Such a system of two forces is referred to as a *couple of forces* or, simply, a *couple.*

LEMMA. *A couple of forces has no resultant.*

Proof. If we supposed that the resultant exists and applied formulas (2.7) and (2.5), this would yield

$$R=0, \quad CA=\frac{AB}{0}$$

The last relations show that the resultant is equal to zero and passes through the point at infinity. This means that the couple has no resultant parallel to the forces forming the couple. Now let us suppose that a couple of forces F, $-F'$ has a resultant R which is not parallel to the forces of the couple (see Fig. 2.5). Then adding the force $-R$ to the force system F, $-F'$ we arrive at the system F, $-F'$, $-R$ of three balanced forces. By the theorem on three forces (see Sec. 2.6 of Chap. 1), the lines of action of these three forces must meet at one point, which is impossible. This contradiction shows that there exists no resultant which is not parallel to the forces of the couple. The lemma is proved.

This lemma is equivalent to the assertion that a couple of forces cannot be balanced by one force. The forces F and $-F'$ forming

the couple are not balanced themselves either because they do not lie in one straight line (see Axiom 1 in Sec. 2.5 of Chap. 1). *A couple applied to a rigid body produces rotation of the body provided that the constraints imposed on the body do not prevent the rotation.* A couple of forces, like a force, is an independent mechanical quantity and is irreducible to one force. The quantitative characteristic of couples and the operations on them will be studied in the next section.

§ 2. Theory of Couples in the Plane

2.1. Moment of a Force about a Point. The moment of a force is one of the basic notions of statics. Given a force $\boldsymbol{F}$ and a point O (Fig. 2.6), let us draw a plane through the point O and the force $\boldsymbol{F}$ and then drop the perpendicular OP from the point O on the line of action AB of the force $\boldsymbol{F}$. The length of this perpendicular is called the *arm of the force* **F** *with respect to the point* (*centre*) O.

By definition, *the moment of the force* **F** *about the point* O *is equal to the product of the modulus* F *of the force by the arm* h *taken with plus or minus sign*:

$$\mathrm{mom}_O\, \boldsymbol{F} = \pm Fh \tag{2.9}$$

The moment of a force is considered *positive* if the plane of the force rotates counterclockwise under the action of the force about the centre (for instance, see Fig. 2.6). If otherwise, that is if the plane rotates clockwise, the moment of the force is considered *negative* and in formula (2.9) the minus sign is taken. The dimension of the moment of a force is the product of unit force by unit length.

In the International System of Units (SI) (see Sec. 2.3 of Chap. 1) the unit measure for the moment of a force is N·m.

REMARKS.

(a) From the definition it follows that the moment of a force does not change when the point of application of that force is transferred along the line of action of the force.

(b) If $h = 0$ then $\mathrm{mom}_O\, \boldsymbol{F} = 0$; it follows that the moment of a nonzero force about a point is equal to zero if and only if the line of action of the force passes through the centre O.

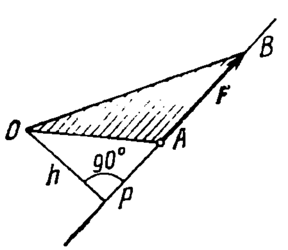

Fig. 2.6

(c) The absolute value of the moment of a force can be represented as twice the area of the triangle with the force as a side and the centre as a vertex:

$$|\mathrm{mom}_O\, \boldsymbol{F}| = Fh = 2S_{\Delta OAB}$$

(d) Using the notion of the moment of a force we can easily determine the point of application C of the resultant of two

parallel forces. Proportion (2.1) can be written in the form

$$F_1 \cdot AC = F_2 \cdot BC$$

which means that

$$|\mathrm{mom}_C \boldsymbol{F}_1| = |\mathrm{mom}_C \boldsymbol{F}_2|$$

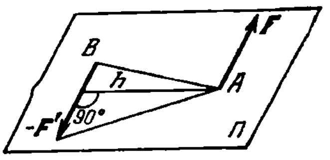

Fig. 2.7

Taking into account the signs of the moments of the forces we obtain

$$\mathrm{mom}_C \boldsymbol{F}_1 + \mathrm{mom}_C \boldsymbol{F}_2 = 0 \tag{2.10}$$

In the case of antiparallel forces we obtain from proportion (2.6) the equality

$$F_1 \cdot AC = F_2 \cdot BC$$

which again leads to equality (2.10). Thus, the point of application of the resultant of two parallel (or antiparallel) forces can be found from the condition that the sum of the moments of the given forces about the sought-for point should turn into zero.

2.2. Moment of a Couple. Let us consider a couple of forces $\boldsymbol{F}$ and $-\boldsymbol{F}'$ (Fig. 2.7); the plane Π in which the forces forming the couple lie will be referred to as the *plane of action of the couple*. The distance h between the lines of action of the forces of the couple is called the *arm of the couple*.

By the moment m of the couple $\boldsymbol{F}$, $-\boldsymbol{F}'$ is meant *the product of the modulus of one of the forces of the couple by the arm taken with plus or minus sign*:

$$m = \pm Fh \tag{2.11}$$

If the couple tends to turn the plane of its action counterclockwise in formula (2.11) the sign + is taken. If otherwise, that is if the rotation is clockwise, the sign — is taken. For instance, the moment of the couple shown in Fig. 2.5 is negative while the moment of the couple in Fig. 2.7 is positive.

From Fig. 2.7 and formulas (2.9) and (2.11) we conclude that

$$m = \mathrm{mom}_A (-\boldsymbol{F}') = \mathrm{mom}_B \boldsymbol{F} \tag{2.12}$$

This means that the moment of a couple is equal to the moment of one of the forces forming the couple about the point of application of the other force. The absolute value of the moment of a couple is equal to twice the area of the triangle whose base is one of the forces of the couple and whose height is the arm of the couple. The moment of a couple is measured in the same units as the moment of a force.

2.3. Theorem on Equivalent Couples Lying in One Plane. Here we shall prove an important theorem which, together with the corol-

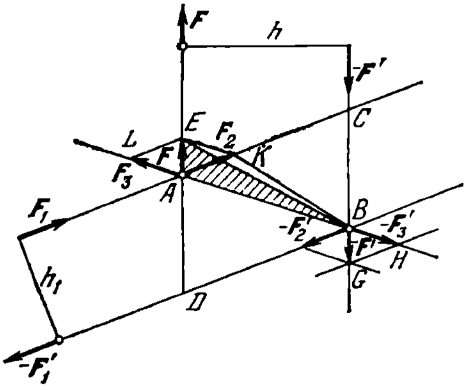

Fig. 2.8

laries it implies, determines the basic properties of the couples in the plane.

THEOREM. *Two couples lying in one plane with the same absolute values of their moments and rotating the plane in the same direction are equivalent.*

Proof. We shall consider two couples $\boldsymbol{F}$, $-\boldsymbol{F}'$ and $\boldsymbol{F}_1$, $-\boldsymbol{F}_1'$ lying in one plane. Let the moments of these couples be equal, that is

$$F_1h_1 = Fh \tag{2.13}$$

and let the directions in which they tend to rotate the plane be the same.

(a) We shall first suppose that the forces $\boldsymbol{F}$ and $\boldsymbol{F}_1$ are not parallel (Fig. 2.8). To prove the equivalence of the couples we transform the couple $\boldsymbol{F}$, $-\boldsymbol{F}'$ into the couple $\boldsymbol{F}_1$, $-\boldsymbol{F}_1'$ using elementary operations (see Sec. 2.4 of Chap. 1). Let us transfer the forces $\boldsymbol{F}$ and $-\boldsymbol{F}'$ to the points A and B and resolve them into the components $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ and $-\boldsymbol{F}_2'$, $-\boldsymbol{F}_3'$ respectively. It is evident that $\boldsymbol{F}_2 = \boldsymbol{F}_2'$ and $\boldsymbol{F}_3 = \boldsymbol{F}_3'$. Since the forces $\boldsymbol{F}_3$ and $-\boldsymbol{F}_3'$ are mutually balanced they can be discarded, which implies that the couple $\boldsymbol{F}$, $-\boldsymbol{F}'$ is equivalent to the couple $\boldsymbol{F}_2$, $-\boldsymbol{F}_2'$:

$$\{\boldsymbol{F}, -\boldsymbol{F}'\} \sim \{\boldsymbol{F}_2, -\boldsymbol{F}_2'\} \tag{2.14}$$

It now remains to prove that $F_2 = F_1$. To this end we shall show that the moment of the couple $\boldsymbol{F}_2$, $-\boldsymbol{F}_2'$ is equal to that of the couple $\boldsymbol{F}$, $-\boldsymbol{F}'$. Indeed, the absolute value of the moment of the couple $\boldsymbol{F}$, $-\boldsymbol{F}'$ is equal to twice the area of the triangle BAE (this triangle is shaded in Fig. 2.8), and that of the moment of the couple $\boldsymbol{F}_2$, $-\boldsymbol{F}_2'$ is equal to twice the area of the triangle BAK. Since these triangles have a common base AB and equal altitudes (because the vertices E

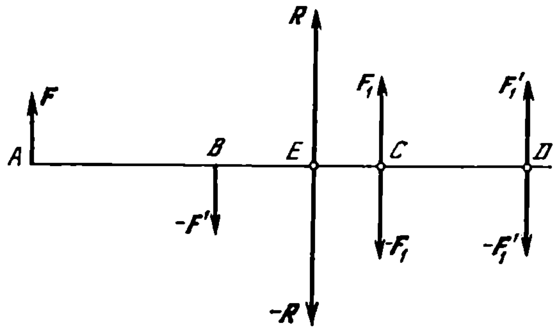

Fig. 2.9

and K lie on a straight line parallel to the base) they are of the same area.

Expressing the moments under consideration in terms of the products of the moduli of the forces by the arms we obtain

$$F_2 h_1 = Fh$$

The comparison of the last equality with (2.13) yields

$$F_2 h_1 = F_1 h_1, \quad \text{that is } F_2 = F_1$$

which means that

$$\{\boldsymbol{F}_2, -\boldsymbol{F}_2'\} \sim \{\boldsymbol{F}_1, -\boldsymbol{F}_1'\}$$

and hence, by virtue of (2.14), we have

$$\{\boldsymbol{F}, -\boldsymbol{F}'\} \sim \{\boldsymbol{F}_1, -\boldsymbol{F}_1'\}$$

For Case (a) the theorem is proved.

(b) Now let us suppose that the forces $\boldsymbol{F}$ and $\boldsymbol{F}_1$ are parallel (Fig. 2.9); as before, the moments of the couples $\boldsymbol{F}, -\boldsymbol{F}'$ and $\boldsymbol{F}_1, -\boldsymbol{F}_1'$ are equal:

$$F \cdot AB = F_1 \cdot CD \tag{2.15}$$

In order to prove the equivalence of the couples we shall again transform the couple $\boldsymbol{F}, -\boldsymbol{F}'$ into the couple $\boldsymbol{F}_1, -\boldsymbol{F}_1'$ using elementary operations (see Sec. 2.4 of Chap. 1). To this end let us apply at the points C and D the balanced forces $\boldsymbol{F}_1, -\boldsymbol{F}_1$ and $\boldsymbol{F}_1', -\boldsymbol{F}_1'$ respectively. Adding together the forces $\boldsymbol{F}$ and $\boldsymbol{F}_1'$ we obtain the force $\boldsymbol{R}$ ($R = F + F_1$) applied at the point E, and

$$F \cdot AE = F_1 \cdot ED$$

Let us subtract equality (2.15) from the last equality:

$$F\,(AE - AB) = F_1\,(ED - CD), \quad \text{that is } F \cdot BE = F_1 \cdot EC \tag{2.16}$$

Now we add together the forces $-\boldsymbol{F}'$ and $-\boldsymbol{F}$. The modulus of their resultant is equal to that of $\boldsymbol{R}$; the direction of the resultant is opposite to that of $\boldsymbol{R}$ and the point of its application is at E, which fol-

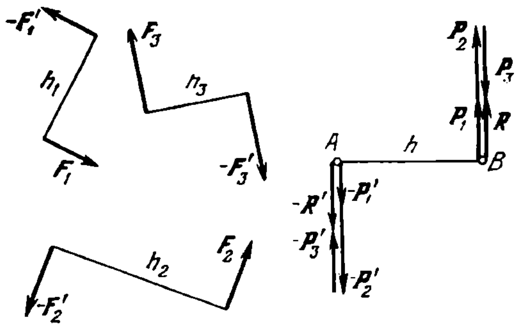

Fig. 2.10

lows from equality (2.16). Since the forces R and $-R$ are mutually balanced, we can discard them; then the remaining forces F_1 and $-F_1'$ form the required couple.

The operations we have performed can be written in the form

$$\{F, -F'\} \sim \{F, -F'\} + \{F_1, -F_1\} + \{F_1', -F_1'\} \sim \{F, F_1'\}$$
$$+ \{-F', -F_1\} + \{F_1, -F_1'\} \sim \{R, -R\} + \{F_1, -F_1'\} \sim \{F_1, -F_1'\}$$

Thus, Case (b) is proved, which completes the proof of the theorem.

We see that a couple of forces considered as an independent mechanical quantity is characterized in its plane of action only by the magnitude of its moment (considered with the corresponding sign).

From the above theorem it follows that: (1) a couple can be transferred in its plane (as a "rigid construction") without changing its action on a body; (2) the action of a couple does not change when the magnitudes of the forces forming the couple and of the arms are varied so that the absolute value of the moment (that is the product of the force by the arm) and the direction of rotation of the couple remain unchanged.

2.4. Composition of Couples Lying in One Plane. Equilibrium Condition for a Plane System of Couples. Now we shall consider the problem of composition of couples in the plane.

Let there be three couples F_1, $-F_1'$; F_2, $-F_2'$ and F_3, $-F_3'$ with arms h_1, h_2 and h_3 respectively (Fig. 2.10). The moments of these couples are denoted by m_1, m_2 and m_3:

$$m_1 = F_1h_1, \quad m_2 = F_2h_2, \quad m_3 = -F_3h_3$$

Now we construct an arbitrary line segment $AB = h$ and reduce the given couples to it. The forces P_1, P_2 and P_3 forming the couples having the arm h and equivalent to the given couples are found from the equalities

$$m_1 = P_1h, \quad m_2 = P_2h, \quad m_3 = -P_3h$$

whence

$$P_1 = \frac{m_1}{h} = \frac{h_1}{h} F_1, \quad P_2 = \frac{m_2}{h} = \frac{h_2}{h} F_2, \quad P_3 = \frac{-m_3}{h} = \frac{h_3}{h} F_3$$

The directions of the forces $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ are along one straight line, and the modulus of their resultant $\boldsymbol{R}$ is

$$R = P_1 + P_2 - P_3$$

Let us denote by $-\boldsymbol{R}'$ the resultant of the forces $-\boldsymbol{P}_1'$, $-\boldsymbol{P}_2'$ and $-\boldsymbol{P}_3'$:

$$R' = P_1' + P_2' - P_3'$$

The modulus of $-\boldsymbol{R}'$ is equal to that of $\boldsymbol{R}$; this force is parallel to $\boldsymbol{R}$ and has opposite direction. Thus, the couples $\boldsymbol{F}_1$, $-\boldsymbol{F}_1'$; $\boldsymbol{F}_2$, $-\boldsymbol{F}_2'$ and $\boldsymbol{F}_3$, $-\boldsymbol{F}_3'$ are reduced to one couple $\boldsymbol{R}$, $-\boldsymbol{R}'$ which is called the *resultant couple of forces.*

The moment m of the resultant couple is

$$m = Rh = (P_1 + P_2 - P_3)\,h = m_1 + m_2 + m_3 \qquad (2.17)$$

Hence, *the moment of the resultant couple is equal to the algebraic sum of the moments of the constituent couples.*

If $P_1 + P_2 - P_3 = 0$ then also $P_1' + P_2' - P_3' = 0$, that is the system of forces applied at the points A and B is balanced. Consequently, the couples $\boldsymbol{F}_1$, $-\boldsymbol{F}_1'$; $\boldsymbol{F}_2$, $-\boldsymbol{F}_2'$ and $\boldsymbol{F}_3$, $-\boldsymbol{F}_3'$ equivalent to that system are also in equilibrium. Since we have $m = 0$, it follows that

$$m_1 + m_2 + m_3 = 0$$

Thus, the given couples are in equilibrium when the algebraic sum of their moments is equal to zero. This is the *equilibrium condition for couples lying in one plane.** The result we have established can easily be generalized to any number of couples.

EXAMPLE 2.1. The left end A of a horizontal beam AB has an immovable hinged support and its right end B lies on a movable roller; the distance between the supports is $AB = l$ (Fig. 2.11). The beam is acted upon by a couple of forces with moment $m_1 < 0$. Determine the reactions of the supports.

Solution. We shall consider the equilibrium of the beam. Let us replace the actions of the constraints by the corresponding reaction forces $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$. Thus, the beam is acted upon by a couple of forces with moment m_1 and the forces $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$. A couple of forces can only be balanced by another couple of forces, and therefore the

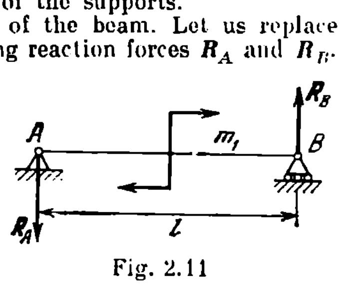

Fig. 2.11

* More precisely, this is the equilibrium condition for couples lying in one plane or in parallel planes; for greater detail see Sec. 1.3 of Chap. 5.

reactions R_A and R_B must form a couple with moment $-m_1$. The reaction R_B of the movable horizontal roller is directed vertically upward, and consequently the reaction R_A must have the same modulus as R_B and be directed vertically downward:

$$R_A = -R_B$$

The moment of the couple R_A, R_B is positive, which is quite natural because this couple must balance the given couple with moment $m_1 < 0$. Now let us write down the equilibrium condition for the couples:

$$\sum m = m_1 + R_B l = 0$$

From this condition we find the moduli of the sought-for reactions:

$$R_A = R_B = -\frac{m_1}{l} > 0$$

Problems

PROBLEM 2.1. At what point A of the tree shown in Fig. 2.12 a rope of length l should be tied in order to fell the tree using the smallest force?

Hint. The arm h of the force of tension T about the base O of the tree, considered as a function of the angle α, must assume its greatest value.

Answer. $\alpha = 45°$, $OA = l \sin \alpha = 0.707l$.

PROBLEM 2.2. Two homogeneous bars AB and BC ($BC = 2AB$) forming an angle of 90° are rigidly connected (Fig. 2.13). The end A is suspended from the thread. The weight of the bar AB is equal to P and the weight of the bar BC is equal to $2P$. Determine the tension T of the thread OA and the angle α for the equilibrium of the system.

Answer. $T = 3P$, $\tan \alpha = \frac{4}{5}$.

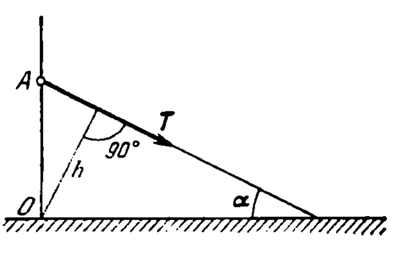

Fig. 2.12

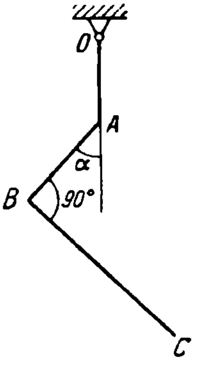

Fig. 2.13

Chapter 3 Plane System of Forces

If all the forces acting on a body lie in one plane the system of these forces is said to be *plane*. Let us consider an arbitrary plane system of forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$, that is a system of forces which are arbitrarily located in a plane. Transferring the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ to the point of intersection of their lines of action and adding them together according to the parallelogram law we obtain their resultant $\boldsymbol{R}_{12} = \boldsymbol{F}_1 + \boldsymbol{F}_2$. If the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are parallel their resultant is found according to the composition rule for parallel (or antiparallel) forces. Further, adding together the forces $\boldsymbol{R}_{12}$ and $\boldsymbol{F}_3$ in the same manner we find their resultant $\boldsymbol{R}_{123}$, etc. Here we must stipulate the special case when the given forces form a couple. In this case all the couples we obtain can be added together according to the rule stated in Sec. 2.4 of Chap. 2.

In the general case, the consecutive composition of all the forces and all the couples results in one resultant force vector $\boldsymbol{R}'$ and one resultant couple $\boldsymbol{P}, -\boldsymbol{P}'$. In particular, it may turn out that the moment of the resultant couple is equal to zero or that the resultant force vector is equal to zero or both the resultant vector and the resultant moment are equal to zero. In these cases the given system of forces is equivalent to the resultant force or to the resultant couple or is equivalent to zero (in the latter case the force system is balanced) respectively.

However, in the general case this method of composition of forces is inconvenient. Instead of it, we shall make use of a more general and convenient method (both in the sense of qualitative results and of mechanical interpretation of the problem).

§ 1. Reduction of a Plane System of Forces to a Given Centre

1.1. Lemma. *A force* $\boldsymbol{F}$ *applied at a point* A *is equivalent to a force* $\boldsymbol{F}'$, *"geometrically"* equal to the force* $\boldsymbol{F}$ *and applied at a point* O, *and the couple* $\boldsymbol{F}, -\boldsymbol{F}'$.

Proof. Let $\boldsymbol{F}$ be the force applied at the point A (Fig. 3.1). Let us apply at the point O two forces $\boldsymbol{F}'$ and $-\boldsymbol{F}'$ which are parallel to the force $\boldsymbol{F}$, have the same moduli as $\boldsymbol{F}$ but are of opposite directions. The system of the three forces $\boldsymbol{F}, \boldsymbol{F}', -\boldsymbol{F}'$ thus obtained, which is equivalent to the force $\boldsymbol{F}$ applied at the point A, can be regarded as consisting of the force $\boldsymbol{F}'$, geometrically equal to the force $\boldsymbol{F}$ and applied at the point O, and of the couple $\boldsymbol{F}, -\boldsymbol{F}'$. Finally, we can

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* In this statement we stress that the quantities $\boldsymbol{F}$ and $\boldsymbol{F}'$ regarded as free vectors are equal; as to the forces $\boldsymbol{F}$ and $\boldsymbol{F}'$, they are not equivalent.

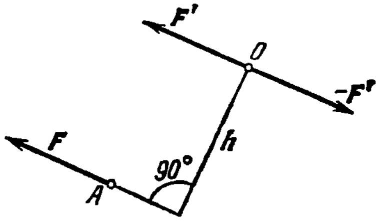

Fig. 3.1

write

$$F \sim F' + \{F, -F'\}$$

The lemma is proved.

The operation of replacing a given force applied at a point A by another force applied at a point O and a couple is called the *reduction of the given force to the point O* (called the *reduction centre*). The couple $\boldsymbol{F}$, $-\boldsymbol{F}'$ is termed the *associated couple*; its moment is equal to $m = \pm Fh$, that is to the moment of the force $\boldsymbol{F}$ with respect to the reduction centre O (see formula (2.12)):

$$m = \mathrm{mom}_O \boldsymbol{F} \tag{3.1}$$

1.2. The Poinsot* Method. Resultant Force Vector and Resultant Moment. For the sake of simplicity we shall suppose that there are three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ applied at points A_1, A_2 and A_3 respectively (Fig. 3.2). Let us take an arbitrary point O in the plane of these forces and apply at O three pairs of (pairwise) mutually balanced forces: $\boldsymbol{F}_1'$ and $-\boldsymbol{F}_1'$; $\boldsymbol{F}_2'$ and $-\boldsymbol{F}_2'$ and $\boldsymbol{F}_3'$ and $-\boldsymbol{F}_3'$. The system of forces obtained in this way which is equivalent to the given force system can be regarded as the collection of three forces $\boldsymbol{F}_1'$, $\boldsymbol{F}_2'$ and $\boldsymbol{F}_3'$ applied at the point O and the three associated couples $\boldsymbol{F}_1$, $-\boldsymbol{F}_1'$; $\boldsymbol{F}_2$, $-\boldsymbol{F}_2'$ and $\boldsymbol{F}_3$, $-\boldsymbol{F}_3'$, that is

$$\{\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3\} \sim \{\boldsymbol{F}_1', \boldsymbol{F}_2', \boldsymbol{F}_3'\} + \{\boldsymbol{F}_1, -\boldsymbol{F}_1'\} + \{\boldsymbol{F}_2, -\boldsymbol{F}_2'\} + \{\boldsymbol{F}_3, -\boldsymbol{F}_3'\}$$

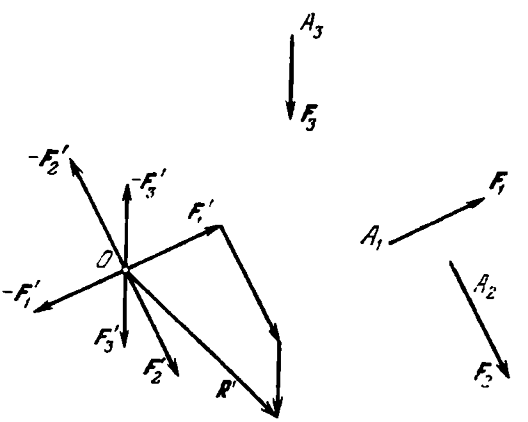

Fig. 3.2

* Poinsot, L. (1777-1859), a French mathematician.

The composition of the forces $\boldsymbol{F}_1'$, $\boldsymbol{F}_2'$ and $\boldsymbol{F}_3'$ according to the polygon law yields the resultant force vector $\boldsymbol{R}'$ applied at the point O:

$$\boldsymbol{R}' = \boldsymbol{F}_1' + \boldsymbol{F}_2' + \boldsymbol{F}_3' = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \sum \boldsymbol{F}_\nu \tag{3.2}$$

The vector $\boldsymbol{R}'$, equal to the vector sum of the given forces and applied at the point O, is called the *resultant* (or *principal*) *force vector* of the given system of forces. The composition of the associated couples according to the rule stated in Sec. 2.4 of Chap. 2 yields the resultant couple $\boldsymbol{P}$, $-\boldsymbol{P}'$ with the moment

$$m_O = m_1 + m_2 + m_3$$
$$= \text{mom}_O \boldsymbol{F}_1 + \text{mom}_O \boldsymbol{F}_2 + \text{mom}_O \boldsymbol{F}_3 = \sum \text{mom}_O \boldsymbol{F}_\nu \tag{3.3}$$

The scalar m_O is called the *resultant* (or *principal*) *moment of the given plane system of forces about the reduction centre O*. Here m_1, m_2 and m_3 are the moments of the associated couples which, according to (3.1), are equal to the moments of the corresponding given forces about the centre O. The operations we have performed can be represented in the form

$$\{\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3\} \sim \boldsymbol{R}' + \{\boldsymbol{P}, -\boldsymbol{P}'\}$$

The resultant couple $\boldsymbol{P}$, $-\boldsymbol{P}'$ is not shown in Fig. 3.2. Figuratively speaking, the resultant couple represents a rotational tendency distributed over the plane, and is not uniquely determined by its moment (see Sec. 2.3 of Chap. 2). By the way, below (in the proof of Varignon's theorem) we shall represent the resultant couple.

All that has been said remains valid for any number of forces. Thus, in the general case a plane force system is equivalent to one resultant force vector $\boldsymbol{R}'$ (see (3.2)), applied at an arbitrary point O, and one resultant couple with a moment equal to the resultant moment m_O (see (3.3)). The method of composition of forces in the plane described here is known as the *Poinsot method of reduction of a plane force system to a given centre.*

The above description of the method implies that the resultant (principal) force vector is independent of the choice of the reduction centre while, generally speaking, the resultant (principal) moment depends on that choice because when the position of the reduction centre is changed the arms of the associated couples and the signs of their moments are also changed. The formula connecting the resultant moment about a new reduction centre with the resultant moment about the old reduction centre will be derived in Sec. 2.2 of Chap. 5. In the present chapter we shall do without this formula, which will not violate the rigorousness of the presentation of the material.

Now we shall pass to some special cases of reduction of a plane system of forces.

1.3. Case of Reduction of a Plane System of Forces to One Couple. We shall begin with the situation when the resultant force vector is equal to zero, that is $R' = 0$ (this means that the force polygon constructed for the given forces is closed), while the resultant moment is different from zero: $m_O \neq 0$. In this case the given force system is equivalent to the resultant couple P, $-P'$ with moment m_O determined by formula (3.3). It is this case when a plane system of forces is reduced to one couple. If we change the reduction centre and take a new centre O_1 then the resultant force vector will be again equal to zero and the moment m_{O_1} of the resultant couple P_1, $-P_1'$ will be again equal to m_O. Indeed, if we had $m_{O_1} \neq m_O$ this would mean that in the special case under consideration the two couples P, $-P'$ and P_1, $-P_1'$, which are equivalent to one and the same given force system, would not be equivalent to each other, which is impossible.

1.4. Varignon's* Theorem. The second special case of reduction of a plane system of forces we are going to consider is the one in which the resultant force vector is different from zero: $R' \neq 0$. Here we must discuss separately the following two possibilities.

(a) For the chosen reduction centre O the *resultant moment is equal to zero*: $m_O = 0$. This means that the given system of forces is equivalent to one force R' applied at the point O. In this case the resultant force vector R' should be called the *resultant* of the given system of forces. Here particular attention should be paid to the essential distinction between the general term "the resultant force vector" and "the resultant of the given system of forces"; to stipulate this distinction for the case under consideration we shall drop the prime and write

$$\{F_1, F_2, \ldots, F_n\} \sim R$$

Thus, if the resultant force vector is different from zero while the resultant moment of the given forces about the point O is equal to zero the system of forces is equivalent to its resultant

$$R = \sum F_\nu$$

whose line of action passes through the point O.

(b) For the chosen reduction centre O the *resultant moment is different from zero*: $m_O \neq 0$. We shall show that in this case as well the given force system is equivalent to the resultant whose line of action, however, no longer passes through the point O. Let us construct the resultant couple so that one of the forces forming the couple, say $-R'$, is applied at the point O, its direction being opposite to that of the resultant force vector R' (Fig. 3.3). The arm h of

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* Varignon, P. (1654-1722), a French mathematician.

the resultant couple is found from the equality

$$m_O = \pm R'h \qquad (3.4)$$

which results in

$$h = \frac{|m_O|}{R'}$$

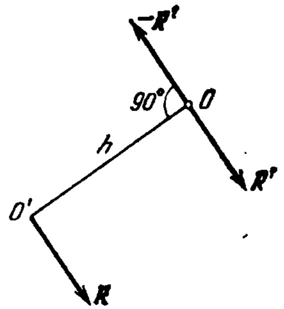


Fig. 3.3

Then the other force $\boldsymbol{R}$ of the resultant couple is applied at a point O' coinciding with the end of the perpendicular of length h to the resultant force vector $\boldsymbol{R}'$ erected at the point O in such a direction that the sense of the rotation produced by the force $\boldsymbol{R}$ corresponds to the sign of the resultant moment m_O. For instance, Fig. 3.3 corresponds to the case when $m_O > 0$. Discarding the forces $\boldsymbol{R}'$ and $-\boldsymbol{R}'$ applied at the point O we conclude that the given system of forces is equivalent to one force, that is to its resultant

$$\boldsymbol{R} = \boldsymbol{R}' = \sum \boldsymbol{F}_\nu$$

applied at the point O'

VARIGNON'S THEOREM (for a plane system of forces). *If the resultant of a plane system of forces exists then its moment about any centre is equal to the algebraic sum of the moments of all the forces, forming that system, about the same centre*:

$$\mathrm{mom}_O\, \boldsymbol{R} = \sum \mathrm{mom}_O\, \boldsymbol{F}_\nu \qquad (3.5)$$

Proof. From Fig. 3.3 we see that the moment of the resultant $\boldsymbol{R}$ about the point O is equal to $\pm Rh$; hence, according to (3.4), we have

$$m_O = \mathrm{mom}_O\, \boldsymbol{R}$$

On the other hand, by formula (3.3), the resultant moment m_O is equal to the algebraic sum of the moments of all the given forces about the same point O:

$$m_O = \sum \mathrm{mom}_O\, \boldsymbol{F}_\nu$$

The comparison of the last two equalities implies formula (3.5), which completes the proof of the theorem.

In Varignon's theorem it is assumed that the resultant of the given plane system of forces exists; as was shown in Secs. 1.3 and 1.4, this is the case if and only if the resultant force vector of the force system is different from zero.

It now remains to consider the third (the last) special case in which both the resultant force vector and the resultant moment of a plane system of forces about a centre O are equal to zero. The whole following section is devoted to that case.

§ 2. Equilibrium Conditions for a Plane System of Forces

2.1. Three Forms of a System of Equilibrium Equations. As was shown in the foregoing section, in the general case a plane system of forces is equivalent to a resultant force vector $\boldsymbol{R}'$ and a resultant couple with moment m_O. If both the resultant force vector $\boldsymbol{R}'$ and the resultant moment m_O are equal to zero then the resultant force and the resultant couple are equivalent to zero and the given system of forces is balanced. If at least one of the two quantities $\boldsymbol{R}'$ and m_O is different from zero then, as was shown in Secs. 1.3 and 1.4, the plane system of forces is equivalent either to the resultant couple or to the resultant force. Consequently, the necessary and sufficient equilibrium conditions for a plane system of forces* are

$$\boldsymbol{R}' = 0, \qquad m_O = 0 \tag{3.6}$$

It should be stressed that in the state of equilibrium the resultant moment with respect to any *arbitrary* point in the plane must be equal to zero. Indeed, if the resultant moment about a point O_1 turned out to be different from zero ($m_{O_1} \neq 0$) and if $\boldsymbol{R}' = \boldsymbol{0}$, then the balanced system of forces would be equivalent to the resultant couple with moment m_{O_1}, which is impossible.

Let us project the resultant force vector $\boldsymbol{R}'$ on the coordinate axes Ox and Oy (in the general case the coordinates must not necessarily be rectangular); then, taking into account (3.2) and (3.3), we can write the equilibrium conditions for the plane system of forces in the analytical form:

$$R'_x \equiv \sum X_\nu = 0, \quad R'_y \equiv \sum Y_\nu = 0, \quad m_O \equiv \sum \mathrm{mom}_O \boldsymbol{F}_\nu = 0 \tag{3.7}$$

Thus, we have proved that

1. *For a plane system of forces to be balanced it is necessary and sufficient that*

(a) *the sum of the projections of all the forces of the system on each of the two arbitrarily chosen coordinate axes should be equal to zero* and

(b) *the sum of the moments of all the forces about any arbitrarily chosen centre in the plane should be equal to zero.*

In (3.7) the equilibrium conditions are expressed by three equations: two of them are written for the projections and the third for the moments. Below we shall derive two other forms of equilibrium conditions for a plane system of forces.

Let us suppose now that the sums of the moments of all the forces about each of two arbitrary points A and B and the sum of the projections of the forces on an axis Ax not perpendicular to AB are equal

* It would be more precise to speak of "equilibrium conditions for a perfectly rigid body under the action of a plane system of forces".

to zero:

$$\sum \text{mom}_A \boldsymbol{F}_\nu = 0$$
$$\sum \text{mom}_B \boldsymbol{F}_\nu = 0, \ \sum X_\nu = 0 \quad (3.8)$$

We shall prove that in this case the system of forces is balanced. Let us suppose that the given system of forces is reduced to a centre A. Then, by the first of the above conditions, the resultant moment about the centre A is equal to zero:

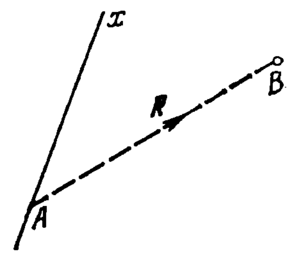


Fig. 3.4

$$m_A = \sum \text{mom}_A \boldsymbol{F}_\nu = 0$$

This means that the given system of forces is equivalent to a resultant $\boldsymbol{R}$ applied at the centre A (Fig. 3.4). We shall show that $\boldsymbol{R} = \boldsymbol{0}$. By Varignon's theorem (3.5) and the second condition (3.8), we have

$$\text{mom}_B \boldsymbol{R} = \sum \text{mom}_B \boldsymbol{F}_\nu = 0$$

Consequently, either $\boldsymbol{R} = \boldsymbol{0}$ or the line of action of $\boldsymbol{R}$ passes through the point B. Let us assume that $\boldsymbol{R} \neq \boldsymbol{0}$ and that the line of action of $\boldsymbol{R}$ passes through B. The vector $\boldsymbol{R}$ is shown by the dash line in Fig. 3.4; this assumption immediately leads to a contradiction to the third condition (3.8) because in this case

$$R_x \equiv \sum X_\nu \neq 0$$

Hence, if conditions (3.8) are fulfilled then $\boldsymbol{R} = \boldsymbol{0}$, and therefore the given system of forces is balanced. From the necessity of conditions (3.6) it follows that conditions (3.8) are not only sufficient but also necessary. Thus, we have proved that

2. *For a plane system of forces to be balanced it is necessary and sufficient that*

(a) *the sum of the moments of all the forces about each of two arbitrary points in the plane should be equal to zero* and

(b) *the sum of the projections of all the forces on any axis* (this axis must not be perpendicular to the straight line joining the points about which the moments are taken) *should be equal to zero.*

In (3.8) the equilibrium conditions are expressed by two equations for the moments and one equation for the projections.

Finally, let us suppose that the sums of the moments of all the forces about three points A, B and C not lying in one straight line are equal to zero:

$$\sum \text{mom}_A \boldsymbol{F}_\nu = 0, \quad \sum \text{mom}_B \boldsymbol{F}_\nu = 0, \quad \sum \text{mom}_C \boldsymbol{F}_\nu = 0 \qquad (3.9)$$

We shall prove that in this case as well the given system of forces is balanced. Let us suppose that the system of forces is reduced to the

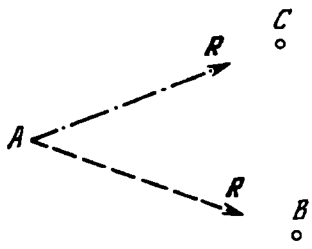


Fig. 3.5

point A. Using the first condition (3.9) we obtain

$$m_A = 0$$

The second condition (3.9) yields

$$\text{mom}_B \boldsymbol{R} = \sum \text{mom}_B \boldsymbol{F}_\nu = 0$$

Consequently, either $\boldsymbol{R} = \boldsymbol{0}$ or $\boldsymbol{R} \neq \boldsymbol{0}$, and the line of action of $\boldsymbol{R}$ passes through the point B. Let us assume that $\boldsymbol{R} \neq \boldsymbol{0}$ and that the line of action of $\boldsymbol{R}$ passes through B; the vector $\boldsymbol{R}$ is shown by the dash line in Fig. 3.5. By virtue of Varignon's theorem (3.5) and the third condition (3.9), we obtain

$$\text{mom}_C \boldsymbol{R} = \sum \text{mom}_C \boldsymbol{F}_\nu = 0$$

and, since $\boldsymbol{R} \neq \boldsymbol{0}$, the line of action of $\boldsymbol{R}$ must also pass through the point C (see the dot-dash line in Fig. 3.5). But this is impossible because the points B and C do not lie in one straight line passing through the point A. We have thus arrived at a contradiction, and consequently $\boldsymbol{R} = \boldsymbol{0}$, that is the given system of forces is balanced. From the necessity of conditions (3.6) it follows that conditions (3.9) are not only sufficient but also necessary for the equilibrium. We have thus proved the following *theorem on three moments:*

3. *For a plane system of forces to be balanced it is necessary and sufficient that the sum of the moments of all the forces about each of three arbitrary points in the plane not lying in one straight line should be equal to zero.*

In (3.9) the equilibrium conditions are expressed by three equations for the moments.

2.2. Special Cases of a Plane System of Forces. Let us consider the equilibrium of a rigid body under the action of a plane system of forces. For every force system we have a definite number of equilibrium equations. In the general case of an arbitrary plane system of forces there are three such equations, and they can be represented in three different forms (see Sec. 2.1). Now we shall consider two important special cases.

(a) *Plane system of concurrent forces.* This is the case when the lines of action of all the forces lying in one plane meet at one point O. The moment of each of the forces about that point O is equal to zero. Therefore the last equation of system (3.7) turns into identity, and the equilibrium conditions for the plane system of concurrent forces are expressed by the first two equations

$$\sum X_\nu = 0, \qquad \sum Y_\nu = 0 \tag{3.10}$$

These equations can also be derived from system of equations (1.28) because the projections of the forces lying in one plane on the axis Oz perpendicular to that plane are equal to zero.

(b) *Plane system of parallel forces.* Let the lines of action of all the forces lying in one plane be parallel to the axis Ox (Fig. 3.6). The projection of each of the forces on the axis Oy perpendicular to Ox is equal to zero. In this case the second equation of system (3.7) turns into identity, and the equilibrium conditions for the plane system of parallel forces are expressed by the first and the third equations

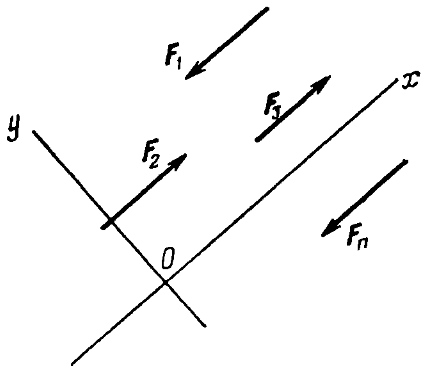


Fig. 3.6

$$\sum X_\nu = 0, \qquad \sum \mathrm{mom}_O \boldsymbol{F}_\nu = 0 \tag{3.11}$$

It should be noted that in this case the left-hand member of the first equation (3.11) is the algebraic sum of the forces. This means that the moduli of the forces parallel to the axis Ox enter into this sum with plus sign and the moduli of the antiparallel forces with minus sign.

EXAMPLE 3.1. Let us consider the crane shown in Fig. 3.7. The weight of the crane without the counterbalance is $\boldsymbol{G}$. The line of action of the force $\boldsymbol{G}$ passes at a distance c from the left rail A. The width of the platform AB is equal to a. The lifting capacity of the crane trolley is $\boldsymbol{P}$, and the boom of the crane arm is equal to l. Determine the smallest counterbalance Q and the greatest distance x from the counterbalance to the vertical passing through the right rail B for which the crane does not overturn in all the positions of the trolley when it is both loaded and unloaded.

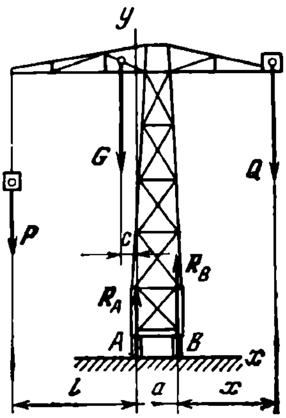


Fig. 3.7

Solution. Let us consider the equilibrium of the crane. To this end we replace the actions of the rails by their reactions $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$. The crane is under the action of a system of antiparallel forces: the forces of weight $\boldsymbol{p}$, $\boldsymbol{G}$ and $\boldsymbol{Q}$ directed vertically downward and the reactions of the rails $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ directed vertically upward.

Case One. The crane trolley with its gib in horizontal position carries the maximum load, that is $p = P$. Since there arises the danger that the crane may overturn to the left (about the point A) we write the second equation

(3.11) in the form

$$\sum \mathrm{mom}_A F = Pl + Gc - Q(x+a) + R_B a = 0$$

From this equation we obtain

$$Pl + Gc - Q(x+a) = -R_B a$$

It should be noted that $R_B \geqslant 0$ and that the reaction R_B turns into zero only at the instant when the crane starts overturning to the left. Therefore we have

$$Pl + Gc - Q(x+a) \leqslant 0, \quad \text{that is} \quad Pl + Gc - Qa \leqslant Qx \qquad (1)$$

Case Two. The crane trolley is unloaded ($p = 0$). In this case there arises the danger that the crane may overturn to the right (about the point B); therefore we write the second equation (3.11) in the form

$$\sum \mathrm{mom}_B F = G(c+a) - Qx - R_A a = 0$$

whence

$$G(c+a) - Qx = R_A a$$

Since $R_A \geqslant 0$ and the sign of equality appears only at the instant when the crane starts overturning to the right, we have

$$G(c+a) - Qx \geqslant 0, \quad \text{that is} \quad Qx \leqslant G(c+a) \qquad (2)$$

Inequalities (1) and (2) can be written as

$$Pl + Gc - Qa \leqslant Qx \leqslant G(c+a)$$

For these inequalities to be compatible the inequality

$$Pl + Gc - Qa \leqslant G(c+a)$$

must hold, whence

$$Q \geqslant \frac{Pl - Ga}{a} = Q_{\min}$$

It is the right-hand member of the last inequality that expresses the smallest magnitude of the counterbalance. In particular, if

$$Pl - Ga \leqslant 0$$

it is possible to do without the counterbalance. In what follows we assume that $Pl > Ga$. From inequality (2) we obtain

$$x \leqslant \frac{G(c+a)}{Q}$$

The substitution of $Q_{\min}$ into the denominator of the right-hand member of the last inequality yields the greatest possible distance from the line of action of the counterbalance to the right rail:

$$x_{\max} = \frac{G(c+a)}{Q_{\min}} = \frac{G(c+a)}{Pl - Ga} a$$

In both the cases we have not used the first equation (3.11) because the determination of the unknown reaction forces is not required in the statement of the problem. For given p, Q and x we have two equations (3.11) at our disposal for determining the moduli of the reactions R_A and R_B; these equations make it possible to find the reactions and the corresponding forces of pressure of the crane on the rails which have the same moduli as the reactions and the opposite directions.

(c) *Equilibrium condition for a lever.* A *lever* is a rigid body which can rotate about a fixed axis O under the action of forces $F_1, F_2, \ldots, F_n$ lying in the plane Oxy perpendicular to the axis of rotation (Fig. 3.8). The simplest examples of levers are the well-known levers of the first and of the second kind. Let us consider the equilibrium of the lever shown in Fig. 3.8; to the given forces we add the components X_O and Y_O of the reactions of the hinge O. These components do not enter into the third equation (3.7), and therefore it is the equality

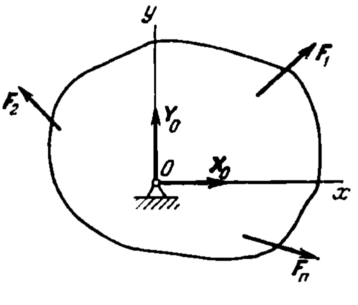


Fig. 3.8

$$\sum_{\nu=1}^{n} \mathrm{mom}_O \boldsymbol{F}_\nu = 0 \qquad (3.12)$$

that expresses the equilibrium condition for the lever. In the first two equations (3.7) let us separate out the projections X_O and Y_O of the reactions of the hinge O:

$$X_O + \sum_{\nu=1}^{n} X_\nu = 0, \qquad Y_O + \sum_{\nu=1}^{n} Y_\nu = 0$$

From these equations we find

$$X_O = -\sum_{\nu=1}^{n} X_\nu, \qquad Y_O = -\sum_{\nu=1}^{n} Y_\nu$$

2.3. Statically Determinate and Statically Indeterminate Systems. Before passing to examples we note that the sequence of operations recommended for the solutions of problems on equilibrium of plane systems of forces is the same as in the case of concurrent forces. First of all we decide for what body the equilibrium should be considered. Then we draw a figure in which both the given and the sought-for forces are shown (including the reactions of constraints) and choose the coordinate system, after which the equilibrium equations are formed. It is advisable to write the equations for the moments about the point at which the greatest number of unknown forces intersect. We can use any of the three forms of equilibrium equations (see Sec. 2.1), but it is desirable to make them as simple as possible.

A system for which the number of the unknown forces (in case the direction of a force is also unknown the components of the force are counted separately) is equal to that of independent equations of equilibrium that can be written for the construction in question is

called *statically determinate*. When the number of the unknown forces exceeds that of independent equations of equilibrium written for the given construction (that is for a rigid body, a system of bodies, etc.; see below Sec. 2.4) the system in question is said to be *statically indeterminate*.

For instance, let us consider the equilibrium of a horizontal beam AB to which two given forces P_1 and P_2 are applied (Fig. 3.9). The left end A has an immovable hinged support and the right end B has a movable hinged support (a roller). It is required to determine the reactions of the supports. Here there are three unknown components of the reactions (see Sec. 2.9 of Chap. 1): X_A, Y_A, and N_B, and we have three equations of equilibrium at our disposal. Consequently the system is statically determinate.

Now let us consider the equilibrium of the horizontal beam AC shown in Fig. 3.10. It is important that the whole beam AC is not built up of two joined parts. At the point B the beam has an additional movable hinged support (roller). Here the number of the unknown components of the reactions is equal to four: X_A, Y_A, N_C and N_B; as to the number of the independent equilibrium equations, it remains equal to three. Therefore this whole beam with three supports represents an example of a statically indeterminate system. The problem of determining the reactions of the supports of such a beam can be solved with the aid of the methods of strength of materials taking into account the bending of the beam.

Determining the number of the unknown forces and comparing it with the number of equilibrium equations we can decide, without solving the problem, whether the system we deal with is statically determinate or, conversely, statically indeterminate.

EXAMPLE 3.2. A ladder AB forming an angle φ with the horizon reclines on a smooth vertical wall, the weight of the ladder being P. At the point D, at a distance of 1/3 of the length of the ladder from its upper end, a person of weight Q stands. Find the forces of pressure the ladder exerts on the wall and on the support A (Fig. 3.11).

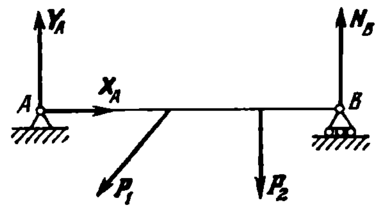


Fig. 3.9

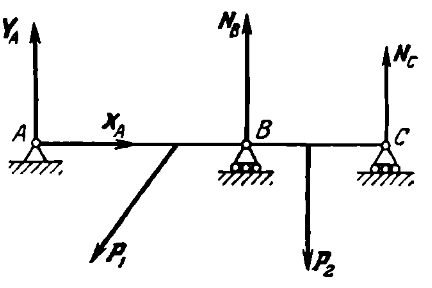


Fig. 3.10

Solution. We shall consider the equilibrium of the ladder AB. Let us replace the action of the constraints by their reactions: the reaction N_B perpendicular to the wall (since the wall is smooth) and the reaction R_A of the support A. Both the modulus and the direction of the reaction R_A are unknown, and therefore we replace R_A by two components X_A and Y_A along the axes of the chosen coordinate system Oxy. We shall write the equilibrium equations in form (3.7). However, before writing them we must find the arms h_1, h_2 and h_3 of the forces N_B, Q and P about the point A (see Fig. 3.11):

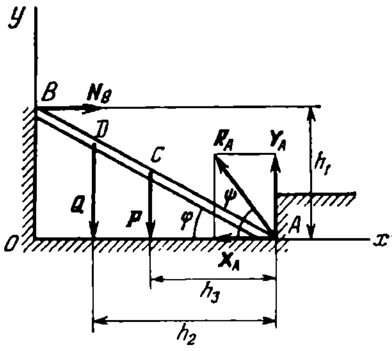


Fig. 3.11

$$h_1 = l \sin\varphi, \quad h_2 = \frac{2}{3} l \cos\varphi, \quad h_3 = \frac{1}{2} l \cos\varphi$$

where $l = AB$ is the length of the ladder.

Thus the equilibrium equations have the form

$$\sum X = N_B + X_A = 0, \qquad \sum Y = -Q - P + Y_A = 0$$

$$\sum \mathrm{mom}_A\, F = Q \cdot \frac{2}{3} l \cos\varphi + P \cdot \frac{1}{2} l \cos\varphi - N_B l \sin\varphi = 0$$

Cancelling the last equation by l we determine the reaction N_B from it:

$$N_B = \left(\frac{2}{4} Q + \frac{1}{2} P\right) \cot\varphi$$

From the first two equations we obtain

$$X_A = -N_B = -\left(\frac{2}{3} Q + \frac{1}{2} P\right) \cot\varphi, \qquad Y_A = Q + P$$

The minus sign in the expression for the X_A component indicates that the projection of the force R_A on the axis Ox is negative, and consequently the component X_A is directed to the left as is shown in Fig. 3.11. The modulus and the direction of the reaction R_A are determined by the formulas

$$R_A = \sqrt{X_A^2 + Y_A^2} = \sqrt{\left(\frac{2}{3} Q + \frac{1}{2} P\right)^2 \cot^2\varphi + (Q+P)^2}$$

$$\tan\psi = \frac{Y_A}{|X_A|} = \frac{6(Q+P)}{4Q+3P} \tan\varphi$$

where ψ is the angle the reaction R_A forms with the negative direction of the axis Ox.

EXAMPLE 3.3. A homogeneous bar AD of length $2l$ and weight P rests on the angle A between the wall and the floor, and at the point B it rests on the angle of another wall (Fig. 3.12). The distances are indicated in the figure, and it is known that $l < \sqrt{a^2 + b^2} < 2l$. Determine the reactions.

Solution. We shall consider the equilibrium of the bar AD. Let us replace the action of the constraints by their reactions: the reaction R_B perpendicular to the bar and the reaction R_A equal to the vector sum of two normal components X_A and Y_A.

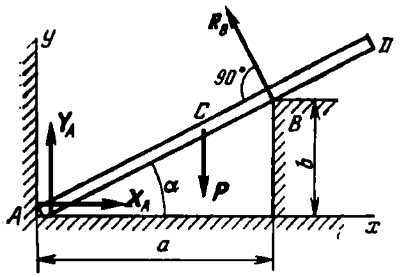


Fig. 3.12

For the coordinate system shown in Fig. 3.12 we form the equilibrium equations:

$$\sum X = X_A + R_B \cos(90° + \alpha) = 0$$

$$\sum Y = Y_A - P + R_B \cos\alpha = 0$$

$$\sum \text{mom}_A F = -PAC\cos\alpha + R_B AB = 0,$$

From the equation of the moments we immediately find the modulus of the reaction R_B:

$$R_B = \frac{AC}{AB} P\cos\alpha = \frac{la}{a^2+b^2} P$$

because

$$AC = l, \quad AB = \sqrt{a^2+b^2}, \quad \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$$

From the first equilibrium equation we find

$$X_A = R_B \sin\alpha = \frac{lab}{(a^2+b^2)\sqrt{a^2+b^2}} P$$

and, finally, the second equilibrium equation yields

$$Y_A = P - R_B\cos\alpha = \left[1 - \frac{la^2}{(a^2+b^2)\sqrt{a^2+b^2}}\right] P$$

The modulus of the reaction R_A and its direction are determined by the formulas

$$R_A = \sqrt{X_A^2 + Y_A^2}, \quad \tan(\widehat{R_A, x}) = \frac{Y_A}{X_A}$$

The problem of Example 3.3 can also be solved graphically by using the theorem on three forces (see Sec. 2.6 of Chap. 1) and then constructing the force triangle on the forces P, R_A and R_B.

EXAMPLE 3.4. One of the ends of a homogeneous bar AB of weight $Q = 200$ N rests on a smooth horizontal floor while its other end lies on a smooth inclined plane forming an angle of 45° with the horizon (Fig. 3.13). The end B of the bar is supported by a rope passing around the pulley C and carrying a weight P; the part BC of the rope is parallel to the inclined plane. Determine the magnitudes of the weight P and of the forces of pressure exerted by the bar on the floor and on the inclined plane neglecting the friction in the pulley.

Solution. We shall consider the equilibrium of the bar AB. Here Q is the given force; it is applied at the midpoint of the bar AB and is directed vertically downward. The sought-for forces are the reactions N_A and N_B and the weight P. Since the horizontal and the inclined planes are supposed to be smooth the reactions N_A and N_B are perpendicular to them. The force P is directed along the part BC of the rope. The chosen directions of the coordinate axes are shown in the figure.

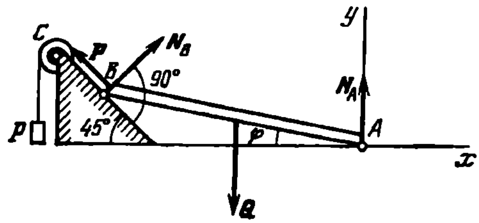


Fig. 3.13

Let us set equilibrium equations (3.7). The sum of the projections of all the forces on the axis Ax is equal to zero:

$$\sum X = N_B \cos 45° + P \cos 135° = 0 \tag{a}$$

The sum of the projections of all the forces on the axis Ay is equal to zero:

$$\sum Y = -Q + N_A + N_B \cos 45° + P \cos 45° = 0 \tag{b}$$

Finally, the sum of the moments of all the forces about the point B is equal to zero:

$$\sum \text{mom}_B\, F = -Q\,\frac{AB}{2} \cos \varphi + N_A AB \cos \varphi = 0 \tag{c}$$

From (c) we find $N_A = Q/2 = 100$ N, and (a) yields $N_B = P$. Substituting these expressions into (b) we obtain

$$2N_B \cos 45° = Q - N_A$$

whence

$$N_B = P = \frac{100}{2 \cos 45°} = 70.7 \text{ N}$$

EXAMPLE 3.5. The ends A and B of a homogeneous bar AB of weight P are supported by smooth horizontal and vertical planes respectively (Fig. 3.14). To the end A of the bar a thread is attached which goes around a pulley and carries a load of weight Q at the end. The friction in the pulley is negligible. Determine the angle α between the bar and the horizontal plane for the state of equilibrium of the bar.

Solution. Let us consider the equilibrium of the bar AB. Here the given forces are the force P applied at the midpoint of the bar AB and the tension T of the thread whose modulus is equal to Q: $T = Q$. Since both supporting planes are smooth their reactions N_A and N_B are perpendicular to them.

From equilibrium equations (3.7) we choose the third; let us take the sum of the moments of the forces about the point of intersection of the lines of action of the reactions N_A and N_B:

$$\sum \text{mom}_C\, F = -P\,\frac{l}{2} \cos \alpha + Ql \sin \alpha = 0$$

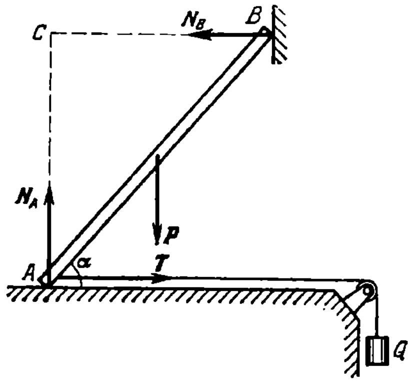


Fig. 3.14

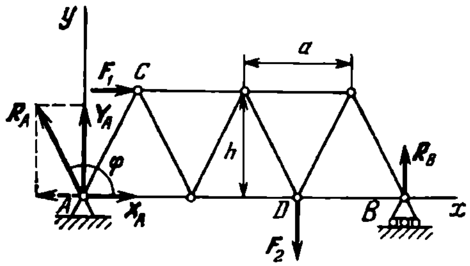


Fig. 3.15

where l denotes the length of the bar. From this equation we find the tangent of the sought-for angle α:

$$\tan\alpha = \frac{P}{2Q}$$

Using the remaining first two equilibrium equations (3.7) we can determine N_A and N_B.

EXAMPLE 3.6. A truss shown in Fig. 3.15 has an immovable hinged support at the point A and a movable hinged support at the point B. The lengths of the horizontal rods are the same and are equal to $a = 8$ m. The height of the truss is $h = 6$ m. To the joints C and D of the truss a horizontal force $F_1 = 2$ kN and a vertical force $F_2 = 10$ kN are applied respectively. Find the reactions of the supports.

Solution. The reaction $\boldsymbol{R}_B$ of the movable support is perpendicular to its plane and is directed upward. The reaction $\boldsymbol{R}_A$ of the immovable support should be replaced by two components $\boldsymbol{X}_A$ and $\boldsymbol{Y}_A$. The coordinate axes are chosen as shown in the figure.

Let us set equilibrium equations (3.8) for the forces acting on the truss. The sum of the projections of all the forces on the axis Ax is equal to zero:

$$\sum X = F_1 + X_A = 0 \tag{a}$$

The sum of the moments of all the forces about the point A is equal to zero:

$$\sum \text{mom}_A\, F = -F_1h - F_2\cdot 2a + R_B\cdot 3a = 0 \tag{b}$$

Finally, the sum of the moments of all the forces about the point B is equal to zero:

$$\sum \text{mom}_B\, F = -F_1h + F_2a - Y_A\cdot 3a = 0 \tag{c}$$

From (a) we find $X_A = -F_1 = -2$ kN. The minus sign indicates that the force $\boldsymbol{X}_A$ has in fact the direction opposite to that shown in the figure. Its real direction is shown by the dash line. From (b) we obtain

$$R_B = \frac{F_1h + 2F_2a}{3a} = 7.17 \text{ kN}$$

Finally, (c) yields

$$Y_A = \frac{F_2a - F_1h}{3a} = 2.83 \text{ kN}$$

The total reaction $\boldsymbol{R}_A$ of the support A has the modulus

$$R_A = \sqrt{X_A^2 + Y_A^2} = 3.47 \text{ kN}$$

and forms the angle

$$\varphi = \arctan \frac{Y_A}{X_A} = \arctan(-1.42) = 125^\circ$$

with the axis Ax.

2.4. Equilibrium of a System of Bodies under the Action of Forces Lying in One Plane. The forces acting on a construction consisting of several perfectly rigid bodies can be divided into two groups. The first group consists of the so-called *external forces* which are the forces with which the bodies not belonging to the given construction act on that construction. This group also includes the reactions of constraints except the reactions appearing in the joints connecting the bodies. The other group consists of the so-called *internal forces* which are the forces of interaction between the bodies entering in the construction. We shall elucidate what has been said by an example.

Let us consider the construction shown in Fig. 3.16; it consists of two bodies: the beams AB and CD lying on the supports A, D and E. The beam AB lies freely on the beam CD. The forces of weight $\boldsymbol{P}$ and $\boldsymbol{Q}$ and also the reactions $\boldsymbol{R}_A$ and $\boldsymbol{R}_D$ of the supports (these reactions are vertical because such are the given active forces) and the reaction $\boldsymbol{R}_E$ are external forces with respect to the whole construction. As to the force of pressure $\boldsymbol{R}_C$ exerted by the beam AB on the beam CD and, similarly, the reaction of the beam CD represented by the force $\boldsymbol{R}_B$, they belong to the group of internal forces. It should be stressed that the force $\boldsymbol{R}_B$ is applied to the beam AB while the force $\boldsymbol{R}_C$ to the beam CD. In such a consideration one should always take into account Axiom 3 (the law of action and reaction; see Sec. 2.5 of Chap. 1); according to this axiom $R_C = R_B$. If we consider the equilibrium of the given construction as a whole then the internal forces do not enter in the equilibrium equations because they "mutually cancel" (this fact is stated more precisely in Axiom 5 in Sec. 2.5 and in the lemma in Sec. 2.7 of Chap. 1).

When considering the equilibrium of a construction consisting of several bodies, besides the study of the state of the construction as a whole, it is also often necessary to partition it and to consider separately the equilibrium of the constituent bodies. The connections

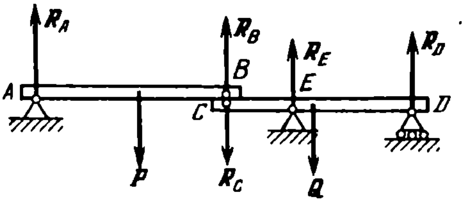


Fig. 3.16

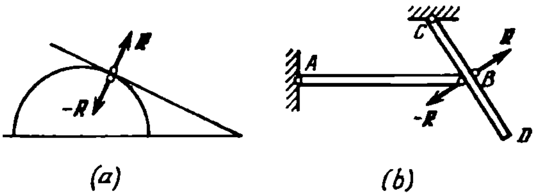


Fig. 3.17

of the bodies in a construction can be of different kinds; below we indicate some of them.

(a) The bodies forming the construction contact one another along smooth surfaces. In this case the internal forces, that is the forces of pressure the bodies exert on one another, are directed along the common normals to the tangents at the points of contact of the bodies (Fig. 3.17*a*) or along the normal to the surface of one body at which it contacts another body (Fig. 3.17*b*).

(b) The bodies are connected together by means of inextensible rods whose ends are attached to the bodies with the aid of hinges or by inextensible flexible threads (Fig. 3.18). If a connecting rod is weightless then the reactions of the rod are directed along it; as to the reaction of a flexible thread, it is always directed along the thread inside it.

(c) The bodies are joined with the aid of a hinge; for instance, see such the joint C in Fig. 3.19. Since the force of pressure exerted by the first body A on the second body B is transmitted by the hinge the direction of this force, that is of the reaction $\boldsymbol{R}_C$ of the hinge (or the force $-\boldsymbol{R}_C$ with which the second body acts on the first) is not known in advance. When solving such problems we usually resolve the reaction of the hinge into two components along the coordinate axes.

EXAMPLE 3.7. A three-hinged arch ABC is under the action of a vertical force $\boldsymbol{P}$. The dimensions are shown in Fig. 3.20*a*. Determine the reactions of the hinges A, B and C.

Solution. The construction whose equilibrium is considered consists of two bodies: the half-arches AB and BC joined by the hinge B. The construction is in equilibrium under the action of the following forces: the external forces, the force $\boldsymbol{P}$ and the reactions $\boldsymbol{R}_A$ and $\boldsymbol{R}_C$ of the hinges A and C, and also the

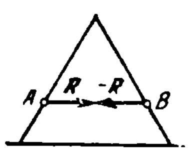


Fig. 3.18

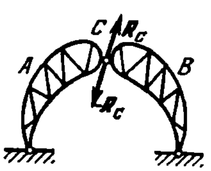


Fig. 3.19

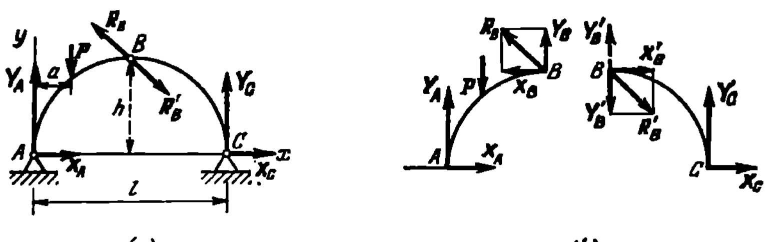


Fig. 3.20

internal forces* R_B and $R'_B = -R_B$, that is the forces of interaction between the half-arches transmitted through the hinge B.

Let us denote by X_A, Y_A and X_C, Y_C the projections of the reactions R_A and R_C on the coordinate axes; then equilibrium equations (3.7) for the whole construction, that is for the three-hinged arch ABC, regarded as a perfectly rigid body have the form

$$\sum X = X_A + X_C = 0, \qquad \sum Y = Y_A + Y_C - P = 0$$

$$\sum \mathrm{mom}_A\, F = -Pa + Y_C \cdot 2l = 0$$

These equations do not involve the internal forces. From the third equation we find

$$Y_C = \frac{Pa}{2l}$$

and then the second equation yields

$$Y_A = P - Y_C = \left(1 - \frac{a}{2l}\right) P$$

The first equation only makes it possible to conclude that

$$X_A = -X_C$$

To determine separately X_A and X_C and also R_B, let us consider the equilibrium of one of the parts of the construction. The scheme of the forces acting on the parts is shown in Fig. 3.20*b*. Usually it is advisable to consider the part of a system which is acted upon by a smaller number of forces. In the case under consideration such is the right half-arch BC. Let us write for this part the equilibrium equations of form (3.7); before the problem has been solved the projections X'_B and Y'_B are supposed to be positive (the component Y'_B is shown by the dash line in Fig. 3.20*b*):

$$\sum X = X'_B + X_C = 0, \qquad \sum Y = Y'_B + Y_C = 0$$

$$\sum \mathrm{mom}_C\, F = -Y'_B l - X'_B h = 0$$

From these equations we find

$$Y'_B = -Y_C = -\frac{Pa}{2l}, \qquad X'_B = -\frac{l}{h}\, Y'_B = \frac{Pa}{2h}, \qquad X_C = -X'_B = -\frac{Pa}{2h}$$

and hence

$$X_A = -X_C = \frac{Pa}{2h}$$

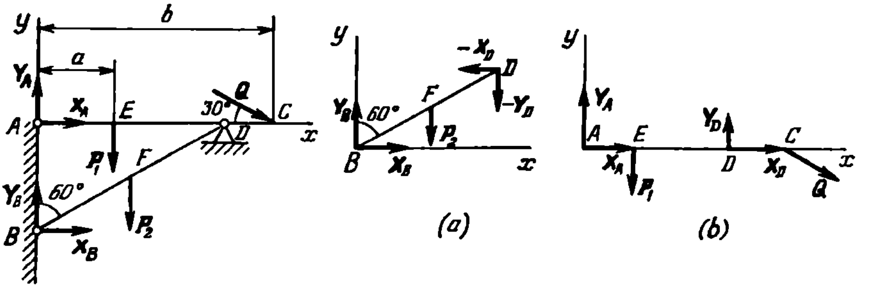


Fig. 3.21

The real direction of the component Y'_B is shown by the continuous line in Fig. 3.20*b*.

We have managed to do without the separate consideration of the equilibrium of the left half-arch AB. We could also consider not the equilibrium of the construction as a whole but that of the left and of the right parts taken separately. Altogether, this would lead to a set of six independent equations involving the six unknowns: X_A, Y_A, X_C, Y_C, X'_B and Y'_B (because the projections of the internal forces satisfy the equalities $X_B = -X'_B$, $Y_B = -Y'_B$).

EXAMPLE 3.8. Two beams AC and BD of equal length are joined with a hinge at the point D and have immovable hinged supports attached to the vertical wall at the points A and B (Fig. 3.21). The beam AC is placed horizontally and the beam BD forms an angle of 60° with the vertical wall. At the point E the beam AC is acted upon by a vertical force $P_1 = 20$ kN and at the point C by a force $Q = 50$ kN inclined at an angle of 30° to the horizon. At the point F the beam BD is acted upon by a vertical force $P_2 = 20$ kN. The other data are: $AE = a = 2$ m and $AC = BD = b = 6$ m. Determine the reactions of the hinges A, B and D.

Solution. The beams AC and BD are in equilibrium under the action of the given forces $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and Q and the unknown forces X_A, Y_A, X_B and Y_B. Since there are four unknowns and since there are only three equilibrium equations that can be written for the whole construction formed by the beams AC and BD, at first glance it may seem that the problem is unsolvable. However, this is not the case.

In order to determine the reactions at the points A and B we use the following procedure. We first write the equilibrium equations for the system of the two beams AC and BD regarded as a whole construction. To this end we project all the forces on the axes Ax and Ay and form the equation for the sum of the moments of the forces about the point A; this yields

$$\sum X = X_A + X_B + Q\cos 30^\circ = 0 \qquad (1)$$

$$\sum Y = -P_1 - P_2 + Q\cos 120^\circ + Y_A + Y_B = 0 \qquad (2)$$

$$\sum \text{mom}_A\, \boldsymbol{F} = -P_1 a - P_2\frac{b}{2}\cos 30^\circ - Qb\sin 30^\circ + X_B b\cos 60^\circ = 0 \qquad (3)$$

To derive one more equation required for the determination of the unknowns we write the equilibrium equation for the beam BD considered separately; to this end we replace the action of the beam AC on the beam BD by the components $\boldsymbol{X}_D$ and $\boldsymbol{Y}_D$ of the reaction $\boldsymbol{R}_D$ applied at the point D (Fig. 3.21*a*). Projecting all the forces applied to the beam BD on the coordinate axes Bx

and By and forming the equation for the sum of the moments of the forces about the point D, we obtain

$$\sum X = X_B - X_D = 0 \tag{4}$$

$$\sum Y = -P_2 + Y_B - Y_D = 0 \tag{5}$$

$$\sum \mathrm{mom}_D F = P_2 \frac{b}{2} \sin 60° + X_B b \sin 30° - Y_B b \sin 60° = 0 \tag{6}$$

Here there are two new unknowns X_D and Y_D which have been added to the former four unknowns; however, there are three new equations of equilibrium that should be added to the former three equations. Thus, we have obtained a system of six equations from which all the six components of the reactions at the points A, B and D can be found.

From equation (3) we obtain

$$X_B = \frac{1}{b \cos 60°} \left(P_1 a + \frac{1}{2} P_2 b \cos 30° + Qb \sin 30° \right) = 80.7 \text{ kN}$$

Equation (1) yields

$$X_A = -X_B - Q \cos 30° = -124 \text{ kN}$$

Further, from equation (6) we find

$$Y_B = \frac{1}{2} P_2 + \frac{\sin 30°}{\sin 60°} X_B = 56.6 \text{ kN}$$

Equation (2) implies

$$Y_A = P_1 + P_2 + Q \sin 30° - Y_B = 8.4 \text{ kN}$$

and, finally, from equations (4) and (5) we obtain

$$X_D = X_B = 80.7 \text{ kN}, \quad Y_D = Y_B - P_2 = 36.6 \text{ kN}$$

A system of equations sufficient for determining the unknown reactions can also be derived in another way by considering separately the equilibrium of the beams BD and AC without using the equilibrium conditions for the whole construction, that is without writing equations (1)-(3). Projecting all the forces applied to the beam AC on the coordinate axes and forming the equation for the sum of the moments of the forces about the point A, we obtain (see Fig. 3.21b):

$$\sum X = X_A + X_D + Q \cos 30° = 0 \tag{7}$$

$$\sum Y = -P_1 + Q \cos 120° + Y_A + Y_D = 0 \tag{8}$$

$$\sum \mathrm{mom}_A F = -P_1 a - Qb \sin 30° + Y_D b \sin 60° = 0 \tag{9}$$

Of course, the solution of system of equations (4)-(9) yields the same values of the reactions as before.

It should be stressed that equations (7)-(9) can be used for verifying the solution of system of equations (1)-(6) we have found; the substitution of this solution into equations (7)-(9) must turn the equations into identities. Let the reader perform that substitution and check that the equations are satisfied identically. If we chose system of equations (4)-(9) for solving the problem then equations (1)-(3) could be used for checking the solution (these equations describing the equilibrium of the whole construction must be satisfied identically by the solution of (4)-(9)).

It should be noted that when we consider the equilibrium of one of the beams (AC or BD) it does not matter how the components X_D and Y_D of the reactions are directed. However, after these directions are chosen, when we consider the equilibrium of the other beam the components of the reactions at the same point should be drawn in opposite directions according to the law of action and reaction.

Problems

PROBLEM 3.1. One of the ends of a homogeneous bar AB of length l and weight P lies on a horizontal plane OB, and at the point D it rests on the smooth surface of a right circular cylinder (Fig. 3.22). It is known that $BD = 2l/3$. Determine the horizontal force S which must be applied at B in order to keep the bar in equilibrium for a given angle α of inclination to the horizon; also find the forces of pressure the bar exerts on the plane and on the cylinder.

Answer. $S = \frac{3}{8} P \sin 2\alpha$, $N_B = \left(1 - \frac{3}{4}\cos^2\alpha\right) P$, $N_D = \frac{3}{4} P \cos \alpha$.

The moduli of the forces of pressure with which the bar acts on the plane and on the cylinder are equal to the moduli of the reactions N_B and N_D respectively but their directions are opposite.

PROBLEM 3.2. Let us consider the arch truss shown in Fig. 3.23 which has an immovable hinged support at the point B and a movable hinged support at the point A, the plane of motion of the roller of the movable support forming an angle 30° with the horizon. The weight of the truss itself is $P = 100$ kN. The resultant F of the forces of pressure of the wind is parallel to AB and its line of action is at a distance of 4 m from AB; the absolute value of F is equal to 20 kN. Determine the reactions of the supports.

Answer. $X_B = -11.2$ kN, $Y_B = 46.0$ kN, $R_A = 62.4$ kN.

PROBLEM 3.3. Two identical homogeneous rods are joined together with a hinge at the point B (Fig. 3.24). At the end A of the first rod there is an immovable hinged support. The free end C of the second rod is under the action of a horizontal force T whose modulus is equal to $\sqrt{3}P/2$, where P is the weight of each of the rods. For the state of equilibrium of the construction determine the angles α and β and also the moduli of the forces of pressure N_A and N_B in the hinges A and B.

Answer. $\alpha = \beta = 30°$, $N_B = \sqrt{7}P/2$, $N_A = \sqrt{19}P/2$.

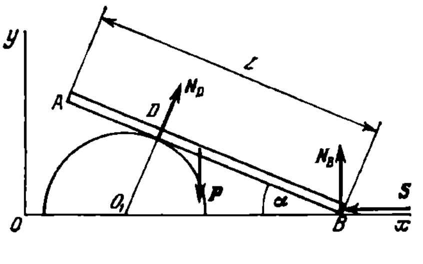


Fig. 3.22

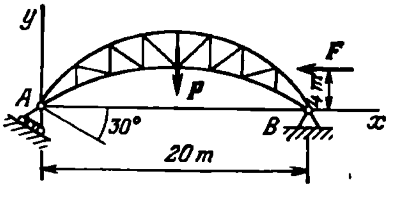


Fig. 3.23

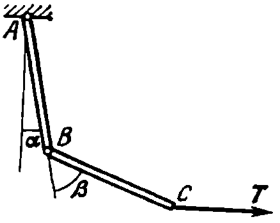


Fig. 3.24

Chapter 4 Plane System of Forces: Friction and Trusses

§ 1. Friction

1.1. Sliding Friction. When rough bodies slide on one another there arises resistance which is referred to as *sliding friction.*

Let a body of weight P lie on a horizontal plane. Suppose that a horizontal force $\boldsymbol{T}$ is applied to that body. In case the surfaces of the contacting bodies are perfectly smooth the normal reaction $\boldsymbol{N}$ of the supporting surface is balanced by the force $\boldsymbol{P}$ while the force $\boldsymbol{T}$ is balanced by no force and therefore imparts motion to the body. Now let us consider the case when the surfaces of the contacting bodies are rough. Then it may turn out that, despite the action of the force $\boldsymbol{T}$, the body remains at rest. In this situation the force $\boldsymbol{T}$ is balanced by another force having the same line of action as $\boldsymbol{T}$ and the opposite direction; it is this force that is called the *force of sliding friction*; we denote it by $\boldsymbol{F}_{\mathrm{fr}}$ (Fig. 4.1). The force of sliding friction is a resistance force; it lies in the plane tangent to both contacting surfaces and its direction is opposite to that of the force.

Let us gradually increase the force $\boldsymbol{T}$. Until the body remains at rest the force of sliding friction balances the force $\boldsymbol{T}$ and increases together with the latter until its maximum value $\boldsymbol{F}_{\max}$ is attained. At this instant there is a limiting situation of the equilibrium of the body when it is on the brink of starting to move.

We thus arrive at the following conclusion: *the modulus of the force of sliding friction in the state of relative rest may assume different values and attains its maximum value at the instant when the relative motion starts*:

$$F_{\mathrm{fr}} \leqslant F_{\max} \tag{4.1}$$

Here the sign of equality corresponds to the limiting case of the equilibrium of the body. The force of friction arising when the body is at rest is referred to as the *force of static friction.* In 1699 the French physicist G. Amonton (1663-1705) established experimentally the law of static friction. Later (1781) this law was confirmed by more precise experiments performed by the French physicist C. A. Coulomb (1736-1806).

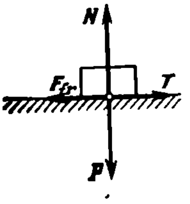


Fig. 4.1

THE AMONTON-COULOMB LAW. *The maximum value of the modulus of the force of static friction is directly proportional to the normal pressure of the body on the supporting surface*:

$$F_{\max} = fN \tag{4.2}$$

The proportionality factor f is called the *coefficient of sliding friction*. This is a dimensionless quantity depending on the materials of the contacting surfaces and on their state (humidity, temperature, roughness, etc.). Inequality (4.1) can now be written in the form

$$F_{fr} \leqslant fN$$

Below are given the values of the coefficient f of sliding friction for some materials:

Steel on ice	0.027
Steel on steel	0.15
Bronze on pig iron	0.16
Bronze on iron	0.19
Leather belt on pig iron	0.28
Oak on oak	0.54-0.62

The friction force arising in the motion of one body on the surface of another body is also proportional to the normal reaction: $F = f'N$. The coefficient f' of sliding friction in motion depends on the velocity of motion of the body and is smaller than the coefficient of static friction $f' < f$.

Let us come back to the force of static friction. When there is friction the total reaction $\boldsymbol{R}$ of the rough surface is determined, both in its magnitude and direction, by the diagonal of the rectangle constructed on the normal reaction and the friction force:

$$\boldsymbol{R} = \boldsymbol{N} + \boldsymbol{F}_{fr}$$

The total reaction $\boldsymbol{R}$ deviates by an angle β from the normal to the supporting surface in the direction opposite to that of the force $\boldsymbol{T}$ (Fig. 4.2*a*). When the force is increased (the force $\boldsymbol{F}_{fr}$ increases simultaneously with $\boldsymbol{T}$) the deviation of the total reaction $\boldsymbol{R}$ from the normal also increases and attains its maximum value when $\boldsymbol{F}_{fr}$ becomes equal to $\boldsymbol{F}_{max}$. The greatest value of the angle β characterizing the angular deviation of the total reaction $\boldsymbol{R}$ from the normal is referred to as the *angle of friction*; we denote it by φ. From Fig. 4.2*b* and formula (4.2) it follows that

$$\tan \varphi = \frac{F_{max}}{N} = f \tag{4.3}$$

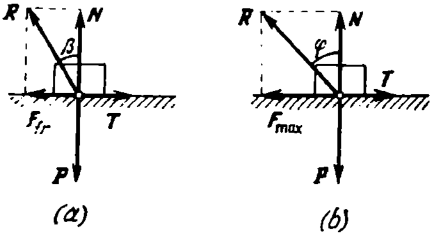
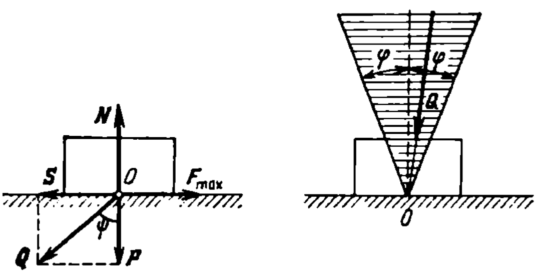


Fig. 4.2



(a) (b)

Fig. 4.3

The smaller the coefficient of sliding friction f the smaller the angle of friction φ; when $f = 0$ then also $\varphi = 0$. The latter case corresponds to an ideal situation in which the surfaces of the contacting bodies are said to be perfectly (absolutely) smooth. The reaction of a perfectly smooth surface is always directed along the normal to that surface (see Sec. 2.9 of Chap. 1).

If a body resting on a rough horizontal plane is acted upon by a force Q forming an angle ψ with the normal to the plane (Fig. 4.3a) then for the body to start moving the modulus of the force S expressed by the formula

$$S = Q \sin \psi$$

must exceed the maximum value of the modulus of the friction force:

$$F_{max} = fN = fQ \cos \psi$$

Thus, there must hold the inequality

$$Q \sin \psi \geqslant fQ \cos \psi$$

that is

$$f = \tan \varphi \leqslant \tan \psi$$

whence

$$\psi \geqslant \varphi \tag{4.4}$$

From inequality (4.4) it follows that if the force Q applied to the body forms an angle with the normal smaller than the angle of friction φ then, however great this force is, the body remains at rest. The region bounded by the rays drawn at the angle φ to the vertical is referred to as the *zone* (*region*) *of friction*; in Fig. 4.3b it is shaded. If the body can move in any direction over the plane the zone of friction is bounded by the surface of a right circular cone* called

* What has been said is valid only under the condition that the contacting surfaces are homogeneous and isotropic; in the general case the cone of friction must not necessarily be circular.

the *cone of friction*; the vertex angle of that cone is 2φ. A force Q passing through the vertex of the cone of friction and lying inside that cone cannot impart motion to the body. From Fig. 4.2*b* we also see that the total reaction $\boldsymbol{R}$ at rest (it is applied at the vertex of the cone) may have any direction inside the cone of friction. It should be noted that in this argument we have not taken into account the weight of the rigid body itself.

The problems involving forces of friction are solved using the ordinary methods with the only difference that when the reactions of constraints are introduced the forces of friction should also be taken into account.

EXAMPLE 4.1. On a rough inclined plane forming an angle α with the horizon there rests a body A of weight P. The body is acted upon by a force Q forming an angle β with the inclined plane (Fig. 4.4). It is known that the angle of friction of the body on the plane is equal to φ, and $\varphi < \beta < 90° - \varphi$. For the state of equilibrium of the body determine the magnitude Q of the force Q.

Solution. We shall consider the equilibrium of the body A. Let us replace the action of the constraint (of the rough inclined plane) by the reactions: the normal reaction N and the force of friction F_{fr}. The axis Ax is drawn upward along the inclined plane. Let us assume that the dimensions of the body are so small that the system of forces we deal with can be considered concurrent (and plane); then for the state of equilibrium we have

$$\sum X = -P \sin\alpha \pm F_{fr} + Q\cos\beta = 0 \tag{1}$$

$$\sum Y = N - P\cos\alpha + Q\sin\beta = 0 \tag{2}$$

From the second equation we find the magnitude of the force of normal pressure of the body on the plane:

$$N = P\cos\alpha - Q\sin\beta \tag{2'}$$

In deriving the equilibrium conditions it is necessary to consider separately the following two cases.

(a) The body A may start moving down the inclined plane. Then the force F_{fr} is directed upward, and equations (1) and (2') yield

$$F_{fr} = P\sin\alpha - Q\cos\beta \leqslant F_{max} = fN = \tan\varphi\,(P\cos\alpha - Q\sin\beta)$$

From this inequality we obtain

$$Q(\cos\beta - \tan\varphi\sin\beta) \geqslant P(\sin\alpha - \tan\varphi\cos\alpha)$$

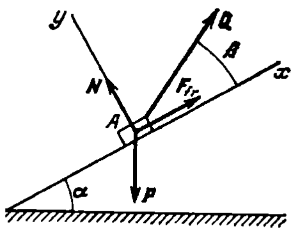


Fig. 4.4

Multiplying both members of the last inequality by $\cos\varphi$ we write

$$Q\cos(\beta+\varphi) \geqslant P\sin(\alpha-\varphi), \quad \text{that is}$$

$$Q \geqslant \frac{\sin(\alpha-\varphi)}{\cos(\beta+\varphi)}P \tag{3}$$

(b) The body A may start moving up the inclined plane. Then the force F_{fr} is directed

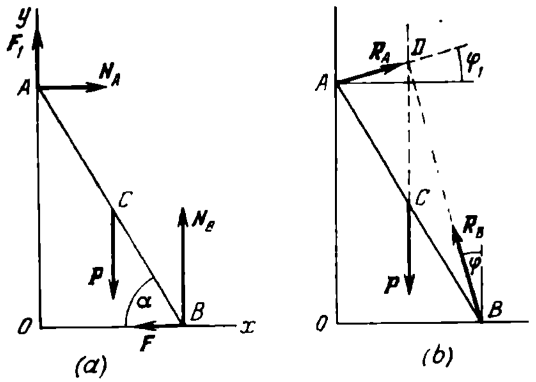


Fig. 4.5

downward, and equations (1) and (2′) yield

$$F_{fr} = Q\cos\beta - P\sin\alpha \leqslant F_{max} = \tan\varphi\,(P\cos\alpha - Q\sin\beta)$$

Transforming this inequality to the form

$$(\cos\beta + \tan\varphi\sin\beta)\,Q \leqslant (\sin\alpha + \tan\varphi\cos\alpha)\,P$$

we obtain

$$Q \leqslant \frac{\sin(\alpha+\varphi)}{\cos(\beta-\varphi)}\,P \tag{4}$$

The combination of inequalities (3) and (4) results in

$$\frac{\sin(\alpha-\varphi)}{\cos(\beta+\varphi)}\,P \leqslant Q \leqslant \frac{\sin(\alpha+\varphi)}{\cos(\beta-\varphi)}\,P$$

When these inequalities are fulfilled the body A is at rest. In particular, in the absence of the force $\boldsymbol{Q}$ there remains only one condition

$$\sin(\alpha-\varphi) \leqslant 0, \quad \text{that is} \quad \alpha \leqslant \varphi$$

and hence in this case the body A is in equilibrium on the rough inclined plane (irrespective of the weight P) provided that the angle of inclination α does not exceed the angle of friction φ.

EXAMPLE 4.2. The ladder AB rests on the wall and on the floor as shown in Fig. 4.5, the surfaces of the wall and the floor being rough. The coefficient of friction between the ladder and the wall is equal to f_1. Determine the coefficient f of friction between the ladder and the floor if it is known that the greatest angle of inclination of the ladder to the horizon for which the ladder remains at rest is equal to α (Fig. 4.5*a*).

Solution. Let us consider the equilibrium of the ladder. It is acted upon by the following forces: the force of weight $\boldsymbol{P}$ of the ladder and the reactions of the supports at the points A and B. By the conditions of the problem, here we deal with the limiting case when the ladder is on the brink of motion; therefore the moduli of the forces of friction are $F_1 = f_1 N_A$ and $F = f N_B$ respectively, these forces themselves being of the directions opposite to those of possible motion of the ladder. Let us draw the coordinate axes as shown in Fig. 4.5*a*. The ladder is under the action of a plane system of forces. Denoting the length of the ladder by $2l$ we write three equilibrium equations:

$$\sum X = N_A - fN_B = 0, \qquad \sum Y = f_1 N_A - P + N_B = 0$$

$$\sum \operatorname{mom}_B \boldsymbol{F} = Pl\cos\alpha - N_A\cdot 2l\sin\alpha - f_1 N_A\cdot 2l\cos\alpha = 0$$

From the third equation we find

$$N_A = \frac{\cos\alpha}{2(\sin\alpha + f_1\cos\alpha)} P$$

The substitution of this value into the second equation yields

$$N_B = P - f_1 N_A = \frac{2\sin\alpha + f_1\cos\alpha}{2(\sin\alpha + f_1\cos\alpha)} P$$

Finally, from the first equilibrium equation we determine the sought-for coefficient of friction:

$$f = \frac{N_A}{N_B} = \frac{\cos\alpha}{2\sin\alpha + f_1\cos\alpha}$$

This problem can also be easily solved graphically. Let us represent the reaction at the point A by one force $\boldsymbol{R}_A$ deviated from the normal by an angle $\varphi_1 = \arctan f_1$ (Fig. 4.5*b*). The ladder is under the action of a plane system of three nonparallel forces $\boldsymbol{P}$, $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$. In the state of equilibrium the lines of action of these forces must meet at one point (see the theorem on three forces in Sec. 2.6 of Chap. 1). Let us extend the lines of action of the forces $\boldsymbol{P}$ and $\boldsymbol{R}_A$ (their positions are known) so that they intersect at a point D. It is the straight line BD that is the line of action of the force $\boldsymbol{R}_B$, and the tangent of the angle φ is equal to the sought-for coefficient of friction. Let the reader obtain the answer to the problem using the geometrical method.

1.2. Rolling Friction. By *rolling friction* is meant resistance appearing when one body rolls on the surface of another body. Let there be a cylindrical roller of weight P and radius R lying on a horizontal plane (Fig. 4.6) to whose centre of gravity a horizontal force $\boldsymbol{T}$ is applied. The action of the weight of the roller produces deformation of the supporting surface, and the point of application of the reactions $\boldsymbol{N}$ and $\boldsymbol{F}_{fr}$ is displaced from A to an intermediate point C. We shall set the equations of equilibrium of the roller beginning with the sums of the projections of the forces on the axes Ox and Oy:

$$\sum X = T - F_{fr} = 0, \quad \sum Y = N - P = 0$$

From these equalities we obtain

$$F_{fr} = T, \; N = P$$

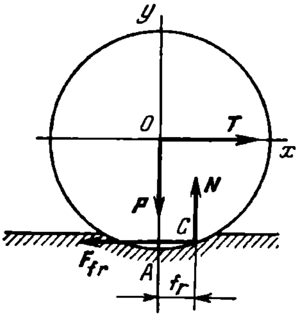


Fig. 4.6

Thus, in the state of equilibrium the roller is acted upon by two balancing couples of forces: the first couple $\boldsymbol{T}$, $\boldsymbol{F}_{fr}$ tends to impart motion to the roller while the other couple $\boldsymbol{P}$, $\boldsymbol{N}$ creates resistance to the motion. The moment of the couple $\boldsymbol{P}$, $\boldsymbol{N}$ is called the *moment of resistance in rolling*; we denote it by m_r. This moment is equal to the moment of the force $\boldsymbol{N}$ about the point A:

$$m_r = \text{mom}_A\, \boldsymbol{N}$$

At any instant when the roller is in equilibrium both the couples are balanced (this specifies the third equilibrium equation):

$$\sum \mathrm{mom}_A \boldsymbol{F} = \mathrm{mom}_A \boldsymbol{N} - TR = 0, \text{ that is } m_r = TR \qquad (4.5)$$

At the instant the roller starts moving, the moment of resistance attains its greatest value. Experiments show that this value is directly proportional to the normal pressure:

$$(m_r)_{max} = f_r N \qquad (4.6)$$

The proportionality factor f_r is called the *coefficient of rolling friction*; it has the dimension of length. The coefficient f_r can be regarded as the greatest magnitude of the line segment by which the point of application of the force $\boldsymbol{N}$ is displaced in the limiting case of equilibrium (see Fig. 4.6).

Given below are the values of the coefficient of rolling friction for some materials:

Steel roller on steel	0.005 cm
Wooden roller on steel	0.03-0.04 cm
Wooden roller on wood	0.05-0.06 cm

When the roller is at rest the moment of the couple $\boldsymbol{P}, \boldsymbol{N}$ does not exceed its maximum value; $m_r \leqslant (m_r)_{max}$; hence, taking into account (4.5) and (4.6), we write

$$TR \leqslant f_r N$$

whence

$$T \leqslant \frac{f_r}{R} N \qquad (4.7)$$

Inequality (4.7) expresses the condition under which the roller does not move. On the other hand, for the roller not to slide the modulus of the force $\boldsymbol{T}$ must be smaller than the maximum value of the modulus of the force of sliding friction:

$$T \leqslant fN$$

Usually the ratio f_r/R is considerably smaller than the coefficient f of sliding friction. Therefore when the state of rest is violated the roller starts rolling on the supporting surface without sliding on it.

§ 2. Flat Trusses

2.1. Notion of a Truss. A construction composed of rectilinear bars connected with one another by hinges and retaining invariable geometrical form is called a *bar system*; bar systems are subjected to the action of external forces and transmit their action to the supports. If the external forces are only applied to the hinges the bar

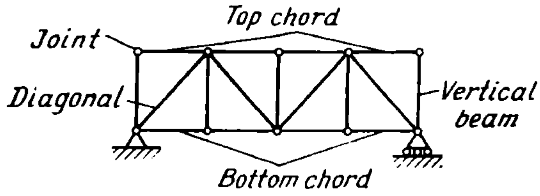


Fig. 4.7

system is called a *truss* and the bars are called its *members*. If the external forces are applied not only to the hinges but also to some points of the bars themselves we speak of a *beam truss*, and in this case its bars are referred to as *beams*. In a beam truss its beams, besides tension and compression, are also subjected to bending. A truss is said to be *flat* when its bars lie in one plane. We shall only consider *flat trusses*.

The points where the members of a truss are connected by hinges are called *joints*. The lower part of a truss is called its *bottom chord* and the upper part is called its *top chord* (Fig. 4.7). The vertical bars of a bar system are spoken of as *vertical beams* and the inclined ones (in case there are vertical beams) are called *diagonals*.

Not every construction composed of bars connected by hinged joints is a truss. By definition, the construction of a truss must guarantee the invariability of its form (rigidity). To construct the simplest truss it is sufficient to connect by joints three bars as shown in Fig. 4.8*a*. The bar triangle thus constructed possesses the invariability of its form (rigidity), and consequently it resists the forces acting on it. If we connect by joints four bars as shown in Fig. 4.8*b* such a construction will change its form under the action of forces applied to its joints, and consequently it is not a truss. In mechanics a construction of that kind is called a mechanism. For this rectangle to become a truss it is sufficient to connect its two opposite vertices by a bar (Fig. 4.8*c*). However, if the other two vertices are also connected by a bar we also obtain a truss (Fig. 4.8*d*). There is an essential distinction between the last two trusses: the truss shown in Fig. 4.8*c* contains no extra bars while the truss in Fig. 4.8*d* contains one such bar.

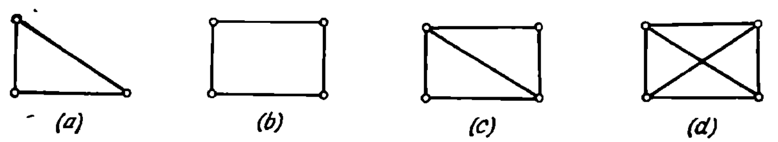
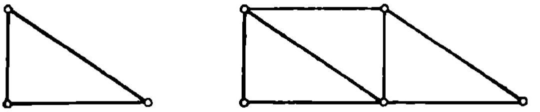


Fig. 4.8



(a) (b)

Fig. 4.9

If it is impossible to remove any of the bars from a truss without violating the property of the geometrical invariability of its form we speak of a *truss without extra members*. In case it is possible to remove one or several bars from a truss without violating the property of the geometrical invariability of its form we speak of a *truss with extra members*. In statics only trusses without extra members are considered.

Now we shall establish the connection between the number of joints and the number of members forming a truss without extra members. Let us consider the simplest truss, that is a truss formed of triangles whose sides are the bars of the truss and whose vertices are the joints. A truss with the smallest number of joints is a bar triangle; it is formed of three bars and has three joints (Fig. 4.9*a*). If a new joint is added then also two new bars must be added, and then the truss remains geometrically invariable and has no extra bars (Fig. 4.9*b*). Thus, if a truss has n joints then it involves $n - 3$ new joints, that is the joints distinct from the three initial bars of the initial bar triangle. Since the addition of every new joint requires the addition of two new bars, the addition of $n - 3$ joints leads to the increase by $2(n - 3)$ of the number of the bars. Adding the three bars of the initial bar triangle we obtain the total number m of the bars in a truss having n joints:

$$m = 2(n - 3) + 3 = 2n - 3 \tag{4.8}$$

If $m < 2n - 3$ we have a construction which may change its geometrical form (a mechanism); if $m > 2n - 3$ we have a truss with extra members.

When considering trusses we must distinguish between statically determinate and statically indeterminate systems. If the reactions of the supports and the stresses in the members of a truss can be found using the methods of statics of a rigid body the truss is said to be *statically determinate*; if otherwise, it is called *statically indeterminate*. It turns out that a truss without extra members having supports of some definite types (see Sec. 2.3 of Chap. 3) is statically determinate. A truss with at least one extra member is always statically indeterminate.

In what follows we shall assume that: (a) all the members of the truss under consideration are rectilinear bars, (b) there is no friction

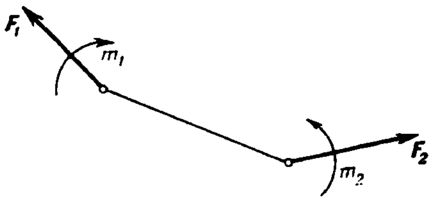


Fig. 4.10

in the joints, and (c) the forces acting on the truss lie in its plane and are applied only at joints. The weight of the members is assumed to be negligibly small.

Under these assumptions the members of the truss can only work in tension or compression and are not subjected to bending. Indeed, since we neglect the weight of the members and since the forces are applied only at joints, every member (bar) is in equilibrium under the action of forces and moments transmitted to it by the corresponding joints (Fig. 4.10). Since there is no friction the moments in the joints are equal to zero. Consequently, every member is in equilibrium under the action of two forces applied at its ends (joints). As we already know, this is only possible when these two forces have equal moduli, lie in one straight line passing through their points of application and have opposite directions. Thus, under the enumerated conditions the *members of the truss only work in tension or compression* under the action of the forces applied to them.

The stresses in the members of a truss depend on the forces applied to the truss. Since the set of these forces includes the reactions of the supports of the truss, before proceeding to the determination of the stresses in the members it is necessary to find the reactions of the supports. After the reactions of the supports have been found we can start determining the stresses in the members.

2.2. Method of Cutting out Joints. Let us determine the stresses in the members of the truss shown in Fig. 4.11*a*. Here $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ are given forces applied at the joints C, D and E. Finding the reactions $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ of the supports, we proceed to the determination of the stresses in the members.

The *method of cutting out joints* lies in the following. Let us mentally cut out the joint A of the truss and replace the action of the

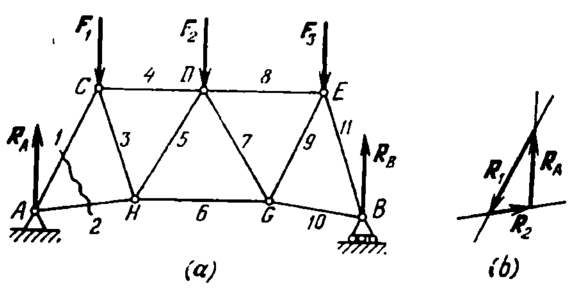


Fig. 4.11

truss on that joint A by the reactions of members *1* and *2*; the directions of these reactions are along the bars; we denote them by $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$. Since the joint A is in equilibrium the force triangle constructed for the forces $\boldsymbol{R}_A$, $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ must be closed. From this force triangle (Fig. 4.11*b*) the moduli and the directions of the forces $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are found. Next we cut out another joint, construct the corresponding force triangle and find from it the moduli and the directions of the reactions of other members, etc. until the reactions of all the members are found.

Generally speaking, the joints can be cut out in an arbitrary order; however, it should be taken into account that the conditions of equilibrium of a joint make it possible to determine the *reactions of only two members*, and therefore in the case under consideration it is advisable to start the calculations beginning with the joint A or B and not with some other joint where three or more members are connected. After the reactions of members *1* and *2* or *10* and *11* have been found we can consider the equilibrium of the joint C or E, but it is impossible to pass to the joint D and consider its equilibrium because at the point D four members with unknown reactions are connected.

It should be stressed that from the equilibrium conditions written for a joint we *find the reactions of the corresponding members*, that is the forces with which these members act on the joint; the forces with which the joint acts on these members have the same moduli as the former forces but opposite directions. For instance (see Fig. 4.11*b*), the reaction $\boldsymbol{R}_1$ of member *1* is directed towards the joint A, and the reaction $\boldsymbol{R}_2$ of member *2* is directed away from that joint. Consequently the joint A acts on member *1* with a force which is opposite to $\boldsymbol{R}_1$ and compresses that member, and member *2* is subjected to the action of a force opposite to $\boldsymbol{R}_2$ and is in tension. Thus, if the reaction of a member is directed *towards a joint* the *member is in compression*, and if the reaction is directed *away from the joint* the *member is in tension*.

To determine the stresses in the members connected by a joint we can also use analytical conditions of equilibrium for the joint (see (3.10)) we have cut out; however, the graphical method is simpler and more visual.

EXAMPLE 4.3. Determine the reactions of the supports and the stresses in the members of the girder frame shown in Fig. 4.12*a*; the forces acting on the frame are indicated in the figure. The lengths of all horizontal and vertical members are the same and are equal to a.

Solution. We begin with the determination of the reactions of the supports; denoting the components of the reactions X_A, Y_A and Y_B (Fig. 4.12*a*) we write the equations of equilibrium for the whole frame:

$$\sum X = 2 - X_A = 0, \quad \sum Y = -6 + Y_A + Y_B = 0$$
$$\sum \text{mom}_A = 2a - 6\cdot 2a + Y_B\cdot 3a = 0$$

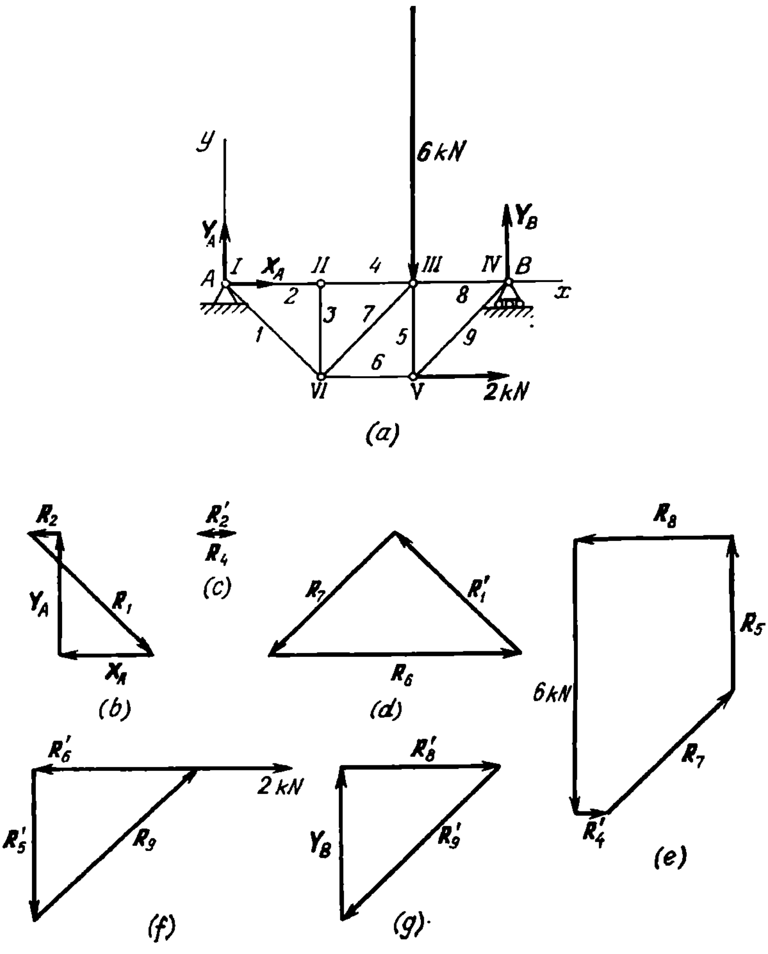


Fig. 4.12

From the first equation we find $X_A = -2$ kN, from the third we find $Y_B = 3.33$ kN, and the second equation yields $Y_A = 2.67$ kN.

Before proceeding to the determination of the stresses in the members let us check whether the frame is statically determinate. Here the number of the joints is $n = 6$ and the number of the members is $m = 9$. The substitution of these numbers into (4.8) results in

$$9 = 2 \cdot 6 - 3$$

which means that the frame is statically determinate. Let us number the joints with Roman numerals and the members with Arabic numerals and choose the scale for the forces: 1 kN $\cong 2/3$ cm.

Now we cut out joint *I* and begin the construction of the force polygon beginning with the known forces; in the case under consideration these are the components of the reactions: X_A (directed to the left) and Y_A. Further, from

the origin of the vector X_A we draw a straight line parallel to one of the members, say to member *1*, and from the terminus of the vector Y_A we draw a line parallel to the other member *2*. Now we close the force quadrilateral as shown in Fig. 4.12*b* and denote the reactions of members *1* and *2* by R_1 and R_2 respectively. Joint *I* acts on member *1* with a force opposite to R_1, this force stretching member *1*; member *2* is subjected to the action of a force, opposite to R_2, which compresses member *2*. The same result can be obtained if we note that the reaction R_1 is directed away from the joint (member *1* is in tension) and that the reaction R_2 is directed towards the joint (member *2* is in compression).

After joint *I* we cannot pass to joint *VI* at which, besides member *1*, three more members (*3*, *6* and *7*) with unknown stresses are connected. Therefore we pass to joint *II*. For joint *II* the known force is the reaction R_2' of member *2* which is directed to the right because member *2* is compressed. Here the force polygon degenerates into line segment (Fig. 4.12*c*). Consequently, the stress in member *3* is equal to zero, while member *4*, like member *2*, is in compression.

After joint *II* we cannot pass to joint *III* (because it connects three members *5*, *7* and *8* with unknown stresses!) and therefore we pass to joint *VI* (Fig. 4.12*d*). Since member *1* is in tension the reaction R_1' of member *1*, that is the force with which member *1* acts on joint *VI*, has a direction opposite to that of the reaction R_1 (see Fig. 4.12*b*). The reaction R_6 is directed away from the joint, and the reaction R_7 is directed towards the joint, that is member *6* is in tension and member *7* is in compression.

Now we can pass to joint *III* or to joint *V*. The force polygons for joints *III* and *V* are shown in Fig. 4.12*e* and 4.12*f* respectively; for joint *IV* the force triangle is shown in Fig. 4.12*g*.

It should be stressed that the reactions R_i' and R_i ($i = 1, 2, \ldots, 9$) are the reactions of one and the same member, numbered with index i, but acting on the joints lying at opposite ends of that member; therefore they have equal moduli and opposite directions. Measuring the unknown forces with the aid of the chosen scale we can compile the following table representing the results of the solution (in the table the compressive stresses are marked with minus sign):

The number of the member	1	2	3	4	5	6	7	8	9
The stress (in kN)	3.8	−0.7	0	−0.7	−3.3	5.3	−3.8	−3.3	4.7

2.3. Method of Sections (Ritter's* Method). The *method of sections* is an analytical method for determining stresses in the members of a truss. We shall demonstrate the application of this method by the example of the truss shown in Fig. 4.13*a* which is acted upon by the forces F_1, F_2 and F_3.

As in the foregoing cases, we start our calculations with the determination of the reactions of the supports X_A, Y_A and Y_B. After they have been determined we pass to the calculation of the stresses in the members. For instance, in order to find the stresses in members *4*, *5* and *6* we mentally draw a section of the truss along the

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* Ritter, A. (1826-1906), a German scientist.

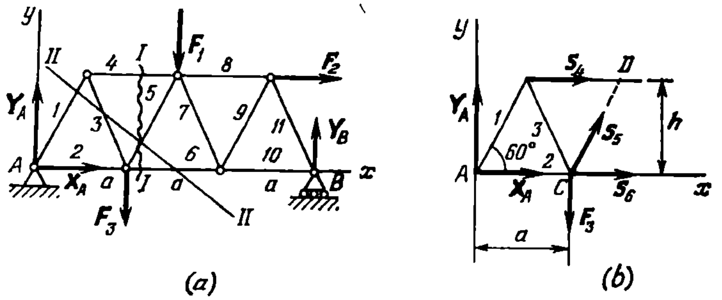

Fig. 4.13

line *I-I* and consider separately the equilibrium of the left (or right) part of the truss. The action of the right part of the truss on its left part is transmitted by the stresses S_4, S_5 and S_6 in the members; in the figure we draw these stresses along the members away from the corresponding joints (as if each of the members were in tension). The unknown stresses S_4, S_5 and S_6 are readily found from the equilibrium conditions for the left part of the truss. After this we mentally draw a section of the truss in some other place and determine the stresses in corresponding members, etc. The order in which these sections are drawn and the places where they are made can be quite arbitrary; however one should bear in mind that the number of the members with unknown stresses in them which are "cut" by the sections must not exceed the number of equilibrium equations that can be formed for the part of the truss in question. For instance, we cannot start determining the stresses in the members beginning with the section along the line *II-II* because here we have four members (*1*, *3*, *5* and *6*) which are "cut", and therefore four unknown stresses S_1, S_3, S_5 and S_6 are involved whereas only three equilibrium equations can be written for their determination, which is insufficient.

As we agreed above, the sought-for stresses in the members are drawn away from the corresponding joint. If the results obtained after the problem has been solved show that the stress in a certain member is positive this means that the stress is in fact directed away from the joint and the member is in tension. If the solution gives a negative value of a stress this means that in reality the stress at the corresponding place of the section is in the direction not away from the joint but towards that joint, that is the corresponding member is in compression. Thus, when we agree to direct the unknown stresses in the members away from the joints and then find their real values by solving the equilibrium equations for the part of the truss under consideration, we cannot only determine the magnitudes of the stresses in the members but also establish their

character: *if a stress is positive the corresponding member is in tension; if it is negative the member is in compression.*

Ritter's method is a modification of the method of sections used for determining stresses in trusses. The specific feature of Ritter's method is that the equilibrium equations for the truss with a "cut" are taken in form (3.9) or (3.8). To determine the stresses (Fig. 4.13*b*) we shall use the theorem on three moments or equations (3.8). As a first centre about which the moments are taken we choose the point C at which the stresses S_5 and S_6 intersect. Due to this choice of the centre the first of equations (3.9) involves only the unknown S_4 and those of the external forces and the reactions of the supports whose moments about the point C are different from zero:

$$\sum \text{mom}_C = -Y_A a - S_4 h = 0$$

From this equation we find

$$S_4 = -\frac{a}{h} Y_A$$

which means that if $Y_A > 0$ then $S_4 < 0$ and member *4* is compressed.

As the second centre we take the point D at which the lines of action of the stresses S_4 and S_5 intersect. Then the second of equilibrium equations (3.9) is written in the form

$$\sum \text{mom}_D = F_3 \cdot \frac{1}{2} a + X_A h - Y_A \cdot \frac{3}{2} a + S_6 h = 0$$

This equation yields

$$S_6 = \frac{a}{2h}(3Y_A - F_3) - X_A$$

Finally, as the third centre for the moments it is advisable to take the point of intersection of the lines of action of the stresses S_4 and S_6. However, in the example under consideration these lines are parallel. Therefore we shall make use of form (3.8) of equilibrium equations and take as the third of equations (3.8) the equation

$$\sum Y = Y_A - F_3 + S_5 \cos 30° = 0$$

(the first two equations (3.8) coincide with those obtained on the basis of the theorem on three moments). From this equation we find the last of the three unknown stresses:

$$S_5 = \frac{1}{\cos 30°}(F_3 - Y_A)$$

The advantage of Ritter's method is that each of the stresses is determined from one equation independently of the other stresses. This advantage is particularly important for those cases when it is required to find not all the stresses but only some of them.

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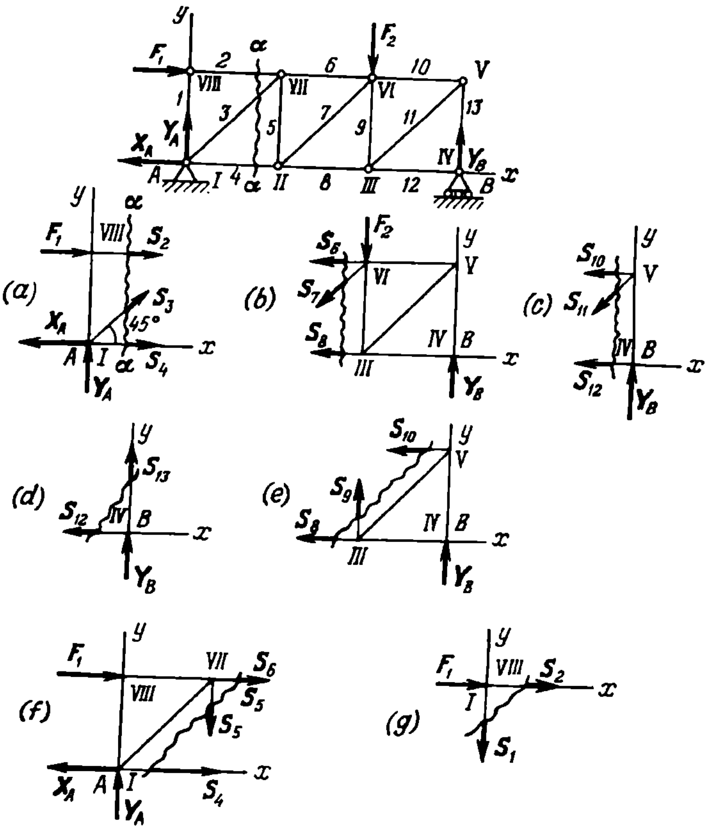


Fig. 4.14

EXAMPLE 4.4. Determine the reactions of the supports and the stresses in the members of the truss shown in Fig. 4.14; the forces applied to the truss are indicated in the figure. The vertical and the horizontal members are of the same length a; the other data are: $F_1 = 3$ kN and $F_2 = 8$ kN.

Solution. Here the number of joints is $n = 8$ and the number of members is $m = 13$. The substitution of these numbers into (4.8) yields the equality $13 = 2\cdot 8 - 3$, that is we deal with a statically determinate system. Next we find the reactions of the supports. Equating to zero the sums of the projections of all the forces on the axes Ax and Ay and forming the equation for the moments of the forces about the centre A, we obtain

$$\sum X = F_1 - X_A = 0, \quad \sum Y = -F_2 + Y_A + Y_B = 0$$

$$\sum \text{mom}_A = -F_1 a - F_2\cdot 2a + Y_B\cdot 3a = 0$$

From this system of equations we find

$$X_A = F_1 = 3 \text{ kN}, \quad Y_B = \frac{1}{3}(F_1 + 2F_2) = 6.33 \text{ kN}, \quad Y_A = F_2 - Y_B = 1.67 \text{ kN}$$

The stresses in the members will be found using the method of sections.

Let us begin with the section along the line α-α; the unknown stresses in the members are drawn away from the joints. The equilibrium conditions for the left part of the truss (Fig. 4.14*a*) give us three equations:

$$\sum X = F_1 - X_A + S_2 + S_3 \cos 45^\circ + S_4 = 0, \quad \sum Y = Y_A + S_3 \cos 45^\circ = 0$$

$$\sum \text{mom}_A = -F_1 a - S_2 a = 0$$

Solving these equations we find

$$S_2 = -F_1 = -3 \text{ kN}, \quad S_3 = -\frac{1}{\cos 45^\circ} Y_A = -2.36 \text{ kN}$$

$$S_4 = -F_1 + X_A - S_2 - S_3 \cos 45^\circ = 4.67 \text{ kN}$$

Now let us draw a section across members *6*, *7* and *8* and write the equilibrium equations for the right part of the truss (Fig. 4.14*b*):

$$\sum X = -S_6 + S_7 \cos 135^\circ - S_8 = 0, \quad \sum Y = -F_2 + Y_B + S_7 \cos 135^\circ = 0$$

$$\sum \text{mom}_{VI} = Y_B a - S_8 a = 0$$

(the equation for the moments is written with respect to joint *VI*). These equations yield

$$S_8 = Y_B = 6.33 \text{ kN}, \quad S_7 = \frac{Y_B - F_2}{\cos 45^\circ} = -2.36 \text{ kN}$$

$$S_6 = -S_8 + S_7 \cos 45^\circ = -4.67 \text{ kN}$$

Further, we draw a section across members *10*, *11* and *12* and consider the equilibrium of the right part of the truss (Fig. 4.14*c*). The equilibrium equations have the form

$$\sum X = -S_{10} + S_{11} \cos 135^\circ - S_{12} = 0, \quad \sum Y = Y_B + S_{11} \cos 135^\circ = 0$$

$$\sum \text{mom}_V = -S_{12} a = 0$$

(here the equation for the moments is written with respect to joint *V*). Solving these equations we obtain

$$S_{12} = 0, \quad S_{11} = \frac{1}{\cos 45^\circ} Y_B = 8.95 \text{ kN}, \quad S_{10} = -S_{12} - S_{11} \cos 45^\circ = -6.33 \text{ kN}$$

To find the stress in member *13* we draw a section across members *13* and *12* and consider the equilibrium of the part of the truss shown in Fig. 4.14*d*, that is of the joint *B*. Projecting all the forces on the axis By we obtain

$$\sum Y = Y_B + S_{13} = 0$$

whence

$$S_{13} = -Y_B = -6.33 \text{ kN}$$

To determine the stress in member *9* we draw a section across members *8*, *9* and *10* and consider the equilibrium of the part of the truss shown in Fig. 4.14*e*. The equation for the projections of all the forces on the axis By is

$$\sum Y = Y_B + S_9 = 0$$

whence

$$S_9 = -Y_B = -6.33 \text{ kN}$$

To determine the stress in member *5* we follow the same procedure (Fig. 4.14*f*). We draw a section across members *4*, *5* and *6*, discard the right part of the truss and equate to zero the sum of the projections of all the forces (acting on

the remaining part) on the axis Ay, which results in

$$\sum Y = Y_A - S_5 = 0$$

whence

$$S_5 = Y_A = 1.67 \text{ kN}$$

Finally, in the same way we consider the equilibrium of joint *VIII* (Fig. 4.14*g*); in this case we obtain

$$S_1 = 0$$

Let us compile the following table for the results of our calculations:

The number of the member	1	2	3	4	5	6	7
The stress (in kN)	0	−3.00	−2.36	4.67	1.67	−4.67	−2.36

The number of the member	8	9	10	11	12	13
The stress (in kN)	6.33	−6.33	−6.33	8.95	0	−6.33

Since we agreed to draw the unknown stresses in the members of the trusses away from the joints, the negative values of some stresses we have found indicate that the corresponding members are in compression; those members in which there are positive stresses are in tension.

In Example 4.4 we did not use Ritter's method because we wanted to demonstrate that, generally speaking, it is possible to do without this method using the method of section of trusses.

Problems

PROBLEM 4.1. A load of weight P is in uniform motion on a rough horizontal surface under the action of a force $\boldsymbol{T}$ whose line of action forms an angle β with the horizon (Fig. 4.15). It is known that the angle of friction is equal to φ. For what value of the angle β the force $\boldsymbol{T}$ has the smallest modulus?

Hint. Introduce the force of friction $\boldsymbol{F}_{fr}$.

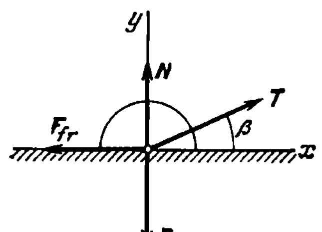


Fig. 4.15

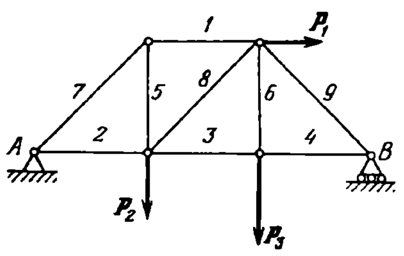


Fig. 4.16

Answer. $\beta = \varphi$, $T_{min} = P \sin \varphi$.

PROBLEM 4.2. A homogeneous beam is supported by a floor and a wall, the coefficient of friction f being the same for both supports. Determine the angle α between the beam and the floor in the state of equilibrium.

Answer. $\tan\alpha = \dfrac{2f}{1-f^2}$.

PROBLEM 4.3. The horizontal and the vertical members of the truss shown in Fig. 4.16 are all of equal length a. The joints of the truss are under the action of the forces $P_1 = 10$ kN, $P_2 = 20$ kN and $P_3 = 30$ kN. Determine the reactions of the supports at the points A and B and the stresses in the members of the truss.

Answer. $X_A = 10$ kN, $Y_A = 20$ kN, $R_B = 30$ kN.

The number of the member	1	2	3	4	5	6	7	8	9
The stress (in kN)	−20	30	30	30	20	30	−28.3	0	−42.4

Chapter 5 Arbitrary Force System

§ 1. Vector Expression for the Moment of a Force and the Theory of Couples in Space

1.1. Vector Expression for the Moment of a Force about a Point. In the study of a plane system of forces (see Sec. 2.4 of Chap. 2) we regarded the moment of a force about a point as an algebraic quantity whose absolute value is equal to the product of the magnitude of the force by the arm or, which is the same, to twice the area of the triangle OAB (Fig. 5.1):

$$\text{mom}_O \boldsymbol{F} = \pm Fh = \pm 2S_{\triangle OAB}$$

This definition is insufficient for the study of an arbitrary (spatial) system of forces because there are cases when the moments of two forces about a point have equal magnitudes but their actions on a body are nevertheless different. In statics in space not only the magnitude of the moment of a force is essential but also the orientation in space of the plane of the triangle OAB. The moment of a force about a point O will completely characterize the action of the given force on a rigid body if we define it as a vector perpendicular to the plane OAB.

The (vector) moment of a force $\boldsymbol{F}$ *about a point O is understood as a vector, applied at the point O, whose modulus is equal to the product of the magnitude of the force by the arm and which is drawn perpendicularly to the plane containing the force and the centre O so that*

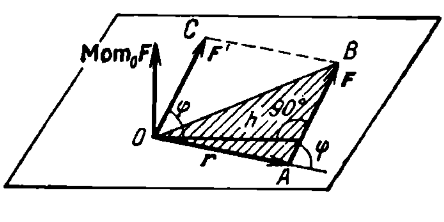


Fig. 5.1

when the rotation of the body by that force is contemplated from the terminus of the vector it is counterclockwise (in other words, this definition corresponds to the right-hand screw rule). The moment of the force $\boldsymbol{F}$ about the point O will be symbolized as $\mathbf{Mom}_O\,\boldsymbol{F}$ (see Fig. 5.1). The modulus Fh of the (vector) moment $\mathbf{Mom}_O\,\boldsymbol{F}$ is equal to twice the area of the triangle OAB or, which is the same, to the area of the parallelogram $OABC$. This parallelogram can be regarded as being constructed on the vectors $\boldsymbol{F}'$ and $\boldsymbol{r}$, where $\boldsymbol{F}'$ is a vector equal to the force vector $\boldsymbol{F}$ and applied at the point O, and $\boldsymbol{r}$ is the radius vector of the point of application of the force $\boldsymbol{F}$. Thus, the modulus of the vector characterizing the moment is

$$\mathrm{Mom}_O\,\boldsymbol{F} = Fh = rF\sin\varphi \tag{5.1}$$

REMARKS.

(a) According to the definition, the vector of the moment of a force remains invariable when the point of application of the force is transferred along its line of action.

(b) From formula (5.1) it follows that the moment of a force $\boldsymbol{F}$ about a point O is equal to zero when $F = 0$ or when $r = 0$ or when $\varphi = 0$ (or $\varphi = 180°$). In the first case the force is equal to zero and in the other cases the line of action of the force passes through the point O.

The modulus of the vector $\mathbf{Mom}_O\,\boldsymbol{F}$ (characterizing the moment of the force $\boldsymbol{F}$ about the point O) is equal to the area of the parallelogram constructed on the vectors $\boldsymbol{r}$ vand $\boldsymbol{F}$. The vector $\mathbf{Mom}_O\,\boldsymbol{F}$ is perpendicular to the plane containing the vectors $\boldsymbol{r}$ and $\boldsymbol{F}$, the rotation from the vector $\boldsymbol{r}$ to the vector $\boldsymbol{F}'$, when observed from the terminus of the vector $\mathbf{Mom}_O\,\boldsymbol{F}$, is counterclockwise. Remembering the definition of the vector product of two vectors (see (1.12)) we see that the vector $\mathbf{Mom}_O\,\boldsymbol{F}$ is equal to the vector product of the vectors $\boldsymbol{r}$ and $\boldsymbol{F}$. Thus, *the (vector) moment of a force* $\boldsymbol{F}$ *about a centre* O *is a bound vector, with point of application at* O, *equal to the vector product of the radius vector* $\boldsymbol{r}$ *of the point of application of the force by the force vector* $\boldsymbol{F}$:

$$\mathbf{Mom}_O\,\boldsymbol{F} = [\boldsymbol{r}, \boldsymbol{F}] \tag{5.2}$$

1.2. Moment of a Force about an Axis. Given a force $\boldsymbol{F}$ and an axis z (Fig. 5.2), let us draw a plane Π perpendicular to the axis z. The point of intersection of the axis z with the plane Π will be denoted by O.

By the moment of the force $\boldsymbol{F}$ *about the axis* z *is meant the moment of the component* $\boldsymbol{f}$ *of that force in the plane* Π (*the vector* $\boldsymbol{f}$ *goes along the projection of the vector* $\boldsymbol{F}$ *on the plane* Π *and is perpendicular to the given axis*) *about the point* O *of intersection of the axis* z *with the plane* Π.

Like in Sec. 2.1 of Chap. 2, the moment of the force $\boldsymbol{F}$ about the axis z is considered positive if the rotation of a body under the action

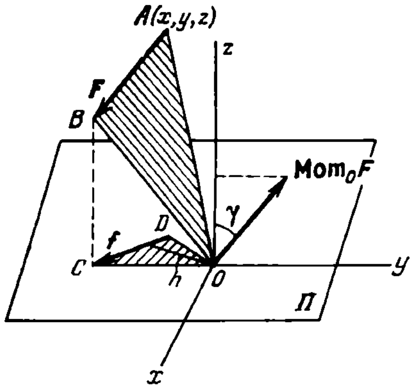


Fig. 5.2

of the force f (which goes along the projection of the vector $\boldsymbol{F}$ on the plane Π), when contemplated from the positive semiaxis z, *is counterclockwise*. If otherwise, the moment is considered negative. The moment of a force $\boldsymbol{F}$ about an axis z will be denoted by $\text{mom}_z \boldsymbol{F}$, where the subscript corresponds to the name of the axis; in the present case we have written the moment of the force $\boldsymbol{F}$ about the axis z. From the definition we have stated it follows that the moment of a force $\boldsymbol{F}$ about an axis z is a scalar whose absolute value equals twice the area of the triangle OCD:

$$\text{mom}_z \boldsymbol{F} = \text{mom}_O f = \pm fh = \pm 2S_{\triangle OCD} \tag{5.3}$$

The moment of a nonzero force about an axis is equal to zero in the following two cases: (a) when the line of action of the force intersects the axis; in this case the arm is equal to zero ($h = 0$) because the line of action of the force vector f (going along the projection of $\boldsymbol{F}$ on Π) passes through the point of intersection of the axis and the plane; (b) when the force is parallel to the axis; in this case the force $\boldsymbol{F}$ is projected on a plane perpendicular to the axis, its projection degenerates into a point and, consequently, $f = 0$. In both cases (a) and (b) the force and the axis lie in one plane.

We shall prove a lemma establishing the relationship between the (vector) moment of a force about a point and the moment of that force about an axis passing through the point.

LEMMA. *The projection of the moment of a force about a point on an axis passing through that point is equal to the moment of the force about that axis.*

Proof. Let us consider a force $\boldsymbol{F}$ represented by the vector $\boldsymbol{AB}$ in Fig. 5.2 and an arbitrary axis z. Let us draw a plane Π perpendicular to the axis, the point O being at the intersection of the axis z and the plane Π. The modulus of the (vector) moment $\textbf{Mom}_O \boldsymbol{F}$ about the

point O is equal to twice the area of the triangle OAB:

$$\mathrm{Mom}_O\,\boldsymbol{F} = 2S_{\triangle OAB}$$

This vector is perpendicular to the plane of the triangle OAB and forms an angle γ with the axis z. Projecting the vector $\mathbf{Mom}_O\,\boldsymbol{F}$ on the axis z we obtain

$$(\mathrm{Mom}_O\,\boldsymbol{F})_z = \mathrm{Mom}_O\,\boldsymbol{F}\cos\gamma = 2S_{\triangle OAB}\cos\gamma \tag{5.4}$$

Further, we have $S_{\triangle OAB}\cos\gamma = \pm S_{\triangle OCD}$ because the area of the projection of a figure is equal to the area of the figure times the cosine of the angle between the plane on which the figure is projected and the plane of the figure itself; the angle between the planes OAB and Π is equal to the angle between the perpendiculars to them, that is to the angle γ. Instead of (5.4) we can write

$$(\mathbf{Mom}_O\,\boldsymbol{F})_z = \pm 2S_{\triangle OCD} \tag{5.5}$$

The comparison of (5.5) and (5.3) shows that

$$(\mathbf{Mom}_O\,\boldsymbol{F})_z = \mathrm{mom}_z\,\boldsymbol{F} \tag{5.6}$$

which is what we had to prove.

Using the relationship we have established and the representation of the (vector) moment of a force about a point in the form of vector product (5.2) we can derive the analytic expressions for the moments of a force about the coordinate axes. To this end we write formula (5.2) in form (1.16) taking into account that the projections of the radius vector of a point on the coordinate axes are equal to the corresponding coordinates of that point:

$$\mathbf{Mom}_O\,\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ X & Y & Z \end{vmatrix} \tag{5.7}$$

Let us place the origin at the point O and project equality (5.7) on the coordinate axes Ox, Oy and Oz (see Fig. 5.2); by analogy with (1.17), this results in

$$\begin{aligned} (\mathbf{Mom}_O\,\boldsymbol{F})_x &= \mathrm{mom}_x\,\boldsymbol{F} = yZ - zY \\ (\mathbf{Mom}_O\,\boldsymbol{F})_y &= \mathrm{mom}_y\,\boldsymbol{F} = zX - xZ \\ (\mathbf{Mom}_O\,\boldsymbol{F})_z &= \mathrm{mom}_z\,\boldsymbol{F} = xY - yX \end{aligned} \tag{5.8}$$

Here x, y and z are the coordinates of the point of application of the force $\boldsymbol{F}$, and X, Y and Z are the projections of the force $\boldsymbol{F}$ on the axes Ox, Oy and Oz respectively.

Attention should be paid to the character of the vectors under consideration. From the definitions of the moment of a force for the case of a plane force system (see Sec. 2.1 of Chap. 2), of the moment of a force about an axis and also of the (vector) moment of a force about

a centre it follows that these scalar and vector quantities do not change when the point of application of the force is transferred along its line of action. Consequently, the notion of a moment applies not only to bound vectors (which is the case in dynamics) but also the sliding vectors. However for free vectors the notion of moment loses sense because we can always make the arm of a given force become equal to zero, which makes the moment become zero.

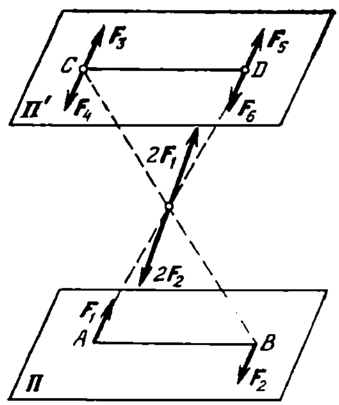


Fig. 5.3

1.3. Theorem on Transferring a Couple into a Parallel Plane. In Sec. 2.3 of Chap. 2 it was shown that two couples lying in one plane are equivalent if they have equal moduli of their moments and specify rotations of same sense. Here we shall prove a theorem on the equivalence of couples in space.

THEOREM. *Two couples are equivalent if they lie in parallel planes, have equal moduli of their moments and specify rotations of same sense.*

Proof. We shall consider a couple $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ ($F_1 = F_2 = F$) with arm AB in a plane Π (Fig. 5.3). Let us take in a plane Π′ parallel to the plane Π a line segment CD equal and parallel to the line segment AB. Next we apply at the points C and D balanced forces $\boldsymbol{F}_3$, $\boldsymbol{F}_4$, $\boldsymbol{F}_5$ and $\boldsymbol{F}_6$ having magnitudes and directions coinciding with those of the forces of the given couples ($F_3 = F_4 = F_5 = F_6 = F$). The resultant of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_5$ is equal to their sum, is parallel to them and is applied at the midpoint of the line segment AD; the resultant of the forces $\boldsymbol{F}_2$ and $\boldsymbol{F}_4$ is equal to their sum, is parallel to them and is applied at the midpoint of the line segment BC. Since the resultants of the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_5$ and $\boldsymbol{F}_2$, $\boldsymbol{F}_4$ have a common point of application (it coincides with the point of intersection of the diagonals of the parallelogram $ABDC$ at which the diagonals bisect) and since the moduli of these resultants are equal while their directions are opposite we can discard them. Then there remain the forces $\boldsymbol{F}_3$, $\boldsymbol{F}_6$ forming a couple the magnitude of whose moment is equal to that of the couple $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ (because both the forces and the arms of both the couples are the same); the couple $\boldsymbol{F}_3$, $\boldsymbol{F}_6$ has the same direction of rotation as the couple $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and lies not in the plane Π but in the parallel plane Π′. According to the theorem proved in Sec. 2.3 of Chap. 2 the couple $\boldsymbol{F}_3$, $\boldsymbol{F}_6$ can be replaced in the plane Π′ by any other couple with the same moment and direction of rotation, whence we conclude that the given couple $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ lying in the plane Π can be replaced by any other couple, lying in an arbitrary plane Π′ parallel

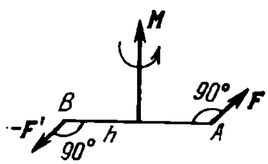


Fig. 5.4

to the plane Π, whose moment equals that of the given couple and whose direction of rotation coincides with that of the given couple. The theorem is proved.

1.4. Vector Expression for the Moment of a Couple. The action of a couple on a rigid body is completely specified if we set the plane of action of the couple, the magnitude of its moment and the direction of rotation. The three factors we have enumerated can be described by representing the moment of the couple by a vector $\boldsymbol{M}$ perpendicular to the plane of action of the couple and having a modulus equal to that of the moment of the couple: $M = Fh$. Let us draw the vector $\boldsymbol{M}$ according to the right-hand screw rule: the rotation of the body produced by the couple, when observed from the terminus of the vector $\boldsymbol{M}$, is counterclockwise (Fig. 5.4). Since a couple can be transferred both in its plane of action and to any other plane parallel to the former without changing the action of the couple on the rigid body, the (vector) moment of a couple can be arbitrarily transferred in space under the only condition that it remains parallel to itself. Consequently, the *(vector) moment of a couple is a free vector.*

From Fig. 5.4 and the definition of the vector specifying the moment of a force about a point (Sec. 1.1) we conclude that

$$\boldsymbol{M} = \mathbf{Mom}_A(-\boldsymbol{F}') = \mathbf{Mom}_B\,\boldsymbol{F}, \quad M = Fh \qquad (5.9)$$

To a fixed (vector) moment of a couple there corresponds not one couple of forces but an infinitude of equivalent couples lying in parallel planes. There is no one-to-one correspondence between the (vector) moments of the couples and the couples themselves: to every couple of forces there corresponds one (vector) moment but the contrary is not true.

1.5. Theorem on Composition of Couples in Space. *Two couples lying in intersecting* planes are equivalent to one couple whose (vector) moment is equal to the vector sum of the moments of the given couples.*

Proof. Let us suppose that there are two couples lying in intersecting planes Π_1 and Π_2 (Fig. 5.5). Let $\boldsymbol{F}_1$ and $-\boldsymbol{F}_1'$ denote the forces forming the first couple with (vector) moment $\boldsymbol{M}_1$, and $F_1 = F_1' = F$. Now we place the first couple so that the force $-\boldsymbol{F}_1'$ is applied at the point A lying on the line of intersection KL of the planes and is directed along the straight line KL; then the arm h_1 of this couple will be perpendicular to KL. Let us transform the couple with (vector) moment $\boldsymbol{M}_2$ lying in the plane Π_2 so that the forces $\boldsymbol{F}_2$ and $-\boldsymbol{F}_2'$ forming it have the moduli equal to F, and then place this couple in

* If two couples lie in parallel planes then, by the theorem proved in Sec. 1.3, they can be transferred to one of these planes and then added together as was indicated in Sec. 2.4 of Chap. 2.

such a way that the force $\boldsymbol{F}_2$ is applied at the point A and is directed along KL opposite to the force $-\boldsymbol{F}_1'$. The arm h_2 of the second couple will also be perpendicular to KL. The forces $-\boldsymbol{F}_1'$ and $\boldsymbol{F}_2$ are mutually balanced and can be discarded. Consequently, the composition of the given couples results in a new couple formed by the forces $\boldsymbol{F}_1$ and $-\boldsymbol{F}_2'$, the arm of that couple being $CB = h$ and the modulus of its (vector) moment being $M = Fh$. The couple $\boldsymbol{F}_1, -\boldsymbol{F}_2'$ is called the *resultant couple*. Let us show that the moment $\boldsymbol{M}$ of the resultant couple is equal to the vector sum of the moments $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ of the given couples.

Making use of the property that the (vector) moment can undergo parallel translation in space, let us place the origins of the vectors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ at the point B; the vector $\boldsymbol{M}_1$ is perpendicular to the plane Π_1, and the vector $\boldsymbol{M}_2$ is perpendicular to the plane Π_2. The moduli of the vectors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are equal to $M_1 = Fh_1$ and $M_2 = Fh_2$ respectively. Adding together the vectors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ according to the parallelogram law, we obtain the vector $\boldsymbol{M}$. We have to show that the vector $\boldsymbol{M}$ is equal to the moment of the couple $\boldsymbol{F}_1, -\boldsymbol{F}_2'$, that is we have to prove that the vector $\boldsymbol{M}$ is perpendicular to the plane containing the vectors $\boldsymbol{F}_1$ and $-\boldsymbol{F}_2'$ and that its modulus is equal to Fh, the direction of $\boldsymbol{M}$ being such that the rotation of the plane produced by the forces $\boldsymbol{F}_1$ and $-\boldsymbol{F}_2'$, when observed from the terminus of $\boldsymbol{M}$, is counterclockwise.

The triangle BCA and the triangle constructed on the vectors $\boldsymbol{M}$ and $\boldsymbol{M}_1$ are similar because the sides $\boldsymbol{M}$ and $\boldsymbol{M}_1$ are proportional to the sides AB and CA:

$$\frac{M_1}{M_2} = \frac{Fh_1}{Fh_2} = \frac{h_1}{h_2}$$

and the angles BAC and $(\widehat{\boldsymbol{M}, \boldsymbol{M}_1})$ are equal (the sides of the triangles are mutually perpendicular). From the similarity of the triangles it follows that

$$\frac{M}{M_1} = \frac{h}{h_1}$$

whence

$$M = \frac{h}{h_1} M_1 = \frac{h}{h_1} Fh_1 = Fh$$

Further, since $\boldsymbol{M}_1 \perp \boldsymbol{F}_1$ and $\boldsymbol{M}_2 \perp \boldsymbol{F}_1$, the plane containing the vectors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ is perpendicular to $\boldsymbol{F}_1$, and consequently we also have $\boldsymbol{M} \perp \boldsymbol{F}_1$. On the other hand, we have $\boldsymbol{M}_1 \perp BA$, and the angles CBA and $\widehat{\boldsymbol{M}, \boldsymbol{M}_1}$ are equal (because they are formed by corresponding sides of similar triangles), whence $\boldsymbol{M} \perp BC$. Consequently, the

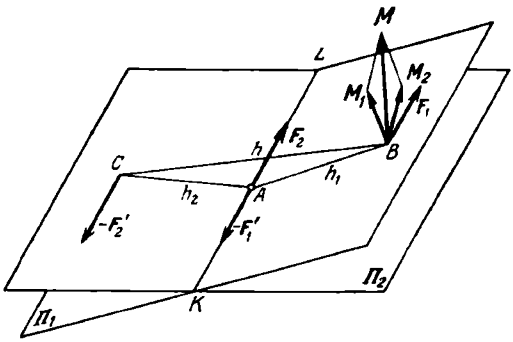


Fig. 5.5

vector $\boldsymbol{M}$ is perpendicular both to $\boldsymbol{F}_1$ and BC, that is perpendicular to the plane containing $\boldsymbol{F}_1$ and $-\boldsymbol{F}_2'$.

Finally, from Fig. 5.5 it is seen that if we observe the rotation of the plane under the action of the forces $\boldsymbol{F}_1$ and $-\boldsymbol{F}_2'$ lying in that plane from the terminus of the vector $\boldsymbol{M}$ it is counterclockwise. Hence, the vector $\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2$ is in fact equal to the moment of the resultant couple. In other words, the moment of the resultant couple is equal to the vector sum of the moments of the constituent couples. The theorem is proved.

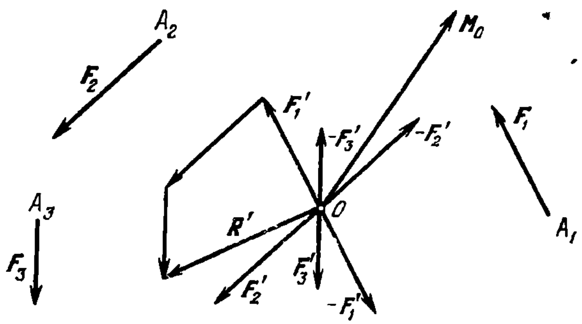
If it is required to add together several couples located arbitrarily in space we can consecutively apply the rule for the composition of two couples we have established, which results in one resultant couple. As in the case of the determination of the resultant of several forces, in order to find the moment of the resultant couple it is simpler to make use of the polygon law: *the (vector) moment of the resultant couple is the closing side of the polygon constructed on the vectors of the moments of the given couples.*

The equilibrium condition for couples can be stated thus: *a collection of couples located arbitrarily in space is balanced if the vector sum of the moments is equal to zero.*

If there are couples lying in one plane or in parallel planes then the above condition implies that the equilibrium takes place when the algebraic sum of the moments of the couples is equal to zero (see the end of Sec. 2.4 of Chap. 2).

§ 2. Reduction of an Arbitrary Force System to a Given Centre

2.1. The Poinsot Method. Resultant Force Vector and Resultant Moment. Let us again enumerate the operations which, according to the axioms of statics (see Secs. 2.4-2.6 of Chap. 1), can be performed on forces applied to a rigid body:



ig. 5.6

(a) Every force can be transferred along its line of action.

(b) Forces applied at one point can be added together according to the polygon law.

(c) Any balanced force system can be added to or subtracted from a given system of forces.

Two force systems which can be transformed into one another using elementary operations (a), (b) and (c) are said to be equivalent. Our immediate aim is to find the simplest possible form of a force system equivalent to the given system of forces. To this end, in the general case, as in the case of a plane system of forces (see Sec. 1.2 of Chap. 3), we shall use the Poinsot method of reduction of a force system to a force and a couple.

We shall consider three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ applied at three points A_1, A_2 and A_3 respectively; in the general case the forces do not lie in one plane (Fig. 5.6). Let us choose an arbitrary point O and transfer to that point the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ using the lemma on the reduction of a given force to an arbitrarily chosen centre (see Sec. 1.1 of Chap. 3). This results in the collection of three forces $\boldsymbol{F}_1'$, $\boldsymbol{F}_2'$ and $\boldsymbol{F}_3'$ applied at the point O and three associated couples $\boldsymbol{F}_1$, $-\boldsymbol{F}_1'$; $\boldsymbol{F}_2$, $-\boldsymbol{F}_2'$ and $\boldsymbol{F}_3$, $-\boldsymbol{F}_3'$ whose (vector) moments are $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ and $\boldsymbol{M}_3$ respectively.

Adding geometrically the forces $\boldsymbol{F}_1'$, $\boldsymbol{F}_2'$ and $\boldsymbol{F}_3'$ applied at the point O we obtain one resultant vector $\boldsymbol{R}'$ applied at the same point O:

$$\boldsymbol{R}' = \boldsymbol{F}_1' + \boldsymbol{F}_2' + \boldsymbol{F}_3' = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \sum \boldsymbol{F}_\nu \tag{5.10}$$

The vector $\boldsymbol{R}'$ is called the *resultant force vector* of the given system of forces.

Further, adding together all the (vector) moments of the associated couples we obtain one *(vector) moment $\boldsymbol{M}_O$ of the resultant couple $\boldsymbol{P}$, $-\boldsymbol{P}'$*; the vector $\boldsymbol{M}_O$ is equal to the vector sum of the moments of the associated couples and is called the *resultant moment* of the given

system of forces:

$$M_O = M_1 + M_2 + M_3 = \sum M_\nu \tag{5.11}$$

Since the moment of an associated couple is equal to the moment of the corresponding force with respect to the reduction centre (see (5.9)), we have

$$M_O = \mathbf{Mom}_O F_1 + \mathbf{Mom}_O F_2 + \mathbf{Mom}_O F_3 = \sum \mathbf{Mom}_O F_\nu \tag{5.12}$$

All that has been said about the composition of three forces remains valid for any number n of forces. Consequently, in the general case, a system of forces which are arbitrarily located in space can be reduced to one resultant, applied at the reduction centre, which is geometrically equal to the resultant force vector and to one resultant couple with moment M_O (the resultant moment) which is equal to* the vector sum of the moments of all the given forces with respect to the reduction centre.

In Fig. 5.6 the resultant couple P, $-P'$ is not shown, and only its (vector) moment M_O (the resultant moment of the given system of forces) is shown. The matter is that, as was mentioned in Sec. 1.4, the (vector) moment of a couple corresponds not to one couple of forces but to an infinitude of equivalent couples lying in parallel planes. Therefore the resultant couple is determined not uniquely but only "to within the equivalence".

Now let us determine analytically the moduli and the directions of the resultant vector and the resultant moment of an arbitrary system of n forces with respect to an arbitrary reduction centre O.

We shall place the origin at the reduction centre; let the coordinates of the point of application of the νth force F_ν ($\nu = 1, 2, \ldots, n$) be x_ν, y_ν and z_ν and let the projections of the νth force F_ν on the coordinate axes be X_ν, Y_ν and Z_ν. Projecting both sides of (5.10) on the coordinate axes we obtain the projections of the resultant vector R' on the axes Ox, Oy and Oz:

$$R'_x = \sum_{\nu=1}^{n} X_\nu, \quad R'_y = \sum_{\nu=1}^{n} Y_\nu, \quad R'_z = \sum_{\nu=1}^{n} Z_\nu \tag{5.13}$$

The modulus of the resultant force vector is found using the formula

$$R' = \sqrt{R'^2_x + R'^2_y + R'^2_z} \tag{5.14}$$

and the cosines of the angles between the resultant vector and the coordinate axes are determined by the formulas

$$\cos(\widehat{R', x}) = \frac{R'_x}{R'}, \quad \cos(\widehat{R', y}) = \frac{R'_y}{R'}, \quad \cos(\widehat{R', z}) = \frac{R'_z}{R'} \tag{5.15}$$

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* One must bear in mind that the moment of a couple is a free vector.

Projecting (5.12) on the coordinate axes and making use of the relationship between the moments of a force about a point and about an axis passing through that point (this relationship is expressed by formula (5.6) proved in Sec. 1.2) we obtain

$$\left.\begin{aligned} M_{Ox} &= \sum_{\nu=1}^{n} \mathrm{mom}_{Ox} \boldsymbol{F}_\nu \\ M_{Oy} &= \sum_{\nu=1}^{n} \mathrm{mom}_{Oy} \boldsymbol{F}_\nu \\ M_{Oz} &= \sum_{\nu=1}^{n} \mathrm{mom}_{Oz} \boldsymbol{F}_\nu \end{aligned}\right\} \tag{5.16}$$

Expressing the moments of the forces about the axes with the aid of formulas (5.8) we derive the formulas

$$\left.\begin{aligned} M_{Ox} &= \sum_{\nu=1}^{n} (y_\nu Z_\nu - z_\nu Y_\nu) \\ M_{Oy} &= \sum_{\nu=1}^{n} (z_\nu X_\nu - x_\nu Z_\nu) \\ M_{Oz} &= \sum_{\nu=1}^{n} (x_\nu Y_\nu - y_\nu X_\nu) \end{aligned}\right\} \tag{5.17}$$

The modulus of the resultant moment and the cosines of the angles between the resultant moment and the coordinate axes are found according to the formulas

$$M_O = \sqrt{M_{Ox}^2 + M_{Oy}^2 + M_{Oz}^2} \tag{5.18}$$

and

$$\cos(\widehat{\boldsymbol{M}_O, x}) = \frac{M_{Ox}}{M_O}, \quad \cos(\widehat{\boldsymbol{M}_O, y}) = \frac{M_{Oy}}{M_O}, \quad \cos(\widehat{\boldsymbol{M}_O, z}) = \frac{M_{Oz}}{M_O} \tag{5.19}$$

2.2. Variation of Resultant Moment due to Change of the Reduction Centre. Invariants of a Force System. As in the case of a plane system of forces, the resultant vector of an arbitrary system of forces is independent of the choice of the reduction centre because it is equal to the vector sum of the force vectors forming the given system. In other words, the *resultant force vector* of a given system of forces is an *invariant* with respect to the choice of the reduction centre.

Now we shall find how the resultant moment of an arbitrary system of forces varies when the reduction centre is changed. Let an arbitrary system of forces be reduced to a resultant force vector $\boldsymbol{R}'$ applied at a point O and a resultant couple whose moment is represented by

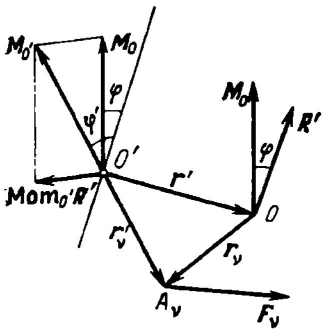

Fig. 5.7

a vector equal to the resultant moment $\boldsymbol{M}_O$ with respect to the same point (Fig. 5.7). Let us take a new reduction centre at a point O'. We shall denote by $\boldsymbol{r}'$ the radius vector of the old reduction centre with respect to the new reduction centre and by $\boldsymbol{r}'_\nu$ the radius vector of the point A_ν of application of the force $\boldsymbol{F}_\nu$ with respect to the new centre; then

$$\boldsymbol{r}'_\nu = \boldsymbol{r}' + \boldsymbol{r}_\nu$$

According to (5.2), the moment of the force $\boldsymbol{F}_\nu$ about the new reduction centre O' is equal to

$$\mathbf{Mom}_{O'}\boldsymbol{F}_\nu = [\boldsymbol{r}' + \boldsymbol{r}_\nu, \boldsymbol{F}_\nu] = [\boldsymbol{r}', \boldsymbol{F}_\nu] + [\boldsymbol{r}_\nu, \boldsymbol{F}_\nu]$$
$$= [\boldsymbol{r}', \boldsymbol{F}_\nu] + \mathbf{Mom}_O\boldsymbol{F}_\nu \qquad (5.20)$$

Adding together the (vector) moments of all the forces of the system with respect to the point O' we obtain the resultant moment with respect to the new reduction centre:

$$\boldsymbol{M}_{O'} = \sum_{\nu=1}^{n} \mathbf{Mom}_{O'}\boldsymbol{F}_\nu = \sum_{\nu=1}^{n} [\boldsymbol{r}', \boldsymbol{F}_\nu] + \sum_{\nu=1}^{n} \mathbf{Mom}_O\boldsymbol{F}_\nu \qquad (5.21)$$

Since the vector $\boldsymbol{r}'$ is one and the same for all the forces $\boldsymbol{F}_\nu$ the first term on the right-hand side of (5.21) is written in the form

$$\sum_{\nu=1}^{n} [\boldsymbol{r}', \boldsymbol{F}_\nu] = [\boldsymbol{r}', \sum_{\nu=1}^{n} \boldsymbol{F}_\nu] = [\boldsymbol{r}', \boldsymbol{R}'] = \mathbf{Mom}_{O'}\boldsymbol{R}'$$

whence we see that the first term on the right-hand side of (5.21) is the moment of the resultant force vector $\boldsymbol{R}'$, applied at the point O about the centre O'. The second term on the right-hand side of (5.21) is the resultant moment of all the forces of the system about the centre O:

$$\sum_{\nu=1}^{n} \mathbf{Mom}_O\boldsymbol{F}_\nu = \boldsymbol{M}_O$$

Finally, we obtain

$$\boldsymbol{M}_{O'} = \mathbf{Mom}_{O'}\boldsymbol{R}' + \boldsymbol{M}_O \qquad (5.22)$$

which means that the *resultant moment of an arbitrary force system about the new reduction centre is equal to the sum of its resultant moment with respect to the old centre and the moment of the resultant vector applied at the old centre with respect to the new centre.*

Projecting equality (5.22) on the direction of the resultant vector and taking into account that $\mathbf{Mom}_{O'} \boldsymbol{R}' \perp \boldsymbol{R}'$ we obtain

$$M_{O'} \cos \varphi' = M_O \cos \varphi \tag{5.23}$$

The projection of the resultant moment of the given force system on the direction of the resultant vector of that force system is a constant quantity independent of the choice of the reduction centre.

The property we have proved can also be stated thus: the scalar product of the resultant moment of a system of forces by the resultant vector of that system is a constant quantity independent of the choice of the reduction centre.

Indeed, multiplying both members of equality (5.23) by R' we obtain

$$R'M_{O'} \cos \varphi' = R'M_O \cos \varphi \tag{5.24}$$

Further, as is known, the product of the moduli of two vectors by the cosine of the angle between them is equal to the scalar product of these two vectors. Hence, for any system of forces there are two quantities independent of the choice of the reduction centre, namely the resultant force vector of the system and the projection of the resultant moment on the direction of the resultant vector (or, which is the same, the scalar product of the resultant force vector by the resultant moment). The first of these quantities is a *vector invariant* and the other is a *scalar invariant.*

2.3. Case of Reduction of a Force System to One Couple. As was shown in Sec. 2.1, in the general case a system of forces located arbitrarily in space can be reduced to one resultant force vector geometrically equal to the resultant vector $\boldsymbol{R}'$ and one resultant couple whose moment is equal to the resultant moment $\boldsymbol{M}_O$ of that system with respect to the reduction centre. Now let us consider some special cases of reduction of an arbitrary system of forces. We shall begin with the case when the resultant vector is equal to zero, that is the force polygon constructed for the given forces is closed while the resultant moment is different from zero:

$$\boldsymbol{R}' = 0, \ \boldsymbol{M}_O \neq 0$$

In this case the given force system is equivalent to the resultant couple $\boldsymbol{P}, -\boldsymbol{P}'$ with moment $\boldsymbol{M}_O$ determined by formula (5.12):

$$\{\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n\} \sim \boldsymbol{P}, -\boldsymbol{P}'$$

It is this case when an arbitrary system of forces is reduced to one couple.

If the position of the reduction centre is changed and the new centre is taken at O' (see Fig. 5.7) then, by virtue of formula (5.22), the resultant moment does not vary, that is

$$\boldsymbol{M}_{O'} = \boldsymbol{M}_O$$

This result can also be confirmed by the following simple argument. The moment $\boldsymbol{M}_O$ of the resultant couple is a free vector and therefore it can be constructed at any point, and the resultant couple itself can be constructed in any plane perpendicular to the vector $\boldsymbol{M}_O$.

2.4. Case of Reduction of a Force System to the Resultant. Varignon's Theorem. The second special case of reduction of an arbitrary system of forces is the one in which the resultant vector is not equal to zero: $\boldsymbol{R}' \neq 0$. Here we must consider separately the following three possibilities.

(a) For the chosen reduction centre O the resultant moment is equal to zero:

$$\boldsymbol{R}' \neq 0, \ \ \boldsymbol{M}_O = 0$$

In this case the given force system is equivalent to one force, that is to the resultant $\boldsymbol{R}$, which is geometrically equal to the resultant force vector and is applied at the reduction centre O:

$$\{\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n\} \sim \boldsymbol{R}$$

In other words, if there appears no resultant couple when the system is reduced to the centre O then the resultant force vector is the resultant of the given system of forces:

$$\boldsymbol{R} = \sum_{\nu=1}^{n} \boldsymbol{F}_\nu$$

(b) For the chosen reduction centre O the resultant moment is perpendicular to the resultant vector:

$$\boldsymbol{R}' \neq 0, \ \ \boldsymbol{M}_O \neq 0, \ \ \boldsymbol{M}_O \perp \boldsymbol{R}' \ \ (\boldsymbol{M}_O \boldsymbol{R}' = 0)$$

Let us show that in this case as well the system of forces can be reduced to a resultant which however does not pass through the point O. By the hypothesis, the resultant force vector $\boldsymbol{R}'$ lies in a plane perpendicular to the resultant moment $\boldsymbol{M}_O$ with respect to the point O. Therefore the resultant couple can be constructed in such a way that one of the forces forming it, say the force $-\boldsymbol{R}'$, is applied at the point O, the direction of that force being opposite to that of the resultant vector $\boldsymbol{R}'$ (Fig. 5.8). The arm h of the resultant couple is found from the equality

$$M_O = R'h$$

whence

$$h = \frac{M_O}{R'}$$

Then the second force $\boldsymbol{R}$ entering into the resultant couple $\boldsymbol{R}$, $-\boldsymbol{R}'$ is applied at the point O' which is at the end of the perpendicular of length h erected at the point O to the plane containing the vectors $\boldsymbol{M}_O$ and $\boldsymbol{R}'$ so that the rotation produced by the force $\boldsymbol{R}$ about the point O, when observed from the terminus of the vector $\boldsymbol{M}_O$ (see Fig. 5.8), is

counterclockwise. Discarding the forces R' and $-R'$ applied at the point O we arrive at the conclusion that the given system of forces is equivalent to one force:

$$\{F_1, F_2, \ldots, F_n\} \sim R$$

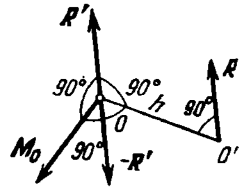

Fig. 5.8

This means that it is equivalent to the *resultant R applied at the point O'*. As before, the resultant is geometrically equal to the resultant force vector:

$$R = R' = \sum_{\nu=1}^{n} F_\nu$$

However, in the present case the line of action of R does not pass through the point O.

Thus, an arbitrary system of forces is reduced to a resultant either when the resultant moment of that force system is equal to zero or when the resultant moment is perpendicular to the resultant vector. In both these cases the scalar invariant (see the end of Sec. 2.2) is equal to zero. Later we shall see that Sec. 2.5 implies that this condition is not only sufficient but also necessary for the existence of the resultant.

Let us prove *Varignon's theorem* on the moment of the resultant for an arbitrary system of forces (for the case of a plane system of forces Varignon's theorem was proved in Sec. 1.4 of Chap. 3).

If the resultant of a force system exists then the (vector) moment of the resultant of that system about any centre is equal to the sum of the vectors representing the moments of all the forces of the given system about the same centre.

Proof. Let the given system of forces be reduced to a resultant R applied at a point O. Now we choose an arbitrary point A and, using the theorem on change of the resultant moment when the reduction centre is changed, write, by analogy with (5.22), the expression for the resultant moment M_A of all the forces of the system about the point A:

$$M_A = \mathrm{Mom}_A\, R + M_O \tag{5.25}$$

Since the resultant moment M_O with respect to the centre O is equal to zero (because the force system is reduced to the resultant R at the point O) we have

$$M_A = \mathrm{Mom}_A\, R \tag{5.26}$$

According to formula (5.12), the resultant moment M_A is equal to the sum of the (vector) moments of all the forces of the system about the point A:

$$M_A = \sum_{\nu=1}^{n} \mathrm{Mom}_A\, F_\nu$$

Finally, we obtain

$$\mathbf{Mom}_A \boldsymbol{R} = \sum_{\nu=1}^{n} \mathbf{Mom}_A \boldsymbol{F}_\nu \tag{5.27}$$

which is what we had to prove.

Now it remains to consider the third possibility for the case when $\boldsymbol{R}' \neq 0$, namely the situation when the resultant moment $\boldsymbol{M}_O$ is not perpendicular to the resultant force vector $\boldsymbol{R}'$.

2.5. Case of Reduction of a Force System to a Wrench. Central Axis. In Secs. 2.3 and 2.4 it was proved that, under certain conditions, a system consisting of n forces is equivalent either to one couple when the resultant vector of that system is equal to zero or to one force (the resultant) when the resultant moment is either equal to zero or perpendicular to the resultant vector. Now we shall find the simplest possible form of a system equivalent to a given system of n forces located arbitrarily in space for the most general case, that is for the case when both the resultant vector and the resultant moment are not equal to zero and form an angle $\varphi \neq 90°$:

$$0 \leqslant \varphi < 90° \text{ or } 90° < \varphi \leqslant 180°$$

Let the given system of n forces be reduced, with respect to a centre O, to a resultant force vector $\boldsymbol{R}'$ and a resultant couple whose moment is represented by a vector equal to the resultant moment $\boldsymbol{M}_O$, the angle between $\boldsymbol{R}'$ and $\boldsymbol{M}_O$ being $\varphi \neq 90°$ (Fig. 5.9). Now we resolve the vector $\boldsymbol{M}_O$ into two components one of which, $\boldsymbol{M}'_O$, goes along $\boldsymbol{R}'$ while the other component, $\boldsymbol{M}''_O$, is perpendicular to $\boldsymbol{R}'$. In the plane which contains the resultant vector $\boldsymbol{R}'$ and is perpendicular to the vector $\boldsymbol{M}''_O$ we replace $\boldsymbol{R}'$ and the couple with moment $\boldsymbol{M}''_O$ by the force $\boldsymbol{R}$ equivalent to them and applied at the point O' (see Sec. 2.4); let us transfer the vector $\boldsymbol{M}'_O$ to the same point. We see that the original system of n forces is equivalent to one force $\boldsymbol{R}$ and one couple $\boldsymbol{P}$, $-\boldsymbol{P}'$ whose plane of action is perpendicular to that force (Fig. 5.10).

The combination of a force and a couple whose plane of action is perpendicular to the force is referred to as a *wrench*.

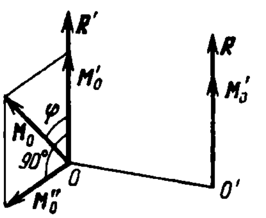

Fig. 5.9

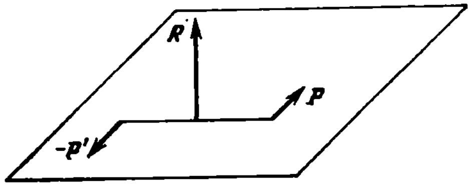

Fig. 5.10

Thus, in the *general case* when the resultant vector is not equal to zero and is not perpendicular to the resultant moment a spatial system of forces *is reduced to a wrench*. A wrench is the simplest form of a system to which a spatial system of forces can be reduced in the general case. The line of action of the force entering into the wrench is called the *central axis* of the given system of forces (the axis of the wrench).

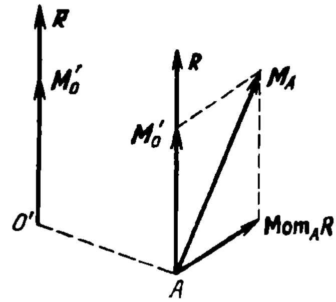

Fig. 5.11

It can be shown that the resultant moment of a system of forces with respect to any centre lying on the central axis has a minimum value. To this end let us take a point A not lying on the central axis and transfer to that point the force $\boldsymbol{R}$ and the couple with (vector) moment $\boldsymbol{M}'_O$; we thus obtain the same force $\boldsymbol{R}$ and some other (vector) moment $\boldsymbol{M}_A$ (Fig. 5.11). The moment $\boldsymbol{M}_A$ is equal to the vector sum of the vector $\boldsymbol{M}'_O$ and the moment of the associated couple which is equal to the moment of the force $\boldsymbol{R}$ about A (see formula (5.22)):

$$\boldsymbol{M}_A = \mathbf{Mom}_A\, \boldsymbol{R} + \boldsymbol{M}'_O$$

Since $\boldsymbol{M}'_O \perp \mathrm{Mom}_A\, \boldsymbol{R}$ we have

$$M_A = \sqrt{M_O'^2 + (\mathrm{Mom}_A\, \boldsymbol{R})^2} > M'_O \tag{5.28}$$

Let us derive formulas making it possible to determine the characteristics of the wrench from the projections X_ν, Y_ν and Z_ν ($\nu = 1, 2, \ldots, n$) of the forces, forming the system, on the coordinate axes and the coordinates x_ν, y_ν and z_ν of the points of application of the forces.

The modulus of the force $\boldsymbol{R}$ which is geometrically equal to the resultant force vector $\boldsymbol{R}'$ of the force system is found using the well-known formula

$$R = \sqrt{R_x'^2 + R_y'^2 + R_z'^2}$$

where

$$R'_x = \sum_{\nu=1}^{n} X_\nu, \quad R'_y = \sum_{\nu=1}^{n} Y_\nu, \quad R'_z = \sum_{\nu=1}^{n} Z_\nu$$

The modulus of the vector $\boldsymbol{M}'_O$ is equal to the absolute value of the projection of the resultant moment on the direction of the resultant force vector, that is

$$M'_O = |M_O \cos\varphi| = \frac{1}{R'} M_O R' \,|\cos\varphi| = \frac{1}{R'} |(\boldsymbol{M}_O, \boldsymbol{R}')| \tag{5.29}$$

where $(\boldsymbol{M}_O, \boldsymbol{R}')$ is the scalar product of the vectors $\boldsymbol{M}_O$ and $\boldsymbol{R}'$.

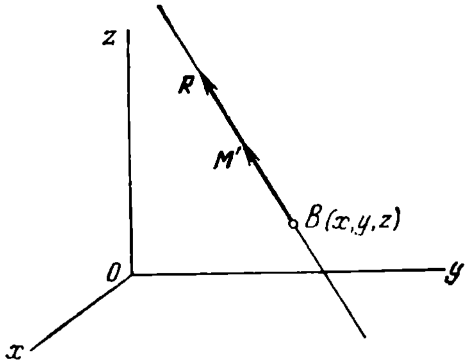

Fig. 5.12

Since the resultant force vector R' and the scalar product of the resultant moment by the resultant vector are static invariants, the moduli of the force R and of the moment M'_O of the couple forming the wrench are independent of the choice of the reduction centre. That is why in what follows we omit the subscript O and write M' instead of M'_O.

The substitution of the expression of the scalar product in terms of the projections of the vector factors into (5.29) results in

$$M' = \frac{1}{R'} \left| M_{Ox}R'_x + M_{Oy}R'_y + M_{Oz}R'_z \right| \tag{5.30}$$

where M_{Ox}, M_{Oy} and M_{Oz} are the sums of the moments of all the forces of the system about the axes Ox, Oy and Oz respectively.

Now let us derive the *equations of the central axis*.

We shall place the origin at an arbitrary point O not lying on the central axis (Fig. 5.12). Further, we take a point B with coordinates x, y and z lying on the central axis and place at that point the origins of the force vector R and of the (vector) moment M' of the couple forming the wrench. Let us form the expression of the resultant moment of the system of forces with respect to the centre O using formula (5.22) describing the relationship between the moments when the reduction centre is changed:

$$M_O = \mathrm{Mom}_O\, R + M'$$

Projecting the last equality on the coordinate axes we obtain

$$M_{Ox} = \mathrm{mom}_{Ox}\, R + M'_x$$

$$M_{Oy} = \mathrm{mom}_{Oy}\, R + M'_y$$

$$M_{Oz} = \mathrm{mom}_{Oz}\, R + M'_z$$

whence

$$
\begin{aligned}
M'_x &= M_{Ox} - \mathrm{mom}_{Ox}\, R \\
M'_y &= M_{Oy} - \mathrm{mom}_{Oy}\, R \\
M'_z &= M_{Oz} - \mathrm{mom}_{Oz}\, R
\end{aligned}
\tag{5.31}
$$

The moments of the force R about the coordinate axes are found by formulas (5.8):

$$
\begin{aligned}
\mathrm{mom}_{Ox}\, R &= yR'_z - zR'_y \\
\mathrm{mom}_{Oy}\, R &= zR'_x - xR'_z \\
\mathrm{mom}_{Oz}\, R &= xR'_y - yR'_x
\end{aligned}
\tag{5.32}
$$

The central axis is the locus of the reduction centres for which the vector R' is parallel to the resultant moment M'. Consequently, the projections of these vectors on the coordinate axes are proportional to one another:

$$\frac{M'_x}{R'_x} = \frac{M'_y}{R'_y} = \frac{M'_z}{R'_z}$$

Substituting the values of M'_x, M'_y and M'_z determined by (5.31) into these proportions and taking into account (5.32) we obtain the equations of the central axis:

$$\frac{M_{Ox}-(yR'_z-zR'_y)}{R'_x} = \frac{M_{Oy}-(zR'_x-xR'_z)}{R'_y} = \frac{M_{Oz}-(xR'_y-yR'_x)}{R'_z} \tag{5.33}$$

Putting in succession $x = 0$, $y = 0$ and $z = 0$ in these equations and solving them we obtain the coordinates of the points of intersection of the central axis with the coordinate planes.

EXAMPLE 5.1. In Fig. 5.13 are shown six forces $P_1 = 3$ kN, $P_2 = 4$ kN, $P_3 = 2$ kN, $P_4 = 5$ kN, $P_5 = 1$ kN and $P_6 = 6$ kN acting along the edges of a rectangular parallelepiped, the lengths of the edges being 4 m, 8 m and 6 m respectively. Reduce this force system to the simplest form and find the coordinates x and y of the point of intersection of the central axis with the plane Oxy.

Solution. In order to reduce the force system to the simplest form we first of all find the projections of the resultant force vector of that system on the coordinate axes and compute the sum of the moments of all the forces about each of the coordinate axes:

$$R'_x = P_3 + P_4 - P_6 = 1 \text{ kN}$$

$$R'_y = 0, \quad R'_z = P_1 + P_2 - P_5 = 6 \text{ kN}$$

$$M_{Ox} = -P_5 \cdot 8 = -8 \text{ kN}\cdot\text{m}$$

$$M_{Oy} = -P_2 \cdot 4 + P_3 \cdot 6 + P_4 \cdot 6 + P_5 \cdot 4 = 30 \text{ kN}\cdot\text{m}$$

$$M_{Oz} = -P_4 \cdot 8 + P_6 \cdot 8 = 8 \text{ kN}\cdot\text{m}$$

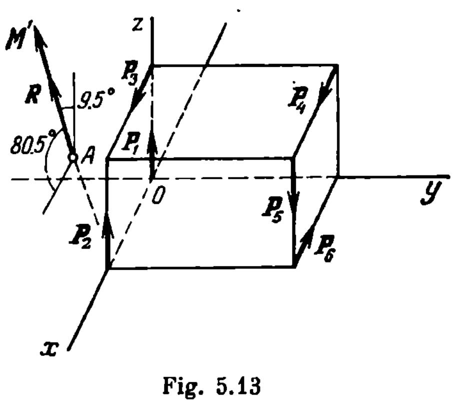

Fig. 5.13

We see that neither the resultant vector nor the resultant moment is equal to zero and therefore, in order to find whether the given system of forces can be reduced to a resultant, it remains to check whether the resultant vector is perpendicular to the resultant moment. To this end we express the scalar product of the resultant vector by the resultant moment in terms of the projections of the vector factors:

$$(\boldsymbol{R}',\ \boldsymbol{M}_O) = R'_x M_{Ox} + R'_y M_{Oy} + R'_z M_{Oz} = 40 \neq 0$$

Hence, the resultant vector and the resultant moment are not equal to zero and are not mutually perpendicular and therefore the given system of forces is reduced to a wrench.

Let us compute the force $\boldsymbol{R}$ and the moment $\boldsymbol{M}'$ of the couple forming the wrench. The modulus of the force $\boldsymbol{R}$ is

$$R = R' = \sqrt{R_x'^2 + R_y'^2 + R_z'^2} = 6.08 \text{ kN}$$

and, according to (5.30), the modulus of the vector $\boldsymbol{M}'$ (representing the moment of the couple) is

$$M' = \frac{1}{6.08}\ |-8\cdot 1 + 30\cdot 0 + 8\cdot 6| = 6.58 \text{ kN}\cdot\text{m}$$

Since the direction of the central axis coincides with that of the resultant force vector, the cosines of the angles between the central axis and the coordinate axes are found from (5.15):

$$\cos(\widehat{\boldsymbol{R},\ x}) = \frac{R'_x}{R'} = \frac{1}{6.08} = 0.164, \qquad (\widehat{\boldsymbol{R},\ x}) = 80.5^\circ$$

$$\cos(\widehat{\boldsymbol{R},\ y}) = \frac{R'_y}{R'} = \frac{0}{6.08} = 0, \qquad (\widehat{\boldsymbol{R},\ y}) = 90^\circ$$

$$\cos(\widehat{\boldsymbol{R},\ z}) = \frac{R'_z}{R'} = \frac{6}{6.08} = 0.987, \qquad (\widehat{\boldsymbol{R},\ z}) = 9.5^\circ$$

The central axis lies in a plane perpendicular to the axis Oy and forms an angle of 80.5° with the plane Oxy.

To determine the coordinates of the point A of intersection of the central axis with the plane Oxy let us substitute $z = 0$ into equations (5.33) of the central axis:

$$\frac{-8-(y\cdot 6-0)}{1} = \frac{30-(0-x\cdot 6)}{0} = \frac{8-(0-y\cdot 1)}{6}$$

From the last relation we derive the following two equations determining the coordinates x and y:

$$30 + 6x = 0, \quad 6(-8-6y) = 8 + y$$

From these equations we find

$$x_A = -5 \text{ m}, \qquad y_A = -\frac{56}{37} \text{ m}$$

The position of the wrench in space is shown in Fig. 5.13.

§ 3. Equilibrium Conditions for an Arbitrary System of Forces

3.1. Vector and Analytical Forms of Equilibrium Conditions. In the special case of reduction of a force system when the resultant vector $\boldsymbol{R}'$ and the resultant moment $\boldsymbol{M}_O$ are equal to zero the given

force system is balanced. Indeed, since the resultant vector is equal to zero all the forces applied at the reduction centre are mutually balanced, and since the resultant moment is equal to zero all the associated couples are balanced. Further, if the resultant vector and the resultant moment do not simultaneously vanish the force system is equivalent either to a resultant or to a couple of forces or to the combination of a resultant vector and a resultant couple, which means that the force system is not balanced.

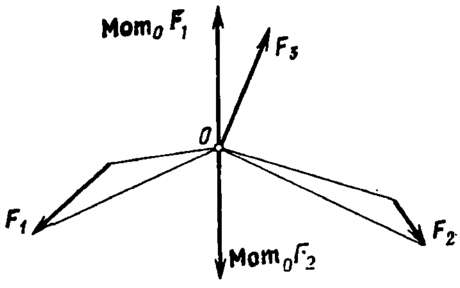

Fig. 5.14

Thus, for a system of forces applied to a rigid body to be balanced it is necessary and sufficient that the resultant force vector of that system and its resultant moment with respect to an arbitrary reduction centre should be equal to zero.

In the vector form the equilibrium conditions are written as

$$\boldsymbol{R}' = 0, \qquad \boldsymbol{M}_O = 0 \tag{5.34}$$

In the case of equilibrium the resultant moment with respect to any reduction centre is equal to zero (see formula (5.22)) and therefore instead of $\boldsymbol{M}_O$ we can write $\boldsymbol{M}$ without the subscript O.

To demonstrate the application of the vector form of equilibrium conditions let us prove the remark at the end of Sec. 2.6 of Chap. 1. We have to prove that if three forces are balanced then they must necessarily lie in one plane. Let us take an arbitrary point O on the line of action of the force $\boldsymbol{F}_3$ (Fig. 5.14). By the second condition (5.34), we have

$$\boldsymbol{M}_O = \mathbf{Mom}_O\, \boldsymbol{F}_1 + \mathbf{Mom}_O\, \boldsymbol{F}_2 = 0$$

Therefore

$$\mathbf{Mom}_O\, \boldsymbol{F}_2 = -\mathbf{Mom}_O\, \boldsymbol{F}_1$$

Consequently, the vectors $\mathbf{Mom}_O\, \boldsymbol{F}_2$ and $\mathbf{Mom}_O\, \boldsymbol{F}_1$ are in one line and have opposite directions. Finally, on the basis of the definition of the vector specifying the moment of a force (see Sec. 1.1) we conclude that the plane Π_1 containing the vector $\boldsymbol{F}_1$ and passing through the point O and the plane Π_2 containing the vector $\boldsymbol{F}_2$ and passing through the point O must coincide. This applies to any point O chosen on the line of action of the force $\boldsymbol{F}_3$, which proves the remark.

The moduli of the vectors $\boldsymbol{R}'$ and $\boldsymbol{M}$ are expressed in terms of their projections on the coordinate axes by the formulas

$$R' = \sqrt{R_x'^2 + R_y'^2 + R_z'^2}, \quad M = \sqrt{M_x^2 + M_y^2 + M_z^2} \tag{5.35}$$

Here R_x', R_y' and R_z' are the projections of the resultant force vector on the axes x, y and z respectively, and M_x, M_y and M_z are the projections of the resultant moment on the same axes.

The vectors $\boldsymbol{R}'$ and $\boldsymbol{M}$ vanish when the following six conditions are fulfilled:

$$R'_x = X_1 + X_2 + \ldots + X_n \equiv \sum_{\nu=1}^{n} X_\nu = 0$$

$$R'_y = Y_1 + Y_2 + \ldots + Y_n \equiv \sum_{\nu=1}^{n} Y_\nu = 0$$

$$R'_z = Z_1 + Z_2 + \ldots + Z_n \equiv \sum_{\nu=1}^{n} Z_\nu = 0$$

$$\tag{5.36}$$

$$M_x = \mathrm{mom}_x \boldsymbol{F}_1 + \mathrm{mom}_x \boldsymbol{F}_2 + \ldots + \mathrm{mom}_x \boldsymbol{F}_n \equiv \sum_{\nu=1}^{n} \mathrm{mom}_x \boldsymbol{F}_\nu = 0$$

$$M_y = \mathrm{mom}_y \boldsymbol{F}_1 + \mathrm{mom}_y \boldsymbol{F}_2 + \ldots + \mathrm{mom}_y \boldsymbol{F}_n \equiv \sum_{\nu=1}^{n} \mathrm{mom}_y \boldsymbol{F}_\nu = 0$$

$$M_z = \mathrm{mom}_z \boldsymbol{F}_1 + \mathrm{mom}_z \boldsymbol{F}_2 + \ldots + \mathrm{mom}_z \boldsymbol{F}_n \equiv \sum_{\nu=1}^{n} \mathrm{mom}_z \boldsymbol{F}_\nu = 0$$

Relations (5.36) express *analytical conditions of equilibrium*.

Thus, for an arbitrary system of forces applied to a rigid body to be balanced it is necessary and sufficient that the sum of the projections of all the forces on each coordinate axis and the sum of the moments of all the forces about each coordinate axis should be equal to zero.

From equilibrium conditions (5.36) it follows that in the general case of an arbitrary system of forces applied to one rigid body the system in question is statically determinate when the number of the unknown forces does not exceed six. When the number of the unknown forces exceeds six the system is statically indeterminate and the corresponding problem cannot be solved using the methods of statics.

3.2. Special Cases of an Arbitrary System of Forces. Here we shall consider the following three special cases.

(a) If the given system of forces is *concurrent* we can place the origin at the point of intersection of the lines of action of the forces. Then we readily note that $M_x = M_y = M_z \equiv 0$ in conditions (5.36), and therefore there remain three analytical conditions of equilibrium for a concurrent system of forces (see (1.28)):

$$\sum_{\nu=1}^{n} X_\nu = 0, \quad \sum_{\nu=1}^{n} Y_\nu = 0, \quad \sum_{\nu=1}^{n} Z_\nu = 0$$

(b) If all the forces *lie in one plane* we can take this plane as the coordinate plane Oxy. In this case we readily see that $R'_z \equiv 0$ and $M_x = M_y \equiv \equiv 0$ in (5.36) (see the remark to formula (5.3)). Therefore there remain three analytical conditions of equilibrium for a plane system of forces (see (3.7)):

$$\sum_{\nu=1}^{n} X_\nu = 0, \qquad \sum_{\nu=1}^{n} Y_\nu = 0$$

$$\sum_{\nu=1}^{n} \mathrm{mom}_O\, \boldsymbol{F}_\nu = 0$$

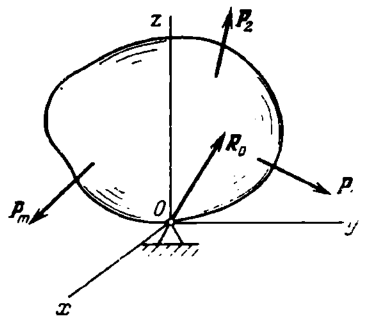

Fig. 5.15

Indeed, by the definition of the moment of a force about an axis, in the case under consideration we have the equality $M_z = m_O$, that is M_z is equal to the algebraic sum of the moments of the given forces about the centre O.

(c) If all the forces are *parallel* to one another and do not lie in one plane we can place the coordinate system so that one of the axes, say the axis z, is parallel to the forces. In this case we see that $R'_x = = R'_y \equiv 0$ and $M_z \equiv 0$ in (5.36), and there remain three analytical conditions of equilibrium for parallel forces:

$$\sum_{\nu=1}^{n} Z_\nu = 0, \qquad \sum_{\nu=1}^{n} \mathrm{mom}_x\, \boldsymbol{F}_\nu = 0, \qquad \sum_{\nu=1}^{n} \mathrm{mom}_y\, \boldsymbol{F}_\nu = 0 \qquad (5.37)$$

In each of Cases (a), (b) and (c) the corresponding system is statically determinate when the number of the unknown forces applied to one rigid body does not exceed three.

3.3. Equilibrium of a Rigid Body with One or Two Fixed Points. Let us consider the following special cases of equilibrium of a constrained rigid body under the action of a spatial system of forces.

(a) *Equilibrium of a rigid body with one fixed point.* Let a body having a ball-and-socket joint O be acted upon by m forces $\boldsymbol{P}_1$, $\boldsymbol{P}_2, \ldots, \boldsymbol{P}_m$ (Fig. 5.15). We shall place the origin of the coordinate system $Oxyz$ at the point O and denote the projections of the reaction of the ball-and-socket joint O (see Sec. 2.9 of Chap. 1) on the axes Ox, Oy and Oz by X_O, Y_O and Z_O respectively. Let us write equilibrium conditions (5.36):

$$\begin{gathered}\sum_{\mu=1}^{m} P_{\mu x} + X_O = 0, \quad \sum_{\mu=1}^{m} P_{\mu y} + Y_O = 0, \quad \sum_{\mu=1}^{m} P_{\mu z} + Z_O = 0 \\ \sum_{\mu=1}^{m} \mathrm{mom}_x\, \boldsymbol{P}_\mu = 0, \quad \sum_{\mu=1}^{m} \mathrm{mom}_y\, \boldsymbol{P}_\mu = 0, \quad \sum_{\mu=1}^{m} \mathrm{mom}_z\, \boldsymbol{P}_\mu = 0\end{gathered} \qquad (5.38)$$

The last three equations do not involve the unknown quantities and express equilibrium conditions. These three equations mean that $M_{Ox} = M_{Oy} = M_{Oz} = 0$, that is $\boldsymbol{M}_O = 0$. Thus, for a rigid body with one fixed point to be in equilibrium it is necessary and sufficient that the resultant moment of the system of forces acting on the body with respect to the fixed point should be equal to zero. In other words, for a rigid body with one fixed point to be in equilibrium the given system of forces acting on the body must reduce to one resultant force $\boldsymbol{R}$:

$$\boldsymbol{R} = -\boldsymbol{R}_O \qquad (5.39)$$

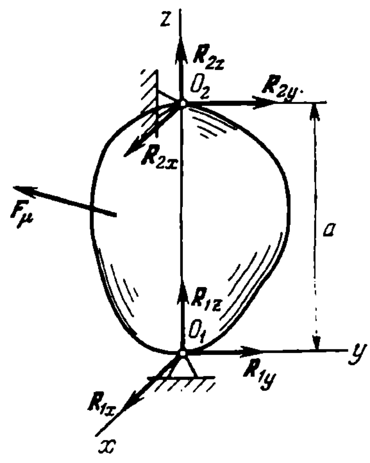

Fig. 5.16

this resultant passing through the fixed point. From the first three equations (5.38) we find the projections of the reaction $\boldsymbol{R}_O$:

$$X_O = -\sum_{\mu=1}^{m} P_{\mu x}, \qquad Y_O = -\sum_{\mu=1}^{m} P_{\mu y}, \quad Z_O = -\sum_{\mu=1}^{m} P_{\mu z}$$

Evidently the last equalities are nothing other than the expression of vector equality (5.39) in terms of the projections.

(b) *Equilibrium of a rigid body with a fixed axis.* Let a rigid body whose two points O_1 and O_2 are fixed with the aid of ball-and-socket joints be acted upon by m forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_m$ (Fig. 5.16). In this case the body can only rotate about the axis passing through the points O_1 and O_2. It is required to determine the reactions at the points O_1 and O_2 and to derive the condition on the forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_m$ under which the body is in equilibrium.

As in the case of the solution of statical problems in the plane, the origin and the direction of the coordinate axes should be chosen so that the equilibrium conditions take the simplest possible form; to this end the axes must be parallel (or perpendicular) to as many unknown forces as possible and must be placed in such a way that they are intersected by as many forces as possible. In the case under consideration we shall place the origin at the point O_1 and draw the axis O_1z along the axis of possible rotation of the body.

Let us resolve the reactions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ at the points O_1 and O_2 into components along the coordinate axes. The body is in equilibrium when the forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_m$ and the components $\boldsymbol{R}_{1x}$, $\boldsymbol{R}_{1y}$, $\boldsymbol{R}_{1z}$ and $\boldsymbol{R}_{2x}$, $\boldsymbol{R}_{2y}$, $\boldsymbol{R}_{2z}$ of the reactions satisfy equilibrium

conditions (5.36). Denoting the distance between the points O_1 and O_2 by a we write

$$
\left.\begin{aligned}
&\sum_{\mu=1}^{m} X_\mu + R_{1x} + R_{2x} = 0\\
&\sum_{\mu=1}^{m} Y_\mu + R_{1y} + R_{2y} = 0\\
&\sum_{\mu=1}^{m} Z_\mu + R_{1z} + R_{2z} = 0\\
&\sum_{\mu=1}^{m} \operatorname{mom}_x \boldsymbol{F}_\mu - R_{2y}a = 0\\
&\sum_{\mu=1}^{m} \operatorname{mom}_y \boldsymbol{F}_\mu + R_{2x}a = 0\\
&\sum_{\mu=1}^{m} \operatorname{mom}_z \boldsymbol{F}_\mu = 0
\end{aligned}\right\} \tag{5.40}
$$

The last equilibrium condition (5.40) does not involve the reactions at the fixed points; consequently, this is nothing other than the necessary and sufficient condition for the body to be in equilibrium under the action of the forces applied to it. The remaining five conditions contain six unknowns[1] and are equilibrium equations.

From the fourth and the fifth equations we find

$$R_{2y} = \frac{1}{a}\sum_{\mu=1}^{m} \operatorname{mom}_x \boldsymbol{F}_\mu, \quad R_{2x} = -\frac{1}{a}\sum_{\mu=1}^{m} \operatorname{mom}_y \boldsymbol{F}_\mu$$

Further, from the first and the second conditions we obtain

$$R_{1x} = \frac{1}{a}\sum_{\mu=1}^{m} \operatorname{mom}_y \boldsymbol{F}_\mu - \sum_{\mu=1}^{m} X_\mu$$

$$R_{1y} = -\frac{1}{a}\sum_{\mu=1}^{m} \operatorname{mom}_x \boldsymbol{F}_\mu - \sum_{\mu=1}^{m} Y_\mu$$

We see that there remains only one equation

$$R_{1z} + R_{2z} = -\sum_{\mu=1}^{m} Z_\mu \tag{5.41}$$

for the components of the reactions along the axis O_1z. However, from equation (5.41) only the sum of the components of the reactions along the axis O_1z can be found, and hence the problem on the deter-

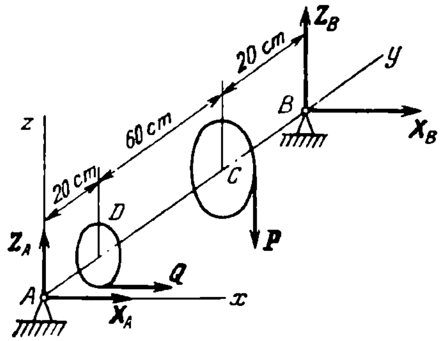

Fig. 5.17

mination of the reactions at the two fixed points involves a statically indeterminate system. Let us suppose that R_{1z} and R_{2z} satisfy equation (5.41); then the quantities $R^*_{1z} = R_{1z} + f$ and $R^*_{2z} = R_{2z} - f$ also satisfy this equation. Thus, there can exist an initial tension between the fixed points. It is the possibility of the existence of such an initial tension that makes the system in question statically indeterminate.

For the system to become statically determinate it is sufficient to realize the support at the point O_2 in the form of a cylindrical bearing. Then the component R_{2z} along the axis O_1z vanishes and there remain five unknowns which can be found from the five equations.

In this case we find from (5.41) the expression

$$R_{1z} = -\sum_{\mu=1}^{m} Z_\mu$$

EXAMPLE 5.2. Let us consider a horizontal shaft AB (Fig. 5.17) to which a wheel C of diameter 1 m and a gear D of radius 10 cm are attached; other dimensions are shown in the figure. The wheel C is acted upon by a vertical force $P = 15$ kN tangent to the wheel; to the gear D a horizontal force Q is applied (the force Q is tangent to the gear). Determine the force Q and the reactions of the bearings A and B for the state of equilibrium of the system.

Solution. For the solution of problems of statics in space it is advisable to follow the same scheme as in the case of plane statics. Namely, after the given and the unknown forces (under whose action the body is in equilibrium) are drawn in the figure representing the system in question and after the position of the coordinate system is chosen (that is the position of its origin and the directions of the coordinate axes are specified), we set the equilibrium equations; the solution of these equations result in the unknown forces (or other unknown quantities).

Let us place the origin at the point A and let the axis Ay be directed along the axis of the shaft; finally, let the axes Ax and Az be perpendicular to Ay and go horizontally and vertically respectively.

The shaft is in equilibrium under the action of the forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ and the reactions $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ of the bearings. Let us resolve the reactions of the bearings into components along the axes Ax and Az: $\boldsymbol{X}_A$, $\boldsymbol{Z}_A$ and $\boldsymbol{X}_B$, $\boldsymbol{Z}_B$. Here there are no components along the axis Ay because there are no constraints preventing the displacement of the shaft along that axis and consequently there are no reactions of the constraints.

To find the five unknowns Q, X_A, Z_A, X_B and Z_B we set equilibrium equations (5.36); in the case under consideration the number of these equations is also equal to five; two of the equations are obtained by equating to zero the sums of the projections of all the forces on the axes Ax and Az, and the other three by equating to zero the sums of the moments of all the forces about the

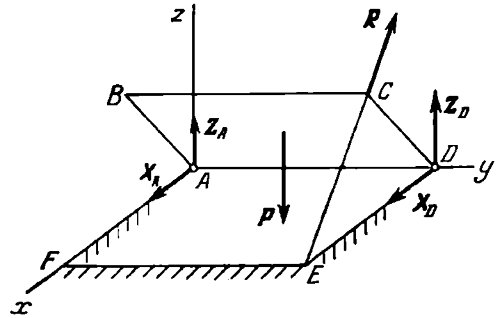

Fig. 5.18

axes Ax, Ay and Az:

$$\sum X = Q + X_A + X_B = 0 \tag{1}$$

$$\sum Z = -P + Z_A + Z_B = 0 \tag{2}$$

$$\sum \mathrm{mom}_x F = -P \cdot 0.8 + Z_B \cdot 1 = 0 \tag{3}$$

$$\sum \mathrm{mom}_y F = -Q \cdot 0.1 + P \cdot 0.5 = 0 \tag{4}$$

$$\sum \mathrm{mom}_z F = -Q \cdot 0.2 - X_B \cdot 1 = 0 \tag{5}$$

From equation (4) we find $Q = 5P = 75$ kN; then from (5) we obtain $X_B = -0.2Q = -15$ kN and from (1) we find $X_A = -Q - X_B = -60$ kN; finally, from (3) we find $Z_B = 0.8P = 12$ kN and from (2) we obtain $Z_A = P - Z_B = 3$ kN.

The minus signs in X_A and X_B mean that the real directions of the corresponding components of the reactions are opposite to those shown in the figure.

EXAMPLE 5.3. The lid of the rectangular box $ABCD$ shown in Fig. 5.18 is supported at the point C by the rod EC. The weight of the lid is $P = 15$ kN. $DC = DE$ and the angle EDC is equal to 60°. Under the assumption that the rod is weightless find the reactions of the cylindrical hinges A and D and also the stress S in the rod.

Solution. Let us place the origin at the point A and draw the axis Ay along the side AD, the axis Az vertically upward and the axis Ax along AF.

The lid is in equilibrium under the action of the force of gravity $\boldsymbol{P}$ applied at its centre of symmetry and directed vertically downward, the reaction $\boldsymbol{R}$ of the rod which goes along the rod in the direction from E to C, and the reactions $\boldsymbol{R}_A$ and $\boldsymbol{R}_D$ of the hinges A and D. Let us resolve the reactions of the hinges into components along the axes Ax and Az: $\boldsymbol{X}_A$, $\boldsymbol{Z}_A$ and $\boldsymbol{X}_D$, $\boldsymbol{Z}_D$. The components along the axis Ay are equal to zero because the cylindrical hinges A and D do not prevent the lid from displacing along the axis Ay. Now we shall write the equilibrium equations.

It should be noted that the angle CED is equal to 60° because $DC = DE$ and $\angle EDC = 60°$. Projecting all the forces on the axes Ax and Az we obtain

$$\sum X = X_A + X_D + R\cos 120° = 0 \tag{1}$$

$$\sum Z = -P + Z_A + Z_D + R\cos 30° = 0 \tag{2}$$

In order to find the moments of the forces about the axis Ax we project the forces on the plane Ayz, construct their vector components along that plane and then compute the moments of these components about the point A. To find the moments of the forces about the axes Ay and Az we first project the forces on the planes Azx and Axy and then compute the moments of the corresponding components about the point A. The equilibrium equations thus found have the form

$$\sum \text{mom}_x \boldsymbol{F} = -P\cdot\frac{1}{2}AD + Z_D\cdot AD + R\cos 30°\cdot AD = 0 \qquad (3)$$

$$\sum \text{mom}_y \boldsymbol{F} = P\cdot\frac{1}{2}AB\cos 60° - R\cdot AB\sin 60° = 0 \qquad (4)$$

$$\sum \text{mom}_z \boldsymbol{F} = -X_D\cdot AD + R\cos 60°\cdot AD = 0 \qquad (5)$$

Finally, cancelling equations (3) and (5) by AD and equation (4) by AB we obtain

$$X_A + X_D - \frac{1}{2}R = 0 \qquad (1')$$

$$Z_A + Z_D + \frac{\sqrt{3}}{2}R = 15 \qquad (2')$$

$$Z_D + \frac{\sqrt{3}}{2}R = 7.5 \qquad (3')$$

$$\frac{\sqrt{3}}{2}R = 3.75 \qquad (4')$$

$$-X_D + \frac{1}{2}R = 0 \qquad (5')$$

From (4′) we find $R = 4.33$ kN, and consequently the compressive stress in the rod EC is $S = -4.33$ kN. Equation (5′) results in $X_D = 2.16$ kN; further, from (1′) we find $X_A = 0$ and from (3′) we obtain $Z_D = 3.75$ kN; finally, (2′) yields $Z_A = 7.5$ kN.

EXAMPLE 5.4. A homogeneous rectangular plate $ABCD$ of weight P is supported by rods whose ends are connected by hinges to the plate and to immovable supports as is shown in Fig. 5.19. Applied to the vertex A of the plate is a force Q lying in the plane $AEHD$ and forming an angle β with the edge AD of the plate. Under the assumption that the weights of the six supporting rods can be neglected determine the stresses in the rods. The dimensions and the angles of the construction are given (they are indicated in Fig. 5.19), and $AM = MB$.

Solution. We shall consider the equilibrium of the plate $ABCD$. Let us place the origin at the point A and let the axes Ax, Ay and Az be drawn as shown in Fig. 5.19. Further, let us replace the action of the constraints, that is of the six supporting rods, by the corresponding reactions $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_3$, $\boldsymbol{R}_4$, $\boldsymbol{R}_5$ and $\boldsymbol{R}_6$ (see Sec. 2.9 (f) of Chap. 1). We draw these reactions, applied to the hinges attached to the plate, inside the corresponding rods. This means that before the problem has been solved each of the rods is supposed to be in tension. The forces $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, . . ., $\boldsymbol{R}_6$ form a spatial force system of general type; the construction dealt with in this problem is a statically determinate system because the number of the unknowns is equal to that of equilibrium equations (5.36).

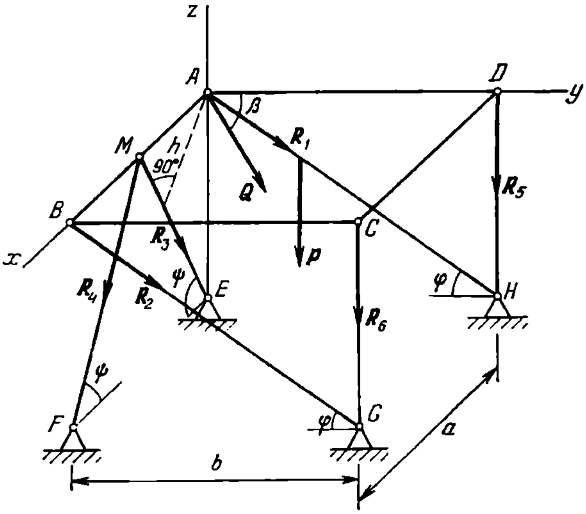

Fig. 5.19

Projecting the given forces P and Q and the reactions of the axes Ax, Ay and Az we obtain the first three equilibrium equations (5.36) of the form

$$\sum X = -R_3 \cos\psi + R_4 \cos\psi = 0 \tag{1}$$

$$\sum Y = Q \cos\beta + R_1 \cos\varphi + R_2 \cos\varphi = 0 \tag{2}$$

$$\sum Z = -P - Q\sin\beta - R_1 \sin\varphi - R_2 \sin\varphi - R_3 \sin\psi - R_4 \sin\psi - R_5 - R_6 = 0 \tag{3}$$

From equation (1) we find

$$R_4 = R_3 \tag{1'}$$

Now we pass to the second group of equilibrium equations (5.36); let us discuss the computation of the moments of the forces about the coordinate axes. We shall begin with finding which of these moments vanish. The forces Q and R_1 pass through the origin A and therefore their moments about each of the three coordinate axes are equal to zero. Further, we note that the forces R_2, R_3 and R_4 pass through the axis Ax, the force R_5 passes through the axis Ay and the lines of action of the forces R_3 and R_4 intersect the axis Az; therefore the corresponding moments are equal to zero. Finally, the forces P, R_5 and R_6 are parallel to the axis Az, and consequently their moments about that axis are equal to zero. Hence, the last three equilibrium equations (5.36) are written in the form

$$\mathrm{mom}_x P + \mathrm{mom}_x R_5 + \mathrm{mom}_x R_6 = 0 \tag{4}$$

$$\mathrm{mom}_y P + \mathrm{mom}_y R_2 + \mathrm{mom}_y R_3 + \mathrm{mom}_y R_4 + \mathrm{mom}_y R_6 = 0 \tag{5}$$

$$\mathrm{mom}_z R_2 = 0 \tag{6}$$

From equation (6) it follows that

$$R_2 = 0 \tag{6'}$$

because the force R_2 does not intersect the axis Az and is not parallel to it and its moment about the axis is equal to zero. In order to compute directly the moments of the forces P, R_5 and R_6 about the axes Ax and Ay it is advisable to draw through these forces the planes perpendicular to the indicated axes (see Sec. 1.2). The moments of the forces R_3 and R_4 about the axis Ay are computed by the second formula (5.8); taking into account that

$$x_M=\frac{1}{2}a,\quad z_M=0,\quad R_{3z}=R_{4z}=-R_3\sin\psi$$

we find

$$\mathrm{mom}_y\, R_3=\mathrm{mom}_y\, R_4=\frac{1}{2}aR_3\sin\psi$$

It should be noted that the same result can also be obtained without resorting to formulas (5.8). Indeed, in the first place, the force R_3 lies in the plane Azx perpendicular to the axis Ay and the arm h (see Sec. 1.2 and Fig. 5.19) is equal to (1/2) $a \sin\psi$, the moment of that force having plus sign because, if we look from the end of the axis Ay, the force R_3 produces counterclockwise rotation about the point A. In the second place (this technique is useful for forces whose direction does not coincide with those of the coordinate axes!), the force R_3 can be resolved into components along the coordinate axes, among which the first one, R_{3x}, passes through the axis Ay and has a zero moment about that axis while the second component is $R_{3z}=-R_3\sin\psi\cdot k$ and its moment is equal to that written above. An analogous argument applies to the force R_4.

Thus, taking into account (1′) and (6′) we can rewrite equations (2)-(5) in the form

$$Q\cos\beta+R_1\cos\varphi=0 \tag{2′}$$

$$-P-Q\sin\beta-R_1\sin\varphi-2R_3\sin\psi-R_5-R_6=0 \tag{3′}$$

$$-\frac{1}{2}Pb-R_5b-R_6b=0 \tag{4′}$$

$$\frac{1}{2}Pa+aR_3\sin\psi+R_6a=0 \tag{5′}$$

From equations (2′) and (4′) we obtain

$$R_1=-\frac{\cos\beta}{\cos\varphi}Q,\quad R_5+R_6=-\frac{1}{2}P$$

The substitution of these expressions into (3′) yields

$$R_3=\frac{1}{4\sin\psi}[2(\cos\beta\tan\varphi-\sin\beta)Q-P]$$

Further, from (5′) we obtain

$$R_6=-\frac{1}{4}P-\frac{1}{2}(\cos\beta\tan\varphi-\sin\beta)Q$$

and from (4′) we find

$$R_5=-\frac{1}{4}P+\frac{1}{2}(\cos\beta\tan\varphi-\sin\beta)Q$$

Thus, we have determined all the six reactions. The expression of R_1 is sure to be negative, and consequently the rod AH is not stretched but is compressed. The signs of the other reactions (and hence the stresses in the rods ME, MF, DH and CG) can be found depending on the quantities P, Q, β and φ. Since the stress in the rod BG is equal to zero this rod can be removed from the construction (of course, for the given force Q).

Problems

PROBLEM 5.1. Reduce to the simplest form the system of four forces with equal moduli $P_1 = P_2 = P_3 = P_4 = P$ applied at the four vertices of the cube with edge a as is shown in Fig. 5.20*a*.

Answer. $R' = \sqrt{2}P\,(j + k)$ and $M_O = \sqrt{2}Pa\,(-j + k)$; since $R' \perp M_O$, the given system of forces is reduced to one resultant $R = R'$ applied at the point A (Fig. 5.20*b*).

PROBLEM 5.2. Reduce to the simplest form the system of four forces $P_1 = 20$ N, $P_2 = 40$ N, $P_3 = 30$ N and $P_4 = 20$ N shown in Fig. 5.21*a* which are applied at the vertices of the cube and are directed along the edges of the cube, the length of the edge being equal to 1 m.

Answer. $R' = 40j + 30k$, $M_O = 20k$, $(\widehat{R', M_O}) \neq 90°$, $R' = 50$ N, $M' = \frac{1}{R'}\,|(M_O, R')| = 12$ N·m. The given system of forces is reduced to a wrench (Fig. 5.21*b*); the equations of the central axis are $x = \frac{8}{25}$, $y = \frac{4}{3}z$.

PROBLEM 5.3. A homogeneous plate $ABDE$ of weight $P = 6$ kN is lifted with the aid of three vertical ropes attached to the points A, K and E

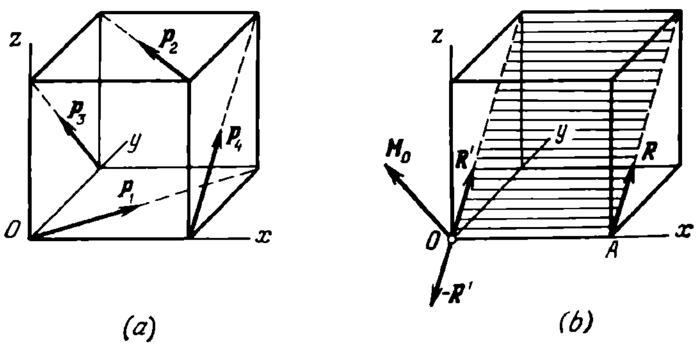

Fig. 5.20

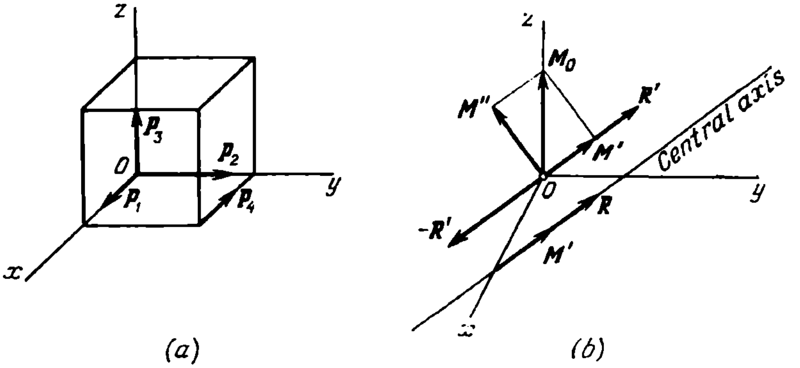

Fig. 5.21

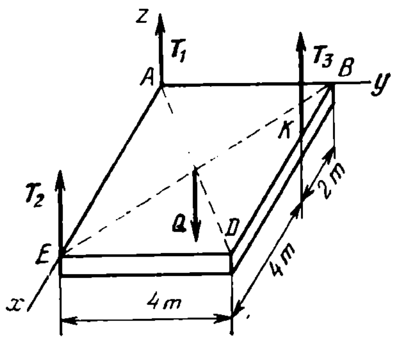

Fig. 5.22

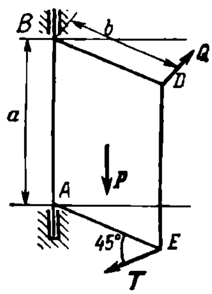

Fig. 5.23

(Fig. 5.22); the dimensions are indicated in the figure. Determine the tensions of the ropes.

Answer. $T_1 = 1$ kN, $T_2 = 2$ kN, $T_3 = 3$ kN.

PROBLEM 5.4. The door of weight P shown in Fig. 5.23 can rotate about the vertical axis AB having a step bearing at its lower end A and a cylindrical bearing at its upper end B. At the point D the door is acted upon by a force Q perpendicular to its plane and at the point E by a horizontal force T forming an angle of 45° with the edge AE. For the equilibrium state of the door determine the modulus of the force T and the reactions of the step bearing and the cylindrical bearing; the dimensions are $AB = a$ and $BD = b$.

Answer. $T = \sqrt{2}\,Q$, $X_A = -Q$, $X_B = Q$, $Y_A = Q + \frac{b}{2a}P$, $Y_B = -\frac{b}{2a}P$, $Z_A = P$.

Chapter 6 Centre of Parallel Forces and Centre of Gravity

§ 1. Centre of Parallel Forces

1.1. Reduction of a System of Parallel Forces to the Resultant. The composition of several parallel forces can be carried out by applying, in succession, the required number of times, the composition rule for two parallel forces (see Sec. 1.1 and Sec. 1.2 of Chap. 2).

Let us consider the problem of composition of four parallel forces applied at points A_1, A_2, A_3 and A_4 and having the same direction (Fig. 6.1). The modulus of the resultant $\boldsymbol{R}_1$ of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is equal to the sum of the moduli of these forces: $R_1 = F_1 + F_2$; the point of application C_1 of the resultant $\boldsymbol{R}_1$ lies on the line segment connecting the points A_1 and A_2 of application of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, is parallel to these forces and has the same direction. The

position of the point C_1 is specified by the relation

$$A_1C_1 = \frac{F_2}{R_1} A_1A_2 \qquad (6.1)$$

Adding together the forces R_1 and F_3 we obtain their resultant R_2 whose modulus is equal to the sum of the moduli of the forces R_1 and F_3: $R_2 = R_1 + F_3 = F_1 + F_2 + F_3$; the force R_2 is applied at the point C_2 lying on the line segment connecting the points C_1 and A_3 of application of the forces R_1 and F_3, is parallel to these forces and has the same direction. The position of the point C_2 is specified by the relation

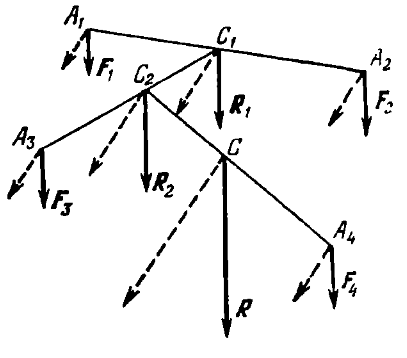

Fig. 6.1

$$C_1C_2 = \frac{F_3}{R_2} C_1A_3$$

Finally, adding together the forces R_2 and F_4 we obtain the sought-for resultant R whose modulus is equal to the sum of the moduli of the forces R_2 and F_4:

$$R = R_2 + F_4 = F_1 + F_2 + F_3 + F_4 \qquad (6.2)$$

The resultant R is parallel to R_2 and F_4, has the same direction and is applied at the point C lying on the line segment C_2A_4, the position of the point C being specified by the relation

$$C_2C = \frac{F_4}{R} C_2A_4$$

In an analogous manner we can find the magnitude and the point of application of the resultant of any number of parallel forces. Thus, the resultant of several parallel forces having the same direction is parallel to these forces and has the same direction as the given forces; its modulus is equal to the sum of the moduli of the given forces, and the position of the point of application of the resultant depends on the magnitudes of the given forces and on the positions of their points of application.

Let us show that the position of the point C is not changed when all the forces are rotated about their points of application so that they remain parallel to one another.

Let us turn the forces F_1, F_2, F_3 and F_4 through one and the same angle about their points of application so that they remain mutually parallel (in Fig. 6.1 the new positions of the forces are shown by dash lines). The resultant R_1 of the forces F_1 and F_2 is again parallel to these forces, has the same direction as the forces, and its modulus

is equal to the sum of the moduli of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$; the position of the point C_1 is again specified by relation (6.1). Since the moduli of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ remain invariable (as well as the positions of the points A_1 and A_2) the line segment A_1C_1 remains the same, that is the position of the point C_1 does not change. Thus, when the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are turned through an angle about the points A_1 and A_2 the resultant of these forces remains parallel to them and turns about the point C_1 through the same angle. Applying the same argument to the forces $\boldsymbol{R}_1$ and $\boldsymbol{F}_3$ and then to $\boldsymbol{R}_2$ and $\boldsymbol{F}_4$ we conclude that when all the forces are turned through an angle without violating their parallelism the resultant of the forces turns through the same angle about the same point C and its modulus remains invariable. The point C of application of the resultant is called the *centre of the system of parallel forces.*

If not all the parallel forces have one direction the course of the argument remains the same but in order to determine the modulus of the resultant and its point of application we must make use of the relations derived in Sec. 1.2 of Chap. 2 for the composition of parallel forces having opposite directions. In this case the modulus of the resultant can be found using a formula analogous to (6.2) in which the moduli of the forces enter into the sum with plus sign when the directions of the forces coincide with that chosen as a positive one and with minus sign when the forces have the opposite direction.

1.2. Centre of Parallel Forces. We shall derive formulas for computing the coordinates of the centre of parallel forces.

Let there be n parallel forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$; for the sake of generality we shall suppose that not all the forces have the same direction. The coordinates of the points of application of the forces will be denoted by

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \ldots, \quad (x_n, y_n, z_n)$$

The resultant of this force system is parallel to the given forces, and its modulus is equal to the algebraic sum of the moduli of the forces:

$$R = \sum_{\nu=1}^{n} \pm F_\nu$$

Here we exclude the case when $\sum \pm F_\nu = 0$. In this special case the system of parallel forces is either equivalent to zero or equivalent to a couple of forces, that is it has no resultant, and the determination of the centre of parallel forces becomes senseless. We shall also assume that $\sum \pm F_\nu > 0$ (if otherwise, the chosen positive direction should simply be changed to the opposite).

The coordinates of the point C of application of the resultant, that is the coordinates of the centre of the system of parallel forces in question, will be denoted by x_C, y_C, z_C.

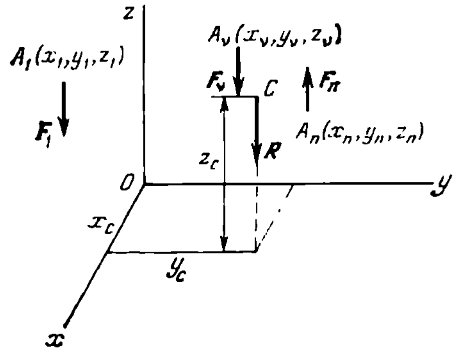

Fig. 6.

Let us turn all the forces so that they become parallel to the axis Oz (Fig. 6.2). The resultant $\boldsymbol{R}$ will also be parallel to Oz. Now we shall compute the moment of the resultant about the axis Oy. By Varignon's theorem (5.27) and formula (5.6), we conclude that the moment of the resultant about the axis Oy is equal to the sum of the moments of the given forces about the same axis. Since in the present case the arms of the forces are equal to the abscissas of the points of their application we have

$$Rx_C = F_1x_1 + F_2x_2 + \ldots + F_nx_n = \sum_{\nu=1}^{n} F_\nu x_\nu \qquad (6.3)$$

whence

$$x_C = \frac{1}{R}\sum_{\nu=1}^{n} F_\nu x_\nu = \frac{\sum\limits_{\nu=1}^{n} F_\nu x_\nu}{\sum\limits_{\nu=1}^{n} F_\nu} \qquad (6.4)$$

(we remind the reader that the values of F_ν in the numerator and in the denominator are taken with plus sign when the directions of the forces coincide with that of the axis and with minus sign when the directions of the forces are opposite to that of the axis).

Further, let us turn all the forces about their points of application so that they become parallel to the axis Ox and then equate the expressions for the moment of the resultant to the sum of the moments of the given forces about the axis Oz:

$$Ry_C = F_1y_1 + F_2y_2 + \ldots + F_ny_n = \sum_{\nu=1}^{n} F_\nu y_\nu$$

whence

$$y_C = \frac{\sum_{\nu=1}^{n} F_\nu y_\nu}{\sum_{\nu=1}^{n} F_\nu} \tag{6.5}$$

Finally, turning all the forces about their points of application so that they become parallel to the axis Oy and equating the expressions for the moment of the resultant to the sum of the moments of the given forces about the axis Ox we obtain

$$z_C = \frac{\sum_{\nu=1}^{n} F_\nu z_\nu}{\sum_{\nu=1}^{n} F_\nu} \tag{6.6}$$

Formulas (6.4)-(6.6) make it possible to determine the coordinates of the centre of parallel forces from the given coordinates of their points of application. Let us multiply formulas (6.4)-(6.6) by the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ respectively and then add them together; this yields one vector formula for the radius vector $\boldsymbol{r}_C = x_C \boldsymbol{i} + y_C \boldsymbol{j} + z_C \boldsymbol{k}$ of the centre of parallel forces:

$$\boldsymbol{r}_C = \frac{F_1 \boldsymbol{r}_1 + F_2 \boldsymbol{r}_2 + \ldots + F_n \boldsymbol{r}_n}{F_1 + F_2 + \ldots + F_n} = \frac{\sum_{\nu=1}^{n} F_\nu \boldsymbol{r}_\nu}{\sum_{\nu=1}^{n} F_\nu} \tag{6.7}$$

where $\boldsymbol{r}_\nu = x_\nu \boldsymbol{i} + y_\nu \boldsymbol{j} + z_\nu \boldsymbol{k}$ $(\nu = 1, 2, \ldots, n)$ is the radius vector of the point of application of the force $\boldsymbol{F}_\nu$.

§ 2. Centre of Gravity

2.1. General Formulas for the Coordinates of the Centre of Gravity. Every rigid body can be regarded as a collection of a large number of very small particles. Each of these particles is attracted to the Earth by a force which is directed vertically downward and is called the *force of gravity of the given particle*. Since in the present course of statics we deal with bodies which are members of various engineering constructions whose dimensions are small in comparison with those of the Earth we can assume that the forces of gravity of separate particles are parallel to one another. The resultant of the forces of gravity of all the particles is equal to their sum: $P = \sum p_\nu$; it is called the *weight of the body*, and the centre of these parallel forces is called the *centre of gravity of the body*.

It should be noted that the centre of gravity of a rigid body occupies a definite position, relative to the body, independent of the

location of the body itself in space. Indeed, when the rigid body is rotated the forces of gravity of separate particles of the body retain their vertically downward direction, these forces rotating with respect to the body about their points of application and remaining mutually parallel. As was shown in the foregoing section, in this case the position of the centre of parallel forces is invariable.

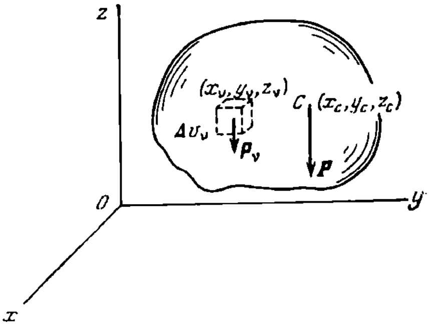

Fig. 6.3

Let us find the coordinates of the centre of gravity of a rigid body of weight P. To this end we mentally break the rigid body into a great number of particles. Let us denote by p_ν the force of gravity of the νth particle, and let the coordinates of a point belonging to the νth particle be (x_ν, y_ν, z_ν) (Fig. 6.3). Then, according to (6.4), (6.5) and (6.6), we have

$$x_C=\frac{1}{P}\sum p_\nu x_\nu,\quad y_C=\frac{1}{P}\sum p_\nu y_\nu,\quad z_C=\frac{1}{P}\sum p_\nu z_\nu \qquad (6.8)$$

Let us express the force of gravity of the νth particle of the body in terms of its mass: $p_\nu = m_\nu g$. We shall consider bodies whose dimensions are not very large; for such bodies g is a constant quantity, one and the same for all the particles, and is equal to the acceleration of gravity at the given place on the Earth. Therefore *formulas* (6.8) *for the coordinates of the centre of gravity of the body* are written in the form

$$x_C=\frac{1}{M}\sum m_\nu x_\nu,\quad y_C=\frac{1}{M}\sum m_\nu y_\nu,\quad z_C=\frac{1}{M}\sum m_\nu z_\nu \qquad (6.9)$$

where $M=\sum m_\nu = P/g$ is the mass of the body. Multiplying formulas (6.9) by the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ respectively and adding them together we obtain one vector formula for the radius vector $\boldsymbol{r}_C = x_C\boldsymbol{i} + y_C\boldsymbol{j} + z_C\boldsymbol{k}$ of the centre of gravity of the body:

$$\boldsymbol{r}_C=\frac{1}{M}\sum m_\nu \boldsymbol{r}_\nu \qquad (6.10)$$

where $\boldsymbol{r}_\nu = x_\nu\boldsymbol{i} + y_\nu\boldsymbol{j} + z_\nu\boldsymbol{k}$ is the radius vector of the νth particle of the body.

Let us consider the special case of a *homogeneous* rigid body. We shall denote the volume of the νth particle of the rigid body by Δv_ν. Then the force of gravity of that particle can be written in the form $p_\nu = \gamma\,\Delta v_\nu$, where γ is the weight of unit volume of the rigid body. Since γ is constant for the homogeneous body the substitution of

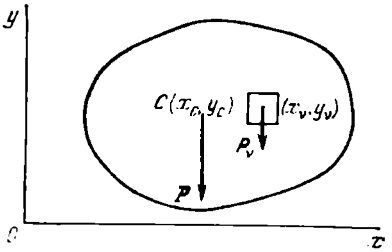

Fig. 6.4

$p_\nu = \gamma\,\Delta v_\nu$ into (6.8) results in

$$x_C = \frac{\sum \gamma\,\Delta v_\nu x_\nu}{\sum \gamma\,\Delta v_\nu} = \frac{1}{V}\sum x_\nu\,\Delta v_\nu$$

$$y_C = \frac{1}{V}\sum y_\nu\,\Delta v_\nu \qquad (6.11)$$

$$z_C = \frac{1}{V}\sum z_\nu\,\Delta v_\nu$$

where $V = \sum \Delta v_\nu$ is the volume of the body.

Expressions given by formula (6.11) are independent of the constant γ and are called the *coordinates of the centre of gravity of the volume*. In other words, by the centre of gravity of a volume is meant the centre of gravity of a homogeneous body occupying that volume.

Let us suppose that the given rigid body is a very thin homogeneous plate of constant width. We denote the area of the νth particle of the plate by Δs_ν and the weight of unit area of the plate by σ; then the force of gravity of the νth particle of the plate is $p_\nu = \sigma\,\Delta s_\nu$. Drawing the coordinate axes Ox and Oy in the plane of the plate (Fig. 6.4) and substituting the value of p_ν into (6.8) we obtain, for the homogeneous plate, the formulas

$$x_C = \frac{\sum \sigma\,\Delta s_\nu x_\nu}{\sum \sigma \Delta s_\nu} = \frac{1}{S}\sum x_\nu\,\Delta s_\nu, \quad y_C = \frac{1}{S}\sum y_\nu\,\Delta s_\nu \qquad (6.12)$$

where $S = \sum \Delta s_\nu$ is the area of the plate.

The quantities x_C and y_C are referred to as the *coordinates of the centre of gravity (centroid) of the area of the given plane figure*; they are the coordinates of the centre of gravity of a homogeneous plate of constant width having the shape of that figure.

The quantity $\sum x_\nu\,\Delta s_\nu$ is called the *static moment of the area S* with respect to the axis Oy; accordingly, $\sum y_\nu\,\Delta s_\nu$ is called the static moment of the area with respect to the axis Ox. We shall denote the static moments by W_y and W_x; then

$$x_C = \frac{1}{S}W_y, \quad y_C = \frac{1}{S}W_x \qquad (6.13)$$

If the axis Oy passes through the centre of gravity of an area then $x_C = 0$ and the static moment is equal to zero: $W_y = 0$. Similarly, if the axis Ox passes through the centre of gravity of the area then $y_C = 0$ and $W_x = 0$. Thus, the *static moment of an area with respect to an axis passing through the centre of gravity of that area is equal to zero.*

The coordinates of the centre of gravity of a curve are defined and computed by analogy with the cases of volume and area. By the *centre of gravity of a curve* is meant the centre of gravity of a thin homogeneous wire of constant cross section whose axis coincides with that of the given curve. Let us denote the weight of unit length of the wire by q; then the force of gravity of the νth particle of the wire of length Δl_ν is $p_\nu = q\,\Delta l_\nu$ (Fig. 6.5). Formulas (6.8) yield

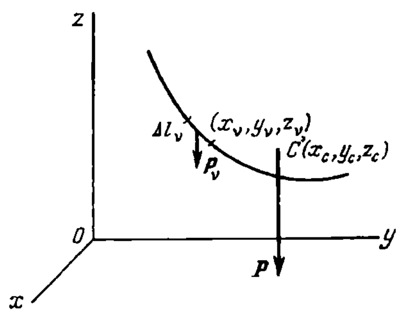

Fig. 6.5

$$x_C = \frac{1}{L}\sum x_\nu\,\Delta l_\nu, \quad y_C = \frac{1}{L}\sum y_\nu\,\Delta l_\nu, \quad z_C = \frac{1}{L}\sum z_\nu\,\Delta l_\nu \qquad (6.14)$$

where L is the length of the curve.

It should be noted that formulas (6.11), (6.12) and (6.14) are approximate. The smaller the particles into which the given volume or area or curve is divided the greater is the accuracy of these formulas. They become exact in the limiting case when the dimensions of the particles of the body tend to zero while the number of the particles increases indefinitely. For instance, the exact value of the coordinate x_C of the centre of gravity of a volume is written in the form

$$x_C = \frac{1}{V}\lim_{\max \Delta v_\nu \to 0} \sum x_\nu\,\Delta v_\nu \qquad (6.15)$$

The numerators of the exact expressions for x_C, y_C and z_C in formulas (6.11), (6.12) and (6.14) involve analogous limiting sums. Such sums are called *integrals*; the rules for their computation are studied in the course of integral calculus.

In some cases the exact determination of the centres of gravity of volumes, areas and curves can be carried out without resorting to integral calculus.

Let us consider a homogeneous body possessing a plane of symmetry. We draw the coordinate axes so that the coordinate plane Oxy coincides with the plane of symmetry. Then to each element of volume with coordinate z_ν there corresponds a similar element of volume with coordinate $-z_\nu$. Consequently,

$$\sum z_\nu\,\Delta v_\nu = 0 \text{ and } z_C = 0$$

Thus, the *centre of gravity of a homogeneous body having a plane of symmetry lies in that plane of symmetry.*

If a homogeneous body has an axis of symmetry the *centre of gravity* of the body lies on that *axis of symmetry*. Indeed, in this case

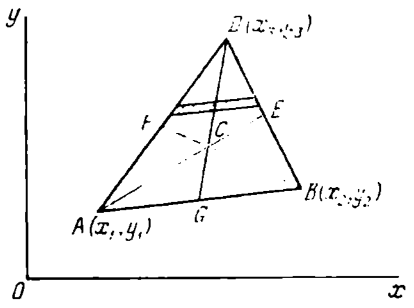

Fig. 6.6

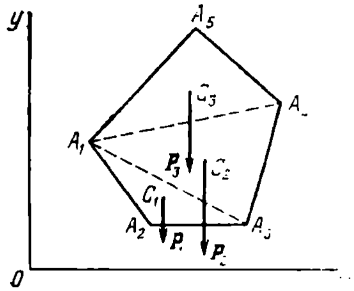

Fig. 6.7

we can draw one of the coordinate axes, for instance, the axis Oz, along the axis of symmetry; then to each element of volume Δv_ν with coordinates (x_ν, y_ν, z_ν) there corresponds a similar element of volume Δv_ν with coordinates $(-x_\nu, -y_\nu, -z_\nu)$ and therefore

$$\sum x_\nu \Delta v_\nu = \sum y_\nu \Delta v_\nu = 0 \text{ and } x_C = y_C = 0$$

Finally, let us consider a homogeneous body possessing the centre of symmetry. If we place the origin O at the centre of symmetry then to each element of volume with coordinates (x_ν, y_ν, z_ν) there corresponds a similar element of volume with coordinates $(-x_\nu, -y_\nu, -z_\nu)$ and therefore

$$\sum x_\nu \Delta v_\nu = \sum y_\nu \Delta v_\nu = \sum z_\nu \Delta v_\nu = 0 \quad \text{and} \quad x_C = y_C = z_C = 0$$

Consequently, the *centre of gravity of a homogeneous body possessing the centre of symmetry is located at that centre of symmetry.*

In the case of a plane figure possessing an axis of symmetry the centre of gravity of the figure lies on that axis; if there are two axes of symmetry the centre of gravity of the figure lies at the point of intersection of these axes.

2.2. Determination of the Centres of Gravity of Simplest Plane Figures, Curves and Bodies. Using the results of Sec. 2.1 we shall find in this section the coordinates of the centres of gravity of some simplest figures, curves and bodies.

(a) *The centre of gravity of a triangle.* Let us consider the triangle ABD shown in Fig. 6.6. We shall divide this triangle into a great number of thin strips with the aid of straight lines parallel to AB. The centre of gravity of each strip is at its midpoint, that is it lies on the median DG of the triangle. Consequently, the centre of gravity of the whole triangle also lies on that median. Breaking the triangle into thin strips parallel to the side BD we similarly conclude that the centre of gravity lies on the median AE; hence, the centre of gravity C of the triangle is at the point of intersection of the medi-

ans of the triangle. Let us denote the coordinates of the vertices of the triangle by (x_1, y_1), (x_2, y_2) and (x_3, y_3); then by the formulas of analytical geometry (see [7]), we find

$$x_C=\frac{1}{3}(x_1+x_2+x_3), \quad y_C=\frac{1}{3}(y_1+y_2+y_3) \tag{6.16}$$

(b) *The centre of gravity of a polygon.* Let there be a thin homogeneous plane plate of constant width having the shape of a polygon, the coordinates of the vertices of the polygon being known (Fig. 6.7). We shall break the polygon into triangles and find the coordinates of the centre of gravity of each triangle using formulas (6.16):

$$x_{C_1}=\frac{1}{3}(x_1+x_2+x_3), \quad y_{C_1}=\frac{1}{3}(y_1+y_2+y_3)$$

$$x_{C_2}=\frac{1}{3}(x_1+x_3+x_4), \quad y_{C_2}=\frac{1}{3}(y_1+y_3+y_4)$$

$$x_{C_3}=\frac{1}{3}(x_1+x_4+x_5), \quad y_{C_3}=\frac{1}{3}(y_1+y_4+y_5)$$

At these centres of gravity the weights P_1, P_2 and P_3 of the triangles are applied. In order to determine the centre of gravity of the whole polygon it only remains to find the centre of these three parallel forces. According to (6.9), we have

$$x_C=\frac{P_1x_{C_1}+P_2x_{C_2}+P_3x_{C_3}}{P_1+P_2+P_3}, \quad y_C=\frac{P_1y_{C_1}+P_2y_{C_2}+P_3y_{C_3}}{P_1+P_2+P_3} \tag{6.17}$$

Since the plate is homogeneous, instead of (6.17) we can write the formulas

$$x_C=\frac{S_1x_{C_1}+S_2x_{C_2}+S_3x_{C_3}}{S_1+S_2+S_3}, \quad y_C=\frac{S_1y_{C_1}+S_2y_{C_2}+S_3y_{C_3}}{S_1+S_2+S_3} \tag{6.18}$$

From (6.18) it follows that if the area of a figure can be broken up into parts whose areas and coordinates of the centres of gravity are known then the centre of gravity of the whole figure can be found using formulas analogous to (6.12):

$$x_C=\frac{1}{S}\sum S_ix_{C_i}, \quad y_C=\frac{1}{S}\sum S_iy_{C_i} \tag{6.19}$$

Here S_i is the area of the ith part of the figure, and x_{C_i} and y_{C_i} are the coordinates of its centre of gravity.

The result we have obtained can be extended to the cases of determining the coordinates of the centres of gravity of volumes and curves composed of parts whose volumes, lengths and also the coordinates of the centres of gravity are known:

$$x_C=\frac{1}{V}\sum V_ix_{C_i}, \quad y_C=\frac{1}{V}\sum V_iy_{C_i}, \quad z_C=\frac{1}{V}\sum V_iz_{C_i} \tag{6.20}$$

and

$$x_C=\frac{1}{L}\sum L_ix_{C_i}, \quad y_C=\frac{1}{L}\sum L_iy_{C_i}, \quad z_C=\frac{1}{L}\sum L_iz_{C_i} \tag{6.21}$$

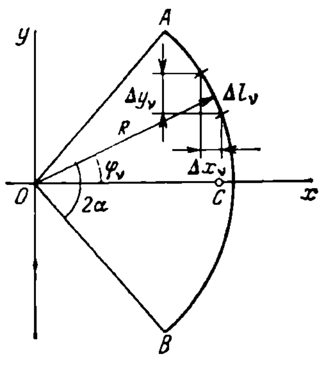

Fig. 6.8

where V_i is the volume of the ith part of the body and L_i is the length of the ith part of the curve.

(c) *The centre of gravity of an arc of a circle.* Let us break the circular arc AB of length L shown in Fig. 6.8 into small parts. Next we replace every part of the arc of length Δl_ν by the corresponding chord and then construct a right triangle with that chord as hypotenuse. Further, in the first formula (6.14) we simultaneously multiply and divide the expression under the summation sign by $\cos\varphi_\nu$:

$$x_C = \frac{1}{L}\sum x_\nu\,\Delta l_\nu = \frac{1}{L}\sum \frac{x_\nu}{\cos\varphi_\nu}\,\Delta l_\nu \cos\varphi_\nu \tag{6.22}$$

From Fig. 6.8 we find $\Delta l_\nu \cos\varphi_\nu = \Delta y_\nu$ and $x_\nu/\cos\varphi_\nu = R$. Substituting the expressions we have obtained into (6.22) we find

$$x_C = \frac{1}{L}\sum R\,\Delta y_\nu = \frac{R}{L}\sum \Delta y_\nu = \frac{R}{L}\,AB \tag{6.23}$$

This result can be rewritten in some other form. Taking into account that the length of the arc is $L = 2\alpha R$ and the length of the chord is $AB = 2R\sin\alpha$, instead of (6.23) we can write the expression

$$x_C = \frac{R}{2\alpha R}\,2R\sin\alpha = \frac{\sin\alpha}{\alpha}\,R \quad (y_C = 0) \tag{6.24}$$

For the arc of a semicircle (Fig. 6.9) we have $2\alpha = \pi$, and formula (6.24) yields

$$x_C = \frac{\sin(\pi/2)}{\pi/2}\,R = \frac{2}{\pi}\,R = 0.637R \quad (y_C = 0) \tag{6.25}$$

(d) *The centre of gravity of a sector of a circle.* Let us divide the sector of a circle OAB of radius R with central angles $\Delta\varphi_\nu$ shown in

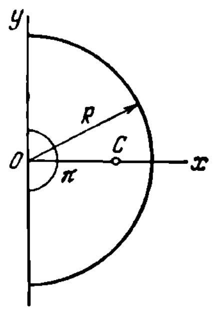

Fig. 6.9

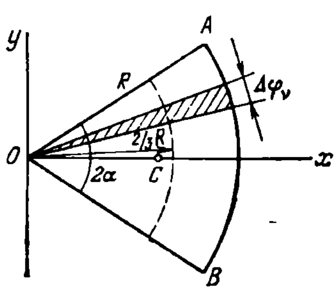

Fig. 6.10

Fig. 6.10 into many elementary sectors. Each of these sectors can be regarded as a triangle with altitude equal to R. The centre of gravity of each triangle lies at a distance of $2R/3$ from the centre O. Consequently, the position of the centre of gravity of the sector coincides with that of the centre of gravity of an arc of a circle with radius $2R/3$ and central angle 2α. From formula (6.24) we obtain

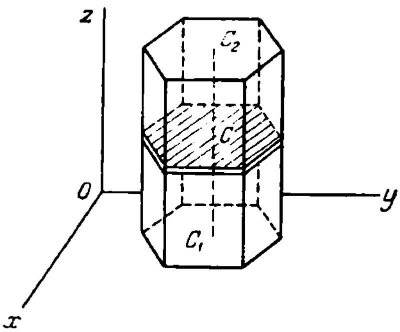

Fig. 6.11

$$x_C = \frac{2}{3}\frac{\sin\alpha}{\alpha}R \quad (y_C = 0) \tag{6.26}$$

For the area of a semicircle (Fig. 6.10) we have $\alpha = \pi/2$, and formula (6.26) yields

$$x_C = \frac{2}{3}\frac{\sin(\pi/2)}{\pi/2}R = \frac{4}{3\pi}R = 0.424R \quad (y_C = 0) \tag{6.27}$$

(e) *The centre of gravity of a homogeneous prism.* Let us divide the prism shown in Fig. 6.11 into thin plates of equal width parallel to the base of the prism. The centres of gravity of these plates lie on the straight line connecting the centres of gravity C_1 and C_2 of the upper and the lower bases. Since the weights of the plates are the same, the centre of gravity of the prism coincides with that of the homogeneous line segment C_1C_2. Consequently, the centre of gravity C of a homogeneous prism is at the midpoint of the line segment connecting the centres of gravity of the upper and the lower bases of the prism.

The same result is obtained if instead of a homogeneous prism we consider a homogeneous cylinder of an arbitrary cross section (both right and oblique).

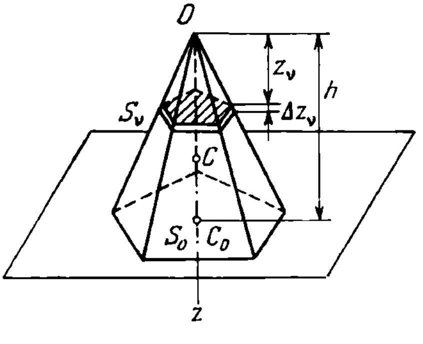

Fig. 6.12

(f) *The centre of gravity of a pyramid.* Arguing as above we readily see that the centre of gravity of a homogeneous pyramid must lie on the straight line joining its vertex O with the centre of gravity C_0 of its base (Fig. 6.12). Now we shall determine the z-coordinate of the centre of gravity of the pyramid, the origin being placed at the vertex.

Let the altitude of the pyramid be h, the area of the base be S_0,

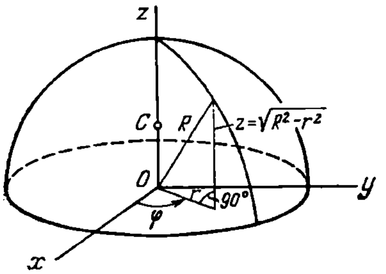

Fig. 6.13

and let the axis Oz go vertically downward from the vertex O. We consider an element of the pyramid of altitude Δz_ν lying between two sections parallel to the base at a distance z_ν from the vertex. The volume of this element is $\Delta v_\nu = S_\nu \Delta z_\nu$, where S_ν is the area of the section at the distance z_ν from the vertex O. On the basis of the theorem on the areas of similar figures we have $S_\nu/S_0 = z_\nu^2/h^2$, whence

$$S_\nu = \frac{z_\nu^2}{h^2} S_0$$

By formula (6.11), we obtain

$$z_C = \frac{1}{V} \sum z_\nu \Delta v_\nu = \frac{S_0}{Vh^2} \sum z_\nu^3 \Delta z_\nu \qquad (6.28)$$

The exact value of z_C is found as the limiting value of the sum in (6.28) for $\max \Delta z_\nu \to 0$, which reduces to the computation of the integral:

$$z_C = \frac{S_0}{Vh^2} \lim_{\max \Delta z_\nu \to 0} \sum z_\nu^3 \Delta z_\nu = \frac{S_0}{Vh^2} \int_0^h z^3\, dz = \frac{S_0 h^2}{4V}$$

The substitution of the value $V = S_0 h/3$ into the last expression results in the sought-for value z_C of the z-coordinate of the centre of gravity of the pyramid reckoned *from the vertex*:

$$z_C = \frac{3}{4} h \qquad (6.29)$$

The argument used in the derivation of formula (6.28) remains applicable to the case of a pyramid with a figure of an arbitrary shape as the base. Consequently, formula (6.29) is also valid for a homogeneous cone with an arbitrary base.

Thus, the centre of gravity of an arbitrary pyramid or cone lies on the line segment connecting the vertex with the centre of gravity of the base, the distance between the centre and the plane of the base being equal to one fourth of the altitude of the pyramid or the cone.

(g) *The centre of gravity of a hemisphere* (Fig. 6.13). By formula (6.15), we have

$$z_C = \frac{1}{V} \iiint_D z\, dv$$

Let us compute the triple integral over the volume D of the hemisphere by passing to cylindrical coordinates r, φ, z (see [1]):

$$\iiint\limits_D z\,dv = \int_0^{2\pi} d\varphi \int_0^R r\,dr \int_0^{\sqrt{R^2-r^2}} z\,dz = 2\pi \int_0^R \frac{1}{2}(R^2-r^2)\,r\,dr$$

$$= \pi\left[R^2\cdot\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^R = \frac{1}{4}\pi R^4$$

Hence, for the z-coordinate of the centre of gravity of the hemisphere reckoned from the diametral plane we obtain

$$z_C = \frac{1}{\frac{2}{3}\pi R^3}\cdot\frac{1}{4}\pi R^4 = \frac{3}{8}R \qquad (6.30)$$

2.3. Determination of the Centres of Gravity of Plane Figures and Solids of Complex Form. When determining the position of the centre of gravity of a figure or a body (solid) we must first of all make use of the *symmetry properties* (see the end of Sec. 2.1) provided there are such. It is also advisable to use formulas (6.19)-(6.21).

Here we shall also mention the so-called *method of negative masses*. As was mentioned in Sec. 1.2, the formulas for the centre of parallel forces remain valid in the case when the directions of some of the forces are opposite to those of the other forces. Therefore when a figure or a solid is divided into parts we may conditionally regard their cavities as having negative areas or volumes. In other words, formulas (6.19) or (6.20) remain valid when some of the quantities S_i or V_i are negative. As to the sum of all S_i or V_i (both positive and negative), it must be equal to the area of the given figure S or to the volume V of the given solid.

EXAMPLE 6.1. Find the centre of gravity of the trapezoid $OABD$ with bases $OA = a$ and $DB = b$ and altitude h (Fig. 6.14).

Solution. By analogy with Sec. 2.2 (a), we readily see that the centre of gravity C lies on the straight line EF connecting the centres of the bases. In order to determine graphically the position of the point C on the straight line EF we break the trapezoid into two triangles OAD and ABD whose centres of gravity are at C_1 and C_2 respectively. The straight line C_1C_2 meets EF at the centre of gravity C of the trapezoid.

In order to determine analytically the position of the centre of gravity C on EF it is sufficient to find one of the coordinates of C (it is more convenient to find y_C). The ordinates of the centre of gravity and the areas of the triangles OAD and ABD are given by the formulas

$$y_{C_1} = \frac{1}{3}h, \quad S_1 = \frac{1}{2}ah;$$

$$y_{C_2} = \frac{2}{3}h, \quad S_2 = \frac{1}{2}bh$$

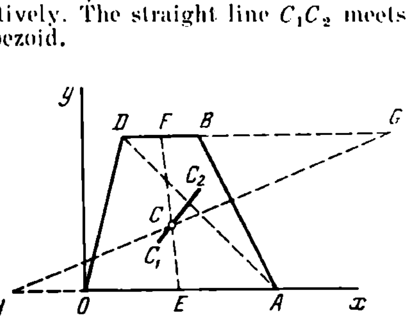

Fig. 6.14

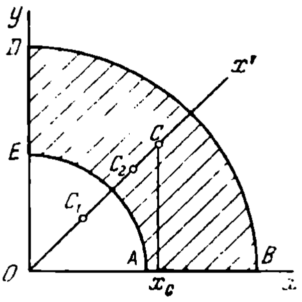

Fig. 6.15

By formula (6.19), for the ordinate of the centre of gravity of the trapezoid we obtain

$$y_C = \frac{1}{\frac{1}{2}(a+b)h}\left[\frac{h}{3}\frac{ah}{2}+\frac{2h}{3}\frac{bh}{2}\right] = \frac{a+2b}{3(a+b)}h$$

The last formula implies a more convenient method for constructing the point C. Let set off the line segments $BG = a$ and $OH = b$ on the extension of the bases (see Fig. 6.14). The line GH intersects EF at the centre of gravity C of the trapezoid. Indeed, the proportionality of the line segments between the parallel lines HE and FG implies

$$\frac{y_C}{h-y_C} = \frac{b+\frac{1}{2}a}{\frac{1}{2}b+a}$$

whence follows the above formula for y_C.

EXAMPLE 6.2. Find the centre of gravity of the quarter of a circular ring with radii R and r (Fig. 6.15).

Solution. Let us place the origin at the point O and let the axis Ox' be directed along the axis of symmetry. We shall regard the figure in question as the sector OBD with the sectorial cavity OAE. The area of the greater sector is

$$S_1 = \frac{1}{4}\pi R^2$$

and the area of the smaller sector is considered negative:

$$S_2 = -\frac{1}{4}\pi r^2$$

The abscissas of the centres of gravity of the sectors reckoned along the axis Ox' are determined by formula (6.26), where $\alpha = \pi/4$ (note that here α denotes the half-angle of the sector):

$$x'_{C_1} = \frac{2}{3}\frac{\sin(\pi/4)}{\pi/4}R = \frac{4\sqrt{2}}{3\pi}R, \qquad x'_{C_2} = \frac{4\sqrt{2}}{3\pi}r$$

Using formulas (6.19) we obtain

$$x'_C = \frac{S_1x'_{C_1}+S_2x'_{C_2}}{S_1+S_2} = \frac{\frac{4\sqrt{2}}{3\pi}R\frac{\pi R^2}{4}-\frac{4\sqrt{2}}{3\pi}r\frac{\pi r^2}{4}}{\frac{1}{4}\pi R^2-\frac{1}{4}\pi r^2} = \frac{4\sqrt{2}}{3\pi}\frac{R^3-r^3}{R^2-r^2}$$

In the coordinate system Oxy (see Fig. 6.15) we have

$$x_C = y_C = x'_C\cos\frac{\pi}{4} = \frac{4}{3\pi}\frac{R^3-r^3}{R^2-r^2} = 0.424\frac{R^2+Rr+r^2}{R+r}$$

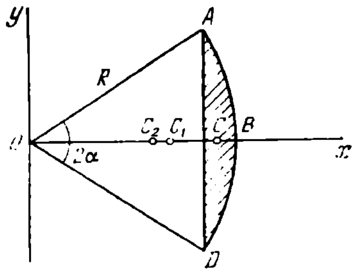

Fig. 6.16

The same expression is also obtained for the abscissa of the centre of gravity of a circular semiring if the axis Ox is its axis of symmetry.

In particular, for $r = 0$ the last formula yields the expression for the coordinates of the centre of gravity of a quarter of a circle with respect to the axes coinciding with its bounding radii or, which is the same, for the abscissa of the centre of gravity of a semicircle if the axis Ox is its axis of symmetry (see (6.27)).

EXAMPLE 6.3. Find the centre of gravity of a segment of a circle, the radius of the circle being R and the central angle being 2α (Fig. 6.16).

Solution. Since the figure in question has an axis of symmetry, namely Ox, we have $y_C = 0$. Let us regard the segment ABD as the sector $OABD$ with a triangular cavity OAD. The area of the sector is

$$S_1 = \frac{1}{2} R^2 \cdot 2\alpha = \alpha R^2$$

and the area of the triangle is considered negative:

$$S_2 = -R \sin\alpha \cdot R \cos\alpha = -R^2 \sin\alpha \cos\alpha$$

The abscissa of the centre of gravity C_1 of the sector is determined by formula (6.26):

$$x_1 = \frac{2}{3} \frac{\sin\alpha}{\alpha} R$$

The centre of gravity C_2 of the triangle is at the point of intersection of its medians, that is

$$x_2 = \frac{2}{3} R \cos\alpha$$

By formula (6.19), we obtain

$$x_C = \frac{S_1 x_1 + S_2 x_2}{S_1 + S_2} = \frac{\alpha R^2 \cdot \frac{2}{3} \frac{\sin\alpha}{\alpha} R - R^2 \sin\alpha \cos\alpha \cdot \frac{2}{3} R \cos\alpha}{\alpha R^2 - R^2 \sin\alpha \cos\alpha}$$

$$= \frac{2}{3} \frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha} R$$

EXAMPLE 6.4. Determine the centre of gravity of a body composed of a cylinder and a hemisphere and having a conic cavity (see Fig. 6.17 where the dimensions are indicated).

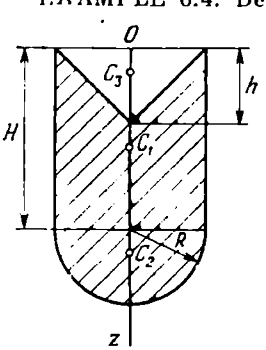

Fig. 6.17

Solution. The axis Oz is the axis of symmetry of the body in question and therefore $x_C = y_C = 0$. Let us compute the volumes V_1, V_2 and V_3 of the cylinder, the hemisphere and the conic cavity respectively:

$$V_1 = \pi R^2 H, \quad V_2 = \frac{2}{3} \pi R^3, \quad V_3 = -\frac{1}{3} \pi R^2 h$$

The z-coordinates of the centres of gravity C_1, C_2 and C_3 are

$$z_1 = \frac{1}{2} H, \quad z_2 = H + \frac{3}{8} R, \quad z_3 = \frac{1}{4} h$$

By formula (6.20), we obtain

$$z_C=\frac{V_1z_1+V_2z_2-V_3z_3}{V_1+V_2-V_3}$$

$$=\frac{\pi R^2H\cdot\frac{1}{2}H+\frac{2}{3}\pi R^3\left(H+\frac{3}{8}R\right)-\frac{1}{3}\pi R^2h\cdot\frac{1}{4}h}{\pi R^2H+\frac{2}{3}\pi R^3-\frac{1}{3}\pi R^2h}$$

$$=\frac{\frac{1}{2}H^2+\frac{2}{3}RH+\frac{1}{4}R^2-\frac{1}{12}h^2}{H+\frac{2}{3}R-\frac{1}{3}h}=\frac{1}{4}\,\frac{6H^2+8RH+3R^2-h^2}{3H+2R-h}$$

Problems

PROBLEM 6.1. Prove that the centre of gravity of a quadrilateral $ABDE$ can be found using the following construction: each of the sides is divided into three equal parts and through the points of division adjoining each of the vertices straight lines are drawn; the resultant figure $FGHK$ is a parallelogram the point of intersection of whose diagonals is the sought-for centre of gravity (Fig. 6.18).

Hint. Draw the diagonal AD dividing the given quadrilateral into two triangles and prove that the centres of gravity of these triangles coincide with those of the corresponding parallelograms into which the straight line AD divides the parallelogram $FGHK$.

PROBLEM 6.2. From a semicircle of radius R a smaller semicircle is cut out, the radius R of the former semicircle serving as the diameter of the latter (Fig. 6.19). Find the centre of gravity C of the remaining part.

Answer. $x_C=\frac{5}{6}R$, $y_C=\frac{14}{9\pi}R=0.495R$.

PROBLEM 6.3. A solid consists of a cylinder and a cone, their bases being equal and adjoining each other. The altitude of the cylinder is equal to H and the altitude of the cone is equal to h. For what value of the ratio h/H does the centre of gravity of the solid coincide with the centre of the base of the cone?

Answer. $\frac{h}{H}=6$.

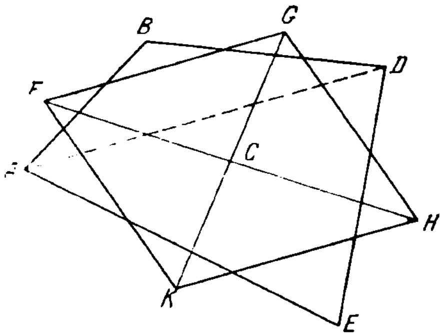

Fig. 6.18

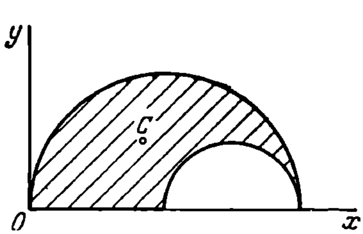

Fig. 6.19

KINEMATICS

Part 2

Introduction to Kinematics

1. Mechanical Motion. Theoretical mechanics treats of mechanical motion and equilibrium of material bodies. By *mechanical motion* is understood the displacement of the bodies relative to one another; it is the simplest form of motion in the material world.

Motion is a form of the existence of matter and one of the results of the interaction between the bodies. In the study of various types of motion, in particular, mechanical motion, we can abstract from some features of the phenomenon and consider a simplified model of the motion.

Kinematics is a branch of theoretical mechanics in which the *mechanical motion is studied from a purely geometrical point of view* without taking into account the interactions causing the motion. Kinematics studies variation of the geometrical configuration of the bodies *with time*.

2. Space and Time. Mechanical motion is caused by the various changes in the material world. All the observed forms of mechanical motion take place in space and time. Space and time are the form of the existence of the material world.

In the course of theoretical mechanics we consider the three-dimensional space whose main properties are homogeneity, isotropy and continuity. For the measurements in space a unit length is chosen. In the International System of Units (SI) the unit of length is the *metre*.

In classical mechanics time is considered universal and formally independent of the motion of material bodies for all the points in space. The mean solar second equal to 1/86 400 of the mean solar day* is taken as unit measure of time. The course of time is assumed to be continuous, and to each arbitrary instant there corresponds a definite point on an infinite straight line—the axis of time. In kinematics the spatial relationships and the division of time into equal intervals are based on the real properties of the moving objects.

* At present the astronomical standard is replaced by the atomic time standard.

At the beginning of the 20th century the principles of classical mechanics were subjected to criticism which resulted in the appearance of relativistic and quantum mechanics. Here we do not go into details and only mention that the principles of the theory of relativity elaborated by J. C. Maxwell (1831-1879), H. A. Lorentz (1853-1928), H. Poincaré (1854-1912) and A. Einstein (1879-1955) essentially changed the usual concepts of space and time. In this theory communication between observers connected with various moving systems is carried out with the aid of light signals, and it is postulated that the speed of light is a universal constant for all the systems and has one and the same value. Relativistic mechanics does not refute classical mechanics but only indicates the limits within which the laws of classical mechanics remain valid; it shows that classical mechanics is applicable when the velocities of the bodies are small in comparison with the speed of light and that these laws are not valid when the velocities are commensurable with the speed of light.

3. Frame of Reference. In theoretical mechanics and in kinematics, its branch, rigid bodies are considered (see Sec. 2.1 of Chap. 1). The simplest rigid body is a particle, a material point, whose dimensions are negligibly small in comparison with the spatial parameters of the motion in question. Hence, depending on the character of the mechanical motion, a given body can be regarded either as a body with finite dimensions (for instance, such is the Earth when its rotation about its own axis is considered) or as a particle (for instance, such is the Earth when its orbital motion is considered).

In order to study the motion of a material body from the geometrical point of view we must know the change of the position of the body in space with time. This, however, is impossible if there are no other bodies (forming a frame or system of reference) relative to which the position of the moving body is determined. If the three-dimensional space in which the motion takes place (for instance, the motion of a particle) were "empty", that is if there were no material bodies other than the particle in question, it would be impossible to specify the position of the particle.

Indeed, since space is homogeneous (this means that any of the points is indistinguishable from all the other points) it is impossible to preferably choose a certain point in order to take it as the origin. Consequently, in such an "empty" space there would be no distinction between motion and rest. Thus, mechanical motion must be understood as a relative motion.

I. Newton (1643-1727) postulated the existence of both absolute space and absolutely fixed bodies in this space (the "fixed" stars) and thus indicated an absolutely immovable frame of reference relative to which the positions of the moving bodies can be determined. I. Newton also postulated the existence of absolute time. All these postulates are necessary for introducing the concept of

absolute motion. However, the works by I. Newton show that he understood that these concepts are only applicable within certain limits.

4. Historical Notes on the Development of Kinematics. Mechanics, a science treating of the motion and equilibrium of material bodies, emerged many centuries ago but kinematics, its branch, was developed comparatively recently. The basic concepts of kinematics such as velocity and acceleration (in rectilinear motion) were introduced by G. Galileo (1564-1642) in the first half of the 17th century. He also stated the composition of velocities law. The general concept of acceleration was introduced by I. Newton. Kinematics of a rigid body was developed by L. Euler (1707-1783).

A. M. Ampère (1775-1836) suggested that the study of mechanical motion from a purely geometrical point of view should be considered as a separate branch of mechanics and introduced the name kinematics (from the Greek κινησισ meaning motion). L. Poinsot (1777-1859) was the first to indicate the possibility of composition and resolution of rotations and introduced the concept of an instantaneous axis of rotation.

In the geometrical studies of L. Poinsot and J. V. Poncelet (1788-1867) the foundation of a very important branch of technical science known as kinematics of machines and mechanisms was created.

N. E. Joukowski (1847-1921) stressed the great advantages of the geometrical method of investigation in mechanics and indicated that kinematics makes it possible to interpret in a visual geometrical way analytical results (formulas).

Kinematics, like theoretical mechanics as a whole, makes the basis of modern engineering. In kinematics the investigation of mechanical motion is carried out with the help of the method of mathematical analysis. However, kinematics is not a branch of mathematics. Its main principle is common for all natural sciences: to state correctly the problem using, when necessary, abstraction and then to return from the abstraction to the real motion on the basis of observation and experiment.

We shall begin the study of kinematics with a brief representation of the theory of differentiation of a variable vector.

5. Differentiation of a Variable Free Vector. A vector whose origin can be placed at any point in space is called a *free vector*. An example of a free vector is the (vector) moment of a couple of forces (see Sec. 1.4 of Chap. 5).

Let us consider a fixed coordinate system $Oxyz$ in space and let a variable free vector be given, that is a free vector which varies with time: $\boldsymbol{a} = \boldsymbol{a}(t)$. Such a variable vector is sometimes referred to as a *vector function* of one scalar argument t. Let us take the vectors $\boldsymbol{a}(t)$ and $\boldsymbol{a}(t + \Delta t)$ corresponding to instants t and $t + \Delta t$ and transfer their origins to the point O; this results in two vectors $\boldsymbol{a}_O(t)$ and

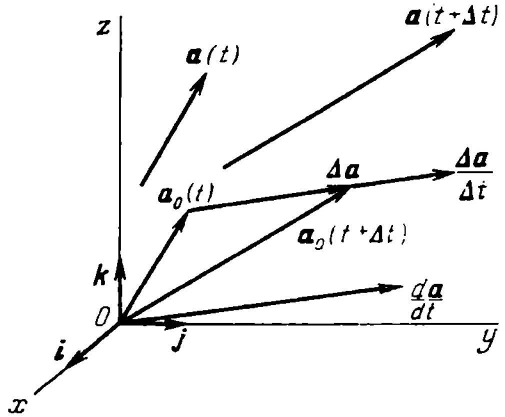

Fig. I.1

a_O $(t + \Delta t)$ (Fig. I.1) equal to the former vectors. The subscript O indicates that the original vectors have been transferred to the fixed point O. Now we construct the vector

$$\Delta \boldsymbol{a} = \boldsymbol{a}_O\,(t + \Delta t) - \boldsymbol{a}_O\,(t)$$

This vector is called the *increment* of the vector $\boldsymbol{a}$ (t). We also construct the vector $\Delta\boldsymbol{a}/\Delta t$ which is collinear to $\boldsymbol{a}$ (t). By the *derivative of the vector* $\boldsymbol{a}$ (t) *with respect to the scalar argument* t is understood a *vector equal (both in its modulus and direction) to the limit of the ratio of the increment of the vector to the increment of its scalar argument*:

$$\frac{d\boldsymbol{a}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{a}}{\Delta t}$$

The derivative $d\boldsymbol{a}/dt$ of a free vector $\boldsymbol{a}$ (t) is also a free vector, and its origin can be placed at any point; in particular, it can be placed at the point O or at the end of the vector $\boldsymbol{a}_O$ (t).

From the definition of the derivative of a vector it follows that the derivative of a constant vector (that is of a vector which does not change with time) is a zero vector and that for the derivative of a sum of vectors and for the derivative of the product of a vector by a constant scalar λ there hold the formulas

$$\frac{d}{dt}(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \ldots + \boldsymbol{a}_n) = \frac{d\boldsymbol{a}_1}{dt} + \frac{d\boldsymbol{a}_2}{dt} + \ldots + \frac{d\boldsymbol{a}_n}{dt}$$

$$\frac{d}{dt}(\lambda \boldsymbol{a}) = \lambda \frac{d\boldsymbol{a}}{dt}$$

A vector $\boldsymbol{a}$ can be represented in the form of the vector sum of its components:

$$\boldsymbol{a} = a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k}$$

where a_x, a_y and a_z are the projections of the vector on the coordinate axes, and $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors along the fixed coordinate axes. For the variable vector

$$\boldsymbol{a}(t) = a_x(t)\,\boldsymbol{i} + a_y(t)\,\boldsymbol{j} + a_z(t)\,\boldsymbol{k}$$

its derivative with respect to the scalar argument t is equal to

$$\frac{d\boldsymbol{a}}{dt} = \frac{da_x}{dt}\,\boldsymbol{i} + \frac{da_y}{dt}\,\boldsymbol{j} + \frac{da_z}{dt}\,\boldsymbol{k} \tag{1}$$

because the vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are constant. From the last formula it follows that the *projections of the derivative of a variable vector on the fixed axes are equal to the derivatives of the corresponding projections of the vector.*

Besides the derivative of a vector function with respect to its scalar argument, we can also consider the differential of the vector function by which the principal part of the increment $\Delta\boldsymbol{a}$ is meant. By analogy with the differential of a scalar function, the differential of the vector function corresponding to time Δt is expressed by the formula

$$d\boldsymbol{a}(t) = \dot{\boldsymbol{a}}(t)\,dt$$

(we remind the reader that the increment Δt of the argument is equal to its differential, i.e. dt). Here and henceforth the dot above a function denotes the derivative with respect to t; two dots denote the second derivative, that is

$$\dot{\boldsymbol{a}}(t) \equiv \frac{d\boldsymbol{a}}{dt}, \quad \ddot{\boldsymbol{a}}(t) \equiv \frac{d\dot{\boldsymbol{a}}}{dt} = \frac{d^2\boldsymbol{a}}{dt^2}$$

The scalar product of two vectors is expressed in terms of the projections of the given vector by means of formula (1.10):

$$(\boldsymbol{a}, \boldsymbol{b}) = a_xb_x + a_yb_y + a_zb_z$$

Therefore, using the formulas for the differentiation of a sum and of a product of scalar functions we obtain the following expression for the derivative of the scalar product of two variable vectors:

$$\frac{d}{dt}(\boldsymbol{a}, \boldsymbol{b}) = \left(\frac{da_x}{dt}\,b_x + \frac{da_y}{dt}\,b_y + \frac{da_z}{dt}\,b_z\right) + \left(a_x\frac{db_x}{dt} + a_y\frac{db_y}{dt} + a_z\frac{db_z}{dt}\right)$$

The expression on the right-hand side of the last formula is nothing other than the sum of the scalar products of the vectors $d\boldsymbol{a}/dt$, $\boldsymbol{b}$ and $\boldsymbol{a}$, $d\boldsymbol{b}/dt$; thus, we obtain the formula

$$\frac{d}{dt}(\boldsymbol{a}, \boldsymbol{b}) = \left(\frac{d\boldsymbol{a}}{dt}, \boldsymbol{b}\right) + \left(\boldsymbol{a}, \frac{d\boldsymbol{b}}{dt}\right) \tag{2}$$

The vector product of two vectors is expressed in terms of the projections of the given vectors by means of determinant (1.16):

$$[\boldsymbol{a}, \boldsymbol{b}] = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

It can also be written in full in the form

$$[\boldsymbol{a}, \boldsymbol{b}] = (a_y b_z - a_z b_y)\, \boldsymbol{i} + (a_z b_x - a_x b_z)\, \boldsymbol{j} + (a_x b_y - a_y b_x)\, \boldsymbol{k}$$

It should be noted that the projections of the vector product on the axes y and z, that is the second and the third brackets, are obtained from the projection on the axis x, that is from the first brackets, by means of circular permutation under which the subscript x goes into y, y goes into z and z into x. Bearing in mind this fact, in the further transformations we shall write only the expression in the first brackets. For the derivative of the vector product of two variable vectors we find the expression

$$\frac{d}{dt}[\boldsymbol{a}, \boldsymbol{b}] = \left(\frac{da_y}{dt} b_z + a_y \frac{db_z}{dt} - \frac{da_z}{dt} b_y - a_z \frac{db_y}{dt} \right) \boldsymbol{i} + \ldots$$

$$= \left(\frac{da_y}{dt} b_z - \frac{da_z}{dt} b_y \right) \boldsymbol{i} + \ldots \quad \left(a_y \frac{db_z}{dt} - a_z \frac{db_y}{dt} \right) \boldsymbol{i}$$

We have obtained the sum of the vector products of the vectors $d\boldsymbol{a}/dt$, $\boldsymbol{b}$ and $\boldsymbol{a}$, $d\boldsymbol{b}/dt$ and thus derived the formula

$$\frac{d}{dt}[\boldsymbol{a}, \boldsymbol{b}] = \left[\frac{d\boldsymbol{a}}{dt}, \boldsymbol{b} \right] + \left[\boldsymbol{a}, \frac{d\boldsymbol{b}}{dt} \right] \tag{3}$$

Since the vector product essentially depends on the order of the factors, this order should be carefully kept when we use formula (3).

In mechanics, besides free vectors, sliding and localized (bound) vectors are also used (see Sec. 1.1 of Chap. 1). Since the definition of the derivative was stated for a free vector some stipulations are needed when we differentiate sliding or localized vectors. We shall agree to differentiate them as if they were free vectors; after the differentiation the mechanical and the geometrical meaning of the derivative must be carefully analysed.

Now we directly pass to the study of kinematics and begin our study with kinematics of a particle.

Chapter 7 Kinematics of a Particle

§ 1. Methods of Describing the Motion of a Particle

1.1. Coordinate Method. The motion of a particle in space can be described using various methods. To specify the position of a particle in space with respect to a rectangular Cartesian system $Oxyz$ (which is conditionally regarded as being fixed) we must specify the coordinates x, y and z of the particle. If these coordinates are known at each instant during the motion in question, that is if the functions

$$x = x(t), \quad y = y(t), \quad z = z(t) \tag{7.1}$$

are known, then the position of the particle in space is known for each instant. For the class of motions we are studying the functions $x(t)$, $y(t)$ and $z(t)$ and also their derivatives at least up to the second order inclusive* are one-valued, finite and continuous. Equations (7.1) are referred to as *equations of motion of the particle*, and this method of describing the motion is called the *coordinate method*. From the point of view of mathematics equalities (7.1) are parametric equations of the curve along which the particle moves in space; this curve is spoken of as the *trajectory* of the moving particle. If the parameter t is cancelled we obtain two equations connecting the coordinates x, y and z of the moving particle, that is equations of the trajectory not involving time t.

1.2. Natural Method. In the general case equations of the trajectory of a particle can be written as

$$\Phi_1(x, y, z) = 0, \quad \Phi_2(x, y, z) = 0 \tag{7.2}$$

It should be noted that each of equations (7.2) determines a surface in space, and the two equations regarded simultaneously specify a curve formed by the intersection of these surfaces. Although equations (7.2) can be derived from equations (7.1), they do not themselves specify the motion because the particle can move along the given trajectory in various ways. In other words, the coordinate s of the particle M reckoned along the trajectory (this coordinate is equal to the arc length of the trajectory, reckoned from an initial position M_0, which is taken with the corresponding sign plus or minus) may change with time in various ways (Fig. 7.1). After the positive direction along the trajectory has been chosen, to each point on the trajectory there corresponds, in a one-to-one way, a positive or a negative coordinate s. The situation here is analogous to the one when a positive or a negative abscissa is associated with each

* In a more general approach the second derivatives may be discontinuous functions of time.

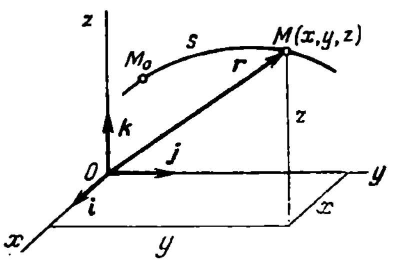

Fig. 7.1

point on the axis of abscissas. It is obvious that if we have equations of a trajectory in a purely geometrical form (that is equations not containing time t) and if, besides these equations, the variation of the coordinate s as function of t is specified, $s = s\ (t)$, then the motion of the particle is completely determined. The equation $s = s\ (t)$ is called the *law of motion*, and the method of representing the motion of a particle with the aid of the equations

$$\Phi_1\ (x,\ y,\ z) = 0, \quad \Phi_2\ (x,\ y,\ z) = 0, \quad s = s\ (t) \qquad (7.3)$$

is referred to as the *natural method*. The graph of the function $s = s\ (t)$ is called the *graph of motion*.

1.3. Vector Method. For the general case of motion of a particle there is one more method of describing the motion which however merely reduces to another way of writing the formulas involved in the first method. If we consider x, y and z as the coordinates of the terminus of the radius vector $\boldsymbol{r} = \boldsymbol{OM}$ issued from the origin O (see Fig. 7.1) then the radius vector can be written in the form $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Since the coordinates of the moving particle vary as functions of time the radius vector of the particle is a vector function of time:

$$\boldsymbol{r} = \boldsymbol{r}\ (t) = x\ (t)\ \boldsymbol{i} + y\ (t)\ \boldsymbol{j} + z\ (t)\ \boldsymbol{k} \qquad (7.4)$$

This *vector method* of representing motion will be conveniently used later when the velocity of a moving particle is defined as a vector.

1.4. Plane Motion. The equations written above are simplified when the moving particle remains in a fixed plane during the whole time of motion. If this plane is taken as the coordinate plane Oxy one of equations (7.1)-(7.3) takes the form $z = 0$. This equation will be taken into account only when the motion in question is regarded as a special case of the general spatial motion of a particle. But if the motion is regarded as a plane motion then the equations of motion have the form

$$x = x\ (t), \qquad y = y\ (t) \qquad (7.1')$$

In the case of the natural representation of a plane motion the equations are

$$\Phi\ (x,\ y) = 0, \qquad s = s\ (t) \qquad (7.3')$$

The first of equations (7.3′) specifies the trajectory of the particle in the plane Oxy and the second describes the law of motion along the trajectory.

1.5. Describing Plane Motion with the Aid of Polar Coordinates. The position of a particle in a plane can also be specified by its polar coordinates. Let us choose a pole O and draw a polar axis Op in the plane of motion. For every point M different from the pole O the length r of the polar radius OM and the polar angle θ are defined; the polar angle is reckoned from the polar axis to the polar radius in the counterclockwise direction (Fig. 7.2). For the correspondence between the points of the plane and the pairs of the polar coordinates to be one-to-one the variation of the coordinates is taken within the intervals

$$0 \leqslant r < \infty, \qquad 0 \leqslant \theta < 2\pi$$

(it should be noted that the polar angle of the pole remains indeterminate). In mechanics we usually prefer continuous variation of the coordinates (in this case the one-to-one character of the correspondence may be violated) and the polar angle is taken as

$$\varphi = \angle MOp + 2k\pi = \theta + 2k\pi$$

where k is an integer.

If the axis Ox of the rectangular Cartesian coordinates in the plane is made to coincide with the polar axis the transformation formulas from the polar coordinates to the Cartesian coordinates are obtained using Fig. 7.2:

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{7.5}$$

The formulas for the inverse transformation are

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$$

To determine the quadrant in which the angle θ lies one must take into account the signs of $\cos\theta$ and $\sin\theta$ specified by (7.5).

The position of a particle moving in the plane can be specified by setting the functions

$$r = r(t), \qquad \varphi = \varphi(t) \tag{7.6}$$

for the time interval under consideration. Equations (7.6) are referred to as *equations of plane motion in polar coordinates.*

In the case of a *rectilinear motion of a particle* it is natural to take the straight trajectory of motion as the axis Ox; then the coordinate method

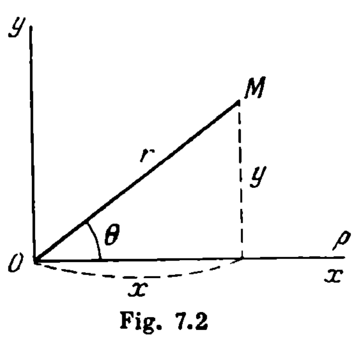

Fig. 7.2

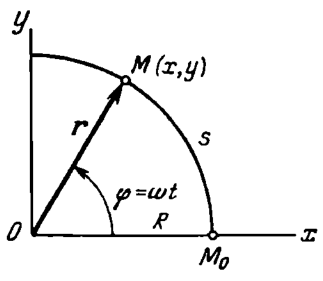

Fig. 7.3

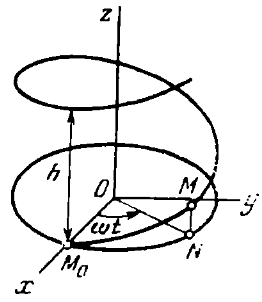

Fig. 7.4

and the natural method of describing the motion reduce to setting the abscissa of the moving particle as function of time:

$$x = x\ (t)$$

In conclusion we shall consider two examples the first of which deals with a plane motion.

EXAMPLE 7.1. Let a point (a particle) M move in a fixed circle of radius R in the counterclockwise direction. Let the angle φ of rotation of the radius vector (reckoned from the axis Ox passing through the centre of the circle and through the initial position M_0 of the particle) vary proportionally to time: $\varphi = \omega t$ (Fig. 7.3). Since $x = R \cos \varphi$ and $y = R \sin \varphi$ the equations of motion are written in the form

$$x = R \cos \omega t, \quad y = R \sin \omega t$$

Squaring each of the equations and adding them together we cancel time t and obtain the equation of the trajectory in the form $x^2 + y^2 = R^2$. By the way, this equation can be written directly as the equation of a circle of radius R with centre at the origin. Since the arc of the circle is $\cup M_0M = s = R\varphi$ the equations

$$x^2 + y^2 = R^2, \quad s = R\omega t$$

correspond to the natural method of representing the motion. Finally, the vector function describing the dependence of the radius vector $\boldsymbol{r} = \boldsymbol{OM}$ on time has the form

$$\boldsymbol{r}\ (t) = x\ (t)\ \boldsymbol{i} + y\ (t)\ \boldsymbol{j} = R\ (\boldsymbol{i} \cos \omega t + \boldsymbol{j} \sin \omega t)$$

This corresponds to the vector method of representing the motion of the particle.

EXAMPLE 7.2. Let the equations of motion of a particle M have the form

$$x = R \cos \omega t, \quad y = R \sin \omega t, \quad z = \frac{\omega h}{2\pi} t \tag{7.7}$$

The coordinates of the projection N of the point M on the plane Oxy are $(x, y, 0)$ (Fig. 7.4). The point N is in the motion described in Example 7.1. Since the angle of rotation of the point N is $\varphi\ (t) = \omega t$ this point makes one revolution around the centre O along the circle of radius R during time $T = 2\pi/\omega$ because $\varphi\ (T) = \omega T = 2\pi$. At the initial instant $t = 0$ the points M and N are at the position M_0. The z-coordinate of the point M is proportional

to time t, and at the instant $t = T$ its value is

$$z(T) = \frac{\omega h}{2\pi} T = h$$

Thus, the distance from the point M to the plane Oxy increases proportionally to time t, and during time T (in which the projection N of M makes one revolution about O) the vertical displacement of M is equal to h. Such a motion of a particle is said to be *helical* (*screw*), and the quantity h is called the *lead of the helix* (*screw*). Equations (7.7) are parametric equations of the *helical line*. From the last equation (7.7) we obtain $t = 2\pi z/(\omega h)$; the substitution of this value into the first two equations (7.7) yields

$$x = R\cos\left(\frac{2\pi}{h} z\right), \qquad y = R\sin\left(\frac{2\pi}{h} z\right)$$

These are equations of the same trajectory, that is of the helical line, written in form (7.2). In order to obtain the expression for $s = s(t)$, which is needed if we want to use the natural method of describing motion, we can apply the corresponding formulas of integral calculus using equations (7.7). Here we shall not perform the calculations; at the end of the following section (Sec. 2.3) we shall derive the formula for computing the arc length of an arbitrary curve needed for these calculations.

In conclusion we write the expression for the radius vector $\boldsymbol{r}(t)$ of the moving particle corresponding to the vector method of representing motion:

$$\boldsymbol{r}(t) = R(\boldsymbol{i}\cos\omega t + \boldsymbol{j}\sin\omega t) + \frac{\omega h}{2\pi} t\boldsymbol{k}$$

§ 2. Velocity of a Particle in Curvilinear Motion

2.1. Velocity Vector of a Particle. Let us suppose that a particle moving along an arc MM' of its trajectory (Fig. 7.5) is at the position $M(x, y, z)$ at instant t and at the position $M'(x + \Delta x, y + \Delta y, z + \Delta z)$ at instant $t + \Delta t$. To the first position there corresponds the radius vector $\boldsymbol{r} = \boldsymbol{OM}$ and to the second position the radius vector $\boldsymbol{r}' = \boldsymbol{OM}'$. The displacement vector of the particle M during time Δt is $\Delta\boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r} = \boldsymbol{MM}'$. Its ratio to the increment of time Δt is called the *average velocity vector* corresponding to the time

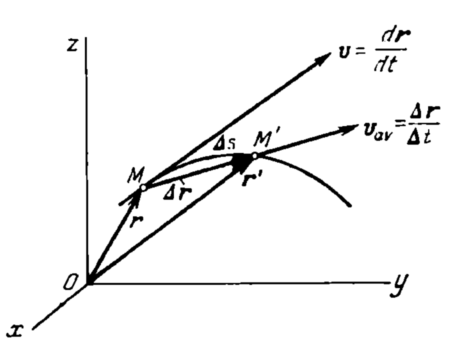

Fig. 7.5

interval $(t,\ t+\Delta t)$:

$$\boldsymbol{v}_{\mathrm{av}}=\frac{\Delta \boldsymbol{r}}{\Delta t}$$

Since the projections of the displacement vector $\Delta \boldsymbol{r}$ are Δx, Δy and Δz we have

$$\boldsymbol{v}_{\mathrm{av}}=\frac{\Delta x}{\Delta t}\boldsymbol{i}+\frac{\Delta y}{\Delta t}\boldsymbol{j}+\frac{\Delta z}{\Delta t}\boldsymbol{k}$$

By *the velocity vector* $\boldsymbol{v}$ *of the particle* M *at instant* t *is meant the limit of the average velocity vector when the time interval* Δt *tends to zero*, that is

$$\boldsymbol{v}=\lim_{\Delta t\to 0}\boldsymbol{v}_{\mathrm{av}}=\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}\boldsymbol{i}+\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}\boldsymbol{j}+\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}\boldsymbol{k}$$

whence

$$\boldsymbol{v}=\frac{dx}{dt}\boldsymbol{i}+\frac{dy}{dt}\boldsymbol{j}+\frac{dz}{dt}\boldsymbol{k} \tag{7.8}$$

The vectors $\boldsymbol{v}_{\mathrm{av}}$ and $\boldsymbol{v}$ are shown in Fig. 7.5. In Sec. 5 of Introduction to Kinematics we considered the differentiation of a variable free vector. Since the displacement vector is $\boldsymbol{MM'}=\Delta\boldsymbol{r}=\boldsymbol{r}'-\boldsymbol{r}$ then *the velocity of the particle* M *is a vector applied at that particle and equal to the derivative of the radius vector* $\boldsymbol{r}$ *with respect to time at the given instant* t:

$$\boldsymbol{v}(t)=\frac{d\boldsymbol{r}(t)}{dt} \tag{7.9}$$

It should be noted that formula (7.8) is nothing other than the expression of formula (7.9) written in full relative to the rectangular Cartesian coordinate system. Here we must stipulate that the radius vector $\boldsymbol{r}(t)$ is a localized (bound) vector with origin at the fixed point O. In contrast to the radius vector, the origin of the velocity vector $\boldsymbol{v}(t)$ is at the moving point M.

In some cases it is advisable to set off from the origin a vector which is geometrically equal to the velocity vector. In this construction the terminus of the vector $\boldsymbol{v}(t)$ describes a curve which is called the *velocity hodograph*. It should be mentioned that the hodograph is useful for differentiating the velocity vector (see Sec. 3.1).

2.2. Velocity Vector in Rectangular Cartesian Coordinates. Now let us determine the direction and the modulus of the velocity vector. Since the displacement vector $\boldsymbol{MM'}$ is directed along the chord MM' of the trajectory, and the limiting position of the chord coincides with that of the tangent (line) to the trajectory, the velocity vector is directed along the tangent to the trajectory in the direction of

motion. The modulus of the velocity is equal to

$$v = \lim_{\Delta t \to 0} \frac{MM'}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{MM'}{\smile MM'} \cdot \frac{\smile MM'}{\Delta t} \right)$$

Denoting the length of the arc MM' by Δs and taking into account that the limit of the ratio of the length of the chord subtending an arc to the length of that arc is equal to unity, that is

$$\lim_{\Delta t \to 0} \frac{MM'}{\smile MM'} = 1$$

we obtain

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Here we have assumed that for the given direction of reckoning the arc length we have $\Delta s > 0$ for $\Delta t > 0$; in the general case we shall write the expression

$$v = \left| \frac{ds}{dt} \right| \tag{7.10}$$

for the *modulus of the velocity*. It should be noted that formula (7.10) makes it possible to directly determine the modulus of the velocity only when the motion is described using the natural method. When the motion is represented by means of the coordinate method we have at our disposal the projections of the velocity on the coordinate axes:

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt}$$

(see formula (7.8) and the remark to formula (1) in Sec. 5 of Introduction to Kinematics). From these expressions there follows the formula for the *modulus of the velocity vector*:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \tag{7.11}$$

In the International System of Units (SI) (see Sec. 2.3 of Chap. 1) the unit for measuring the modulus of the velocity is m/s.

Using formula (7.10) for the modulus of the velocity v and transforming the square root we obtain

$$\left| \frac{ds}{dt} \right| = \frac{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}}{dt}$$

The last expression corresponds to the formula of the differential of arc length:

$$ds = \pm \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

(this formula is established in differential calculus; see [1]).

The direction of the velocity vector and consequently the direction along the tangent to the trajectory are specified by the *direction cosines*:

$$\cos(\widehat{v,\ x}) = \frac{v_x}{v} = \frac{\frac{dx}{dt}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}$$
$$\cos(\widehat{v,\ y}) = \frac{v_y}{v}, \quad \cos(\widehat{v,\ z}) = \frac{v_z}{v} \tag{7.12}$$

Here $(\widehat{v,\ x})$, $(\widehat{v,\ y})$ and $(\widehat{v,\ z})$ are the angles between the velocity vector and the positive directions of the axes Ox, Oy and Oz respectively.

2.3. Algebraic Velocity. Path Length. As was mentioned, the letter s denotes the coordinate along the trajectory, that is the arc length of the trajectory reckoned from a fixed point M_0 on the trajectory (and supplied with the corresponding sign (plus or minus)). The choice of the sign for reckoning the arc length corresponds to the choice of the positive direction along the tangent to the trajectory. Thus, the direction in which the coordinate s of the moving particle increases is taken as the positive direction along the tangent.

Let us agree to assign an algebraic value (positive or negative) to the velocity; to this end we define the *algebraic velocity* v_τ of the particle by means of the formula

$$v_\tau = \frac{ds}{dt} \tag{7.13}$$

Hence, the algebraic velocity is positive in the direction of increase of s (that is when the direction of the velocity vector coincides with the positive direction along the tangent to the curve at the given point) and is negative in the direction of decrease of s (that is when the direction of the velocity vector is opposite to the positive direction along the tangent).

The velocity vector of a moving particle is always directed along the tangent to the trajectory, and therefore the algebraic velocity is equal to the projection of the velocity vector on the direction of the tangent at the given point:

$$v_\tau = \pm v$$

where v is the modulus of the velocity.

If the modulus of the velocity of a particle is known as function of time, that is $v = v(t)$, formula (7.10) makes it possible to find the path length S travelled by the particle for each instant of time. Multiplying both sides of formula (7.10) by $dt > 0$ we obtain

$$|ds| = v(t)\, dt$$

The integration of this equality from 0 to t with respect to time and from 0 to S with respect to the path travelled results in

$$\int_0^S |ds| = \int_0^t v(t)\,dt$$

(we assume that $t = 0$ and $S = 0$ at the initial instant). Finally, the path length is equal to

$$S = \int_0^t v(t)\,dt \tag{7.14}$$

In the special case when the modulus of the velocity of a particle is constant during the whole time of motion, that is $v(t) = V$, the curvilinear motion is said to be *uniform*. For this case formula (7.14) yields

$$S = Vt$$

The last formula expresses the law of variation of path length for a uniform curvilinear motion.

The formulas expressing the velocity of the particle when its plane motion is represented in polar coordinates will be derived in Sec. 1.3 of Chap. 11.

EXAMPLE 7.3. Using equations of motion (7.7) we obtain for Example 7.2 the formulas

$$v_x = \frac{dx}{dt} = -R\omega \sin \omega t, \quad v_y = \frac{dy}{dt} = R\omega \cos \omega t, \quad v_z = \frac{dz}{dt} = \frac{\omega h}{2\pi}$$

The velocity vector $\boldsymbol{v}$ is written in the form

$$\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k} = -\boldsymbol{i} R\omega \sin \omega t + \boldsymbol{j} R\omega \cos \omega t + \frac{\omega h}{2\pi}\boldsymbol{k}$$

and its modulus is expressed thus:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{R^2\omega^2 \sin^2 \omega t + R^2\omega^2 \cos^2 \omega t + \frac{\omega^2 h^2}{4\pi^2}}$$
$$= \frac{\omega}{2\pi}\sqrt{4\pi^2 R^2 + h^2}$$

The direction cosines of the velocity vector are specified by the formulas

$$\cos(\widehat{v, x}) = \frac{v_x}{v} = -\frac{2\pi R}{\Gamma} \sin \omega t$$

$$\cos(\widehat{v, y}) = \frac{v_y}{v} = \frac{2\pi R}{\Gamma} \cos \omega t, \quad \cos(\widehat{v, z}) = \frac{v_z}{v} = \frac{h}{\Gamma}$$

where $\Gamma = \sqrt{4\pi^2 R^2 + h^2}$. It can easily be verified that the sum of the squares of the direction cosines is equal to unity, which agrees with the well-known formula of analytic geometry. Since $\cos(\widehat{\boldsymbol{v}, z}) = \text{const}$ the tangent to the helical

line forms a constant angle with the z-axis at each point:

$$\gamma = \widehat{(v, z)} = \arccos \frac{h}{\Gamma}$$

The modulus of the velocity being constant during the whole time of motion, the motion along the helical line specified by equations (7.7) is uniform. If the path travelled is reckoned from the position M_0 (see Fig. 7.4) then

$$S(t) = vt = \frac{\omega\Gamma}{2\pi} t$$

The path S travelled during the time $T = 2\pi/\omega$ (see Example 7.2) is

$$S = \frac{\omega\Gamma}{2\pi}\,\frac{2\pi}{\omega} = \Gamma = 2\pi R \sqrt{1 + \frac{h^2}{4\pi^2 R^2}}$$

We have computed the arc length of one coil of the helical line using only the formulas of kinematics. The ratio $h^2/(4\pi^2 R^2)$ is usually much smaller than unity. In such a case we can use the expansion

$$\sqrt{1+u} = 1 + \frac{u}{2} + \dots$$

established in the theory of series (see [1]). This gives the following approximate expression for the length of one coil of the helical line:

$$S \approx 2\pi R \left(1 + \frac{h^2}{8\pi^2 R^2}\right)$$

§ 3. Acceleration of a Particle in Curvilinear Motion

3.1. Acceleration Vector of a Particle. Suppose that the velocity vector of a moving particle M at instant t is equal to $v = v(t)$ and that the velocity vector is $v' = v(t + \Delta t)$ at instant $t' = t + \Delta t$, when the point is at a position M' (Fig. 7.6). Now we transfer the origin of the vector v' to the fixed point in space at which the moving particle M is at instant t, that is we construct the vector v'_M. Next we construct the vector Δv (the increment of the vector v):

$$\Delta v = v'_M - v$$

From the particle M we set off the vector $\Delta v/\Delta t$ equal to the ratio of the increment of the velocity vector to the increment of time Δt (see Fig. 7.6). This is called the *average acceleration vector* during the time interval $(t, t + \Delta t)$:

$$w_{\text{av}} = \frac{\Delta v}{\Delta t}$$

By the *acceleration vector* w *of the particle* M *at instant* t *is meant the limit of the average acceleration vector when the time interval* Δt *tends to zero*:

$$w = \lim_{\Delta t \to 0} w_{\text{av}} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

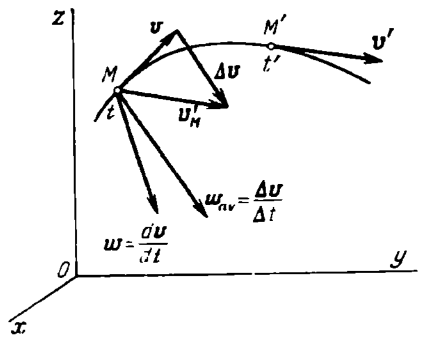

Fig. 7.6

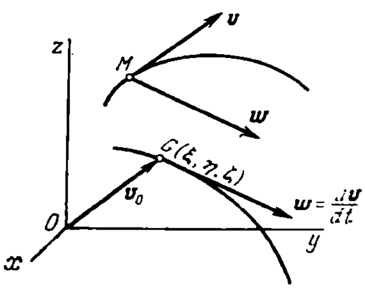

Fig. 7.7

According to Sec. 5 of Introduction to Kinematics, the *acceleration of the particle M is a vector applied at the moving particle and equal to the derivative of the velocity vector* $\boldsymbol{v}$ *with respect to time at the given instant t*:

$$\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt} \tag{7.15}$$

Let us discuss the relationship between the acceleration vector of a particle and the so-called velocity hodograph. Let us transfer the vector $\boldsymbol{v}$ to the origin O of a fixed coordinate system $Oxyz$, that is let us construct at the point O the vector $\boldsymbol{v}_O$ which is geometrically equal to the vector $\boldsymbol{v}$, the terminus of the vector $\boldsymbol{v}_O$ being denoted by G (Fig. 7.7). In the general case the vector $\boldsymbol{v}$ varies with time and the point G moves in space. As has been mentioned, the trajectory of the point G is called the velocity hodograph. Since the projections of the velocity on the axes $Oxyz$ are equal to dx/dt, dy/dt and dz/dt the coordinates of the point G (ξ, η, ζ) moving along the hodograph are obviously expressed by the formulas

$$\xi = v_x = \frac{dx}{dt}, \quad \eta = v_y = \frac{dy}{dt}, \quad \zeta = v_z = \frac{dz}{dt} \tag{7.16}$$

These equations represent parametrically the velocity hodograph. Cancelling the parameter (time t) from equations (7.16) we obtain the equations of the hodograph in the coordinate form.

The velocity vector of the moving point G of the hododraph is

$$\frac{d\xi}{dt}\boldsymbol{i} + \frac{d\eta}{dt}\boldsymbol{j} + \frac{d\zeta}{dt}\boldsymbol{k} = \frac{dv_x}{dt}\boldsymbol{i} + \frac{dv_y}{dt}\boldsymbol{j} + \frac{dv_z}{dt}\boldsymbol{k} = \frac{d\boldsymbol{v}}{dt}$$

and, by virtue of (7.15), the last vector is equal to the acceleration vector $\boldsymbol{w}$ of the particle M moving along its trajectory (see Fig. 7.7).

3.2. Acceleration Vector in Rectangular Cartesian Coordinates. From formulas (7.15) and (7.9) it follows that

$$w = \frac{dv}{dt} = \frac{d^2r}{dt^2} \tag{7.17}$$

Hence, the acceleration vector of the moving particle M is equal to the second derivative of the radius vector of the particle with respect to time. Since the radius vector can be represented in the form

$$r = r(t) = x(t)\,i + y(t)\,j + z(t)\,k$$

formula (7.17) yields the following expression for the acceleration vector in rectangular Cartesian coordinates:

$$w = \frac{d^2x}{dt^2}\,i + \frac{d^2y}{dt^2}\,j + \frac{d^2z}{dt^2}\,k \tag{7.18}$$

Formula (7.18) can also be derived from formulas (7.15) and (7.8). For the projections of the acceleration vector of the particle on the axes of the rectangular Cartesian coordinate system we have

$$w_x = \frac{d^2x}{dt^2} = \frac{dv_x}{dt}, \quad w_y = \frac{d^2y}{dt^2} = \frac{dv_y}{dt}, \quad w_z = \frac{d^2z}{dt^2} = \frac{dv_z}{dt} \tag{7.19}$$

where x, y and z are the coordinates of the moving particle specified by equations of motion (7.1).

For the *modulus of the acceleration* we have the formula

$$w = \sqrt{w_x^2 + w_y^2 + w_z^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2} \tag{7.20}$$

and the direction cosines of the acceleration vector are expressed by the formulas

$$\cos(\widehat{w,\ x}) = \frac{w_x}{w} = \frac{\dfrac{d^2x}{dt^2}}{\sqrt{\left(\dfrac{d^2x}{dt^2}\right)^2 + \left(\dfrac{d^2y}{dt^2}\right)^2 + \left(\dfrac{d^2z}{dt^2}\right)^2}}$$
$$\cos(\widehat{w,\ y}) = \frac{w_y}{w}, \quad \cos(\widehat{w,\ z}) = \frac{w_z}{w} \tag{7.21}$$

In the International System of Units (SI) the unit for measuring the modulus of acceleration is m/s^2.

Formulas (7.19) determine the projections of the acceleration vector on the axes of a fixed rectangular Cartesian coordinate system $Oxyz$. As we shall see, it will also be useful to find the projections of the acceleration vector on the axes of a movable rectangular coordinate system whose origin is at the moving point M and the directions of whose axes are in a certain manner determined by the trajectory itself. Before passing to this question we shall remind the reader of some geometrical notions concerning spatial curves (for greater petail see [1]).

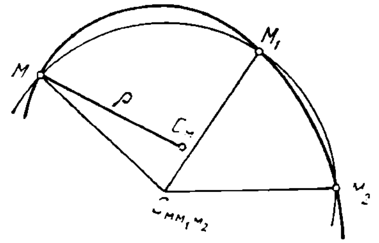

Fig. 7.8

3.3. Coordinate Axes Connected with a Natural Trihedron. A circle is the simplest plane curve and therefore it is natural to try to represent separate parts of a spatial curve as arcs of certain circles.

Let us consider a spatial curve and three points M, M_1 and M_2 on it lying close to one another (Fig. 7.8). Through these points we can draw a plane (which will be denoted by MM_1M_2) and a circle in that plane with centre at some point $C_{MM_1M_2}$. Both the plane and the circle are specified uniquely provided that the points M, M_1 and M_2 do not lie in one straight line. When the points M_1 and M_2 tend to the point M the plane MM_1M_2 tends to its limiting position, that is to a certain plane; this limiting plane is called the *osculating plane* of the curve at the point M. The limiting position to which the circle passing through the points M, M_1 and M_2 tends is a circle MM_1M_2 lying in the osculating plane (it is called the *circle of curvature* at the point M). Let C_M be the centre of the circle of curvature at the point M, that is the limiting position of the point $C_{MM_1M_2}$ when the points M_1 and M_2 tend to the point M. The radius $\rho = C_M M$ of the circle of curvature is called the *radius of curvature* of the curve at the point M and the point C_M is called the *centre of curvature.*

The tangent $M\tau$ to the curve at the point M occupies the limiting position of the chord MM_1 and therefore lies in the osculating plane. The plane which passes through the point M and is perpendicular to the tangent $M\tau$ is called the *normal plane*, and each straight line lying in that plane and passing through the point M is called the *normal* to the curve at the point M. Among all the normals at the point M there are two playing an important role: the *principal normal* Mn lying in the osculating plane and the *binormal* Mb perpendicular to the principal normal. Since the centre of curvature C_M also lies in the osculating plane and since the radius of curvature $C_M M$ is perpendicular to the tangent $M\tau$ the principal normal is nothing other than the straight line coinciding with the radius of curvature.

The tangent, the principal normal and the binormal at the point M form the so-called *natural trihedron* of the curve at the point M. The plane containing the tangent and the binormal is called the *rectifying plane.*

Now, as the axes of a movable coordinate system we take the following lines: (1) the tangent $M\tau$ with unit vector $\boldsymbol{\tau}$ in the direction of increase of the coordinate s (see Sec. 2.2); (2) the principal normal Mn with unit vector $\boldsymbol{n}$ directed to the centre of curvature (this means that the vector $\boldsymbol{n}$ goes in the direction of concavity of the curve); (3) the binormal Mb whose unit vector $\boldsymbol{b}$ is taken in accordance with

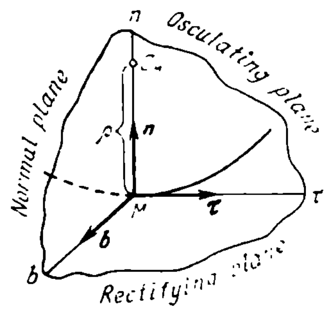

Fig. 7.9

the right-hand screw rule for the rotation from the unit vector of the tangent to the unit vector of the principal normal (see Fig. 7.9).

Let us denote by $\Delta\beta$ the angle between the tangents drawn at two points M and M' of the curve lying close to each other (Fig. 7.10) and let Δs be the length of the arc MM'; it can be shown (see [1]) that the radius of curvature ρ of the curve at the point M is expressed by the formula

$$\rho = \frac{1}{k} \tag{7.22}$$

where $k = \lim\limits_{\Delta s \to 0} \frac{\Delta\beta}{\Delta s}$; the quantity k is called the *curvature* of the given curve at the point M.

3.4. Tangential and Normal Accelerations of a Particle. In this section we shall derive the formulas for the projections of the acceleration vector on the coordinate axes connected with the natural trihedron.

Let us take a definite instant t; then the position of the point M corresponding to t is fixed. Let us construct the velocity vector $\boldsymbol{v} = \boldsymbol{v}(t)$ at the point M and the velocity vector $\boldsymbol{v}' = \boldsymbol{v}(t + \Delta t)$ at the point M' corresponding to instant $t' = t + \Delta t$ (Fig. 7.11). Next we transfer the vector $\boldsymbol{v}'$ to the point M and construct the vector

$$\boldsymbol{LN} = \Delta\boldsymbol{v} = \boldsymbol{v}' - \boldsymbol{v}$$

At the point M we construct the vector $\boldsymbol{MK}$ in the direction of the vector $\boldsymbol{v}$, the modulus of $\boldsymbol{MK}$ being equal to that of the vector v': $MK = v'$. The vector $\boldsymbol{LN}$ is equal to the vector sum of $\boldsymbol{LK}$ and $\boldsymbol{KN}$:

$$\Delta\boldsymbol{v} = \boldsymbol{LK} + \boldsymbol{KN}$$

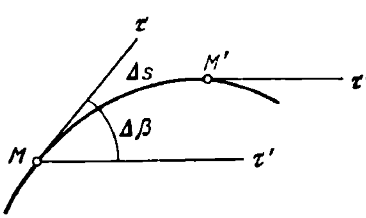

Fig. 7.10

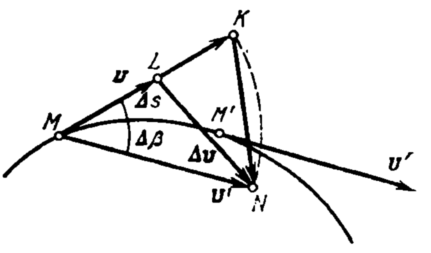

Fig. 7.11

By definition, the acceleration of the point M is equal to

$$\boldsymbol{w} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{LK}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\boldsymbol{KN}}{\Delta t} \tag{7.23}$$

We shall begin with finding the limit of the first term on the right-hand side of (7.23).

The vector $\boldsymbol{LK}$ being directed along the vector $\boldsymbol{v}$, the direction of the vector $\boldsymbol{LK}/\Delta t$ is invariable for fixed t and variable Δt; this direction, like that of the velocity, is specified by the unit vector $\boldsymbol{\tau}$ along the tangent. The modulus of the vector $\boldsymbol{LK}$ is equal to

$$LK = MK - ML = v' - v = \Delta v$$

where Δv is the increment of the modulus of the velocity (see Fig. 7.11 which corresponds to the case when the modulus of the velocity of the point M increases and the point moves in the positive direction along the tangent to the trajectory). Thus,

$$\lim_{\Delta t \to 0} \frac{\boldsymbol{LK}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \boldsymbol{\tau} = \frac{dv}{dt} \boldsymbol{\tau}$$

In order to find the modulus of the limit of the second term of the right-hand side of (7.23) let us construct the arc KN of the circle of radius $MK = MN = v'$. The central angle of this arc is equal to the angle between two tangents drawn closely to each other (earlier we denoted this angle by $\Delta\beta$). Let us transform the ratio $KN/\Delta t$:

$$\frac{KN}{\Delta t} = \frac{KN}{\smile KN} \frac{\smile KN}{\Delta t} = \frac{KN}{\smile KN} \frac{v' \Delta\beta}{\Delta t} = \frac{KN}{\smile KN} v' \frac{\Delta\beta}{\Delta s} \frac{\Delta s}{\Delta t}$$

In these calculations we have taken into account that $\smile KN$ is the arc length of the circle and is therefore equal to $v' \Delta\beta$. As we know,

$$\lim_{M' \to M} \frac{KN}{\smile KN} = 1$$

Therefore, taking into account formulas (7.10) and (7.22), we obtain

$$\lim_{\Delta t \to 0} \frac{KN}{\Delta t} = vkv = \frac{v^2}{\rho}$$

It now remains to determine the direction of the limiting position of the vector $\boldsymbol{KN}/\Delta t$. Since

$$\angle MKN = \frac{\pi}{2} - \frac{\Delta\beta}{2} \xrightarrow[\Delta t \to 0]{} \frac{\pi}{2}$$

this (limiting) direction is perpendicular to the tangent and lies in the osculating plane, that is it coincides with the direction of the principal normal Mn. Thus,

$$\lim_{\Delta t \to 0} \frac{\boldsymbol{KN}}{\Delta t} = \frac{v^2}{\rho} \boldsymbol{n}$$

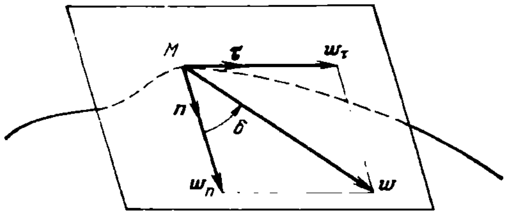

Fig. 7.12

Now we can write equality (7.23) in the form

$$w = \frac{dv}{dt}\,\boldsymbol{\tau} + \frac{v^2}{\rho}\,\boldsymbol{n} \tag{7.24}$$

Hence, the projections of the acceleration vector of the point M on the tangent and on the principal normal are equal to

$$w_\tau = \frac{dv}{dt} \tag{7.25}$$

and

$$w_n = \frac{v^2}{\rho} \tag{7.26}$$

respectively, and the projection w_b on the binormal is identically equal to zero. Thus, the acceleration vector lies in the osculating plane $M\tau n$ (Fig. 7.12). The last two formulas imply the expression for the *modulus of the acceleration vector*:

$$w = \sqrt{w_\tau^2 + w_n^2}$$

The direction of the acceleration vector can be determined by computing the angle $\delta = \widehat{(\boldsymbol{w}, \boldsymbol{n})}$:

$$\tan\delta = \frac{w_\tau}{w_n} = \frac{\rho}{v^2}\,\frac{dv}{dt}$$

Formula (7.25) also remains valid for the intervals of time during which the modulus v of the velocity of the particle decreases when the particle moves in the positive direction along the tangent to the trajectory. In these cases $dv/dt < 0$, that is the projection of the acceleration vector of the particle on the tangent is negative; this means that when the modulus of the velocity decreases, the direction of the *tangential acceleration*

$$\frac{dv}{dt}\,\boldsymbol{\tau}$$

of the particle is opposite to that of the motion of the particle. We can thus state the following test for determining whether the motion of the point along the curve is accelerated or decelerated: in the first case the direction of the tangential acceleration of the particle coin-

cides with that of the motion and in the second case it is opposite to that of the motion.

REMARK 1. If we take into account the cases when the direction of motion of a particle along a curve is opposite to that in which the coordinate s is reckoned (to the direction of unit vector $\boldsymbol{\tau}$) or when this direction changes during the time of motion, formula (7.25) for the projection of the acceleration vector on the tangent should be written in the form

$$w_\tau = \frac{dv_\tau}{dt} = \frac{d^2s}{dt^2} \tag{7.25a}$$

Here v_τ is the algebraic velocity (that is the projection of the velocity vector on the tangent) specified by formula (7.13). For these cases formula (7.24) takes the form

$$\boldsymbol{w} = \boldsymbol{w}_\tau + \boldsymbol{w}_n = \frac{d^2s}{dt^2}\boldsymbol{\tau} + \frac{v^2}{\rho}\boldsymbol{n} \tag{7.24a}$$

The test for determining whether the motion of a particle along a curve is accelerated or decelerated remains valid in this general case as well.

The projection of the acceleration vector of a particle on the principal normal is always nonnegative, that is the *normal acceleration*

$$\boldsymbol{w}_n = \frac{v^2}{\rho}\boldsymbol{n}$$

of the particle is directed towards the centre of curvature (or is equal to zero) during the whole time of motion.

If during the time interval under consideration the particle M moves with velocity having a constant modulus (such a motion is called uniform) when $w_\tau = dv/dt \equiv 0$. In this case the total acceleration $\boldsymbol{w}$ of the particle is equal to its normal acceleration: $\boldsymbol{w} \equiv \boldsymbol{w}_n$.

If a particle moves along a straight line then, since the curvature $k = 1/\rho$ of the straight line is equal to zero, the normal acceleration of the particle vanishes. The acceleration of a particle in rectilinear motion is equal to its tangential acceleration.

In the case when a particle is in uniform rectilinear motion (and in this case only!) the acceleration vector $\boldsymbol{w}$ of the particle is identically equal to zero.

REMARK 2. If we use the first Frenet-Serret formula

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{1}{\rho}\frac{ds}{dt}\boldsymbol{n}$$

for the derivative of unit tangent vector $\boldsymbol{\tau}$ with respect to time (see [1]), formula (7.24a) can be derived directly. Indeed, by virtue of (7.13), we have

$$\boldsymbol{v} = \frac{ds}{dt}\boldsymbol{\tau}$$

whence follows

$$w=\frac{dv}{dt}=\frac{d^2s}{dt^2}\tau+\frac{ds}{dt}\frac{d\tau}{dt}=\frac{d^2s}{dt^2}\tau+\frac{v^2}{\rho}n$$

because $v^2 = (ds/dt)^2$.

When the motion of a particle is specified by equations of motion in Cartesian coordinates we can find the moduli of the velocity vector and of the acceleration vector as functions of time using formulas (7.11) and (7.20). After this we can determine the projection w_τ of the acceleration vector on the tangent with the aid of formulas (7.25) or (7.25a), and then the projection w_n of the acceleration vector on the principal normal can be found:

$$w_n=\sqrt{w^2-w_\tau^2} \tag{7.27}$$

Finally, from formula (7.26) the radius of curvature of the trajectory is found as function of t (for any instant):

$$\rho=\frac{v^2}{w_n} \tag{7.28}$$

Now, what are the projections of the velocity vector on the coordinate axes connected with the natural trihedron? The answer to this question is quite simple because the velocity vector is directed along the tangent to the trajectory, and hence $v_\tau = \pm v$ and $v_n = v_b \equiv 0$. If the direction of the vector v coincides with that of unit vector τ then $v_\tau = v$, and if these directions are opposite then $v_\tau = -v$.

The formulas expressing the acceleration of a particle for the case when its plane motion is described using polar coordinates will be derived in Sec. 1.3 of Chap. 11.

EXAMPLE 7.4. Let the equations of motion of a particle be

$$x=\frac{1}{2p}t^2, \quad y=t \quad (p>0)$$

Find the equation of the trajectory and also the velocity, the path length, the acceleration and the radius of curvature for time $t = p$.

Solution. Eliminating t from the equations of motion we obtain

$$x=\frac{1}{2p}y^2, \text{ that is } y^2=2px$$

However, for $t = 0$ we have $x = y = 0$ and for $t > 0$ we have $x > 0$, $y > 0$, and therefore the trajectory is not the whole parabola but only its upper half (Fig. 7.13). For $t = p$ the coordinates of the point M are $(p/2, p)$. Let us compute the projections of the velocity vector on the coordinate axes:

$$v_x=\frac{dx}{dt}=\frac{1}{p}t, \quad v_y=\frac{dy}{dt}=1$$

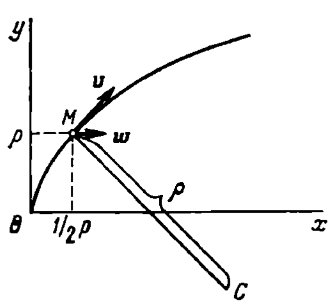

Fig. 7.13

The modulus of the velocity vector is

$$v(t) = \sqrt{v_x^2 + v_y^2} = \frac{1}{p} \sqrt{t^2 + p^2}$$

The direction cosines of the velocity vector are found by formulas (7.12):

$$\cos(\widehat{v,\ x}) = \frac{v_x}{v} = \frac{t}{\sqrt{t^2+p^2}}, \qquad \cos(\widehat{v,\ y}) = \frac{v_y}{v} = \frac{p}{\sqrt{t^2+p^2}}$$

At time $t = p$ the modulus of the velocity vector and its direction cosines are

$$v(p) = \sqrt{2}, \quad \cos(\widehat{v(p),\ x}) = \cos(\widehat{v(p),\ y}) = \frac{\sqrt{2}}{2}$$

that is

$$\angle (v(p),\ x) = \angle (v(p),\ y) = \frac{\pi}{4}$$

The path travelled by the moving particle during time $t = p$ is determined by formula (7.14):

$$S(p) = \frac{1}{p} \int_0^p \sqrt{t^2+p^2}\, dt = \frac{1}{2p} [t \sqrt{t^2+p^2} + p^2 \ln (t + \sqrt{t^2+p^2})]_0^p$$

$$= \frac{1}{2} p [\sqrt{2} + \ln (1 + \sqrt{2})] = 1.15p$$

The projections of the acceleration vector on the axes Ox and Oy are

$$w_x = \frac{dv_x}{dt} = \frac{1}{p}, \qquad w_y = \frac{dv_y}{dt} = 0$$

This means that the modulus of the acceleration vector w is constant ($w = 1/p$) and the vector itself is parallel to the axis Ox during the whole time of motion (see Fig. 7.13). The projection of the acceleration vector on the tangent is found with the aid of formula (7.25):

$$w_\tau = \frac{dv}{dt} = \frac{t}{p\sqrt{t^2+p^2}}$$

Next we find the other projection (on the principal normal):

$$w_n = \sqrt{w^2 - w_\tau^2} = \sqrt{\frac{1}{p^2} - \frac{t^2}{p^2(t^2+p^2)}} = \frac{1}{\sqrt{t^2+p^2}}$$

Finally, using formula (7.28) we compute the radius of curvature of the trajectory for an arbitrary instant t:

$$\rho(t) = \frac{v^2}{w_n} = \frac{1}{p^2} (t^2+p^2) \sqrt{t^2+p^2}$$

For time $t = p$ we have

$$\rho(p) = 2\sqrt{2}p$$

In Fig. 7.13 the centre of curvature C for the point M of the trajectory is shown ($CM = \rho(p)$).

EXAMPLE 7.5. For Examples 7.2 and 7.3 we have

$$w_x = \frac{dv_x}{dt} = -R\omega^2 \cos \omega t, \quad w_y = \frac{dv_y}{dt} = -R\omega^2 \sin \omega t, \quad w_z = \frac{dv_z}{dt} = 0$$

The acceleration vector $\boldsymbol{w}$ is written in the form

$$\boldsymbol{w} = w_x\boldsymbol{i} + w_y\boldsymbol{j} + w_z\boldsymbol{k} = -\boldsymbol{i}R\omega^2 \cos \omega t - \boldsymbol{j}R\omega^2 \sin \omega t$$

and its modulus

$$w = \sqrt{w_x^2 + w_y^2 + w_z^2} = \sqrt{R^2\omega^4 \cos^2 \omega t + R^2\omega^4 \sin^2 \omega t} = R\omega^2$$

is a constant quantity. The direction cosines of the acceleration vector are found using formulas (7.21):

$$\cos(\widehat{\boldsymbol{w}, x}) = \frac{w_x}{w} = -\cos \omega t, \quad \cos(\widehat{\boldsymbol{w}, y}) = \frac{w_y}{w} = -\sin \omega t$$

$$\cos(\widehat{\boldsymbol{w}, z}) = \frac{w_z}{w} = 0$$

From the last formula it follows that

$$\angle(\boldsymbol{w}, z) = \frac{\pi}{2}$$

This means that during the motion of the particle M the acceleration vector is always perpendicular to the axis Oz and hence lies in a plane parallel to Oxy. Since the modulus of the velocity is a constant quantity (see Example 7.3) the projection of the acceleration vector of the particle on the tangent is equal to zero:

$$w_\tau = \frac{dv}{dt} = 0$$

When the tangential acceleration is equal to zero the normal acceleration w_n coincides with the total acceleration $\boldsymbol{w}$ of the particle and consequently

$$w_n = w = R\omega^2$$

The radius of curvature of the trajectory (of the helical line) is found from formula (7.27):

$$\rho = \frac{v^2}{w_n} = \frac{\omega^2 (4\pi^2R^2 + h^2)}{4\pi^2R\omega^2} = R + \frac{h^2}{4\pi^2R}$$

Thus, the radius of curvature of a helical line is a constant quantity for each point on the line.

To conclude the example, let us investigate the velocity and the acceleration hodographs. The equations of the velocity hodograph are obtained by eliminating time t from equations (7.16) which have the form

$$\xi = v_x = -R\omega \sin \omega t, \quad \eta = v_y = R\omega \cos \omega t, \quad \zeta = v_z = \frac{\omega h}{2\pi}$$

The elimination yields

$$\xi^2 + \eta^2 = R^2\omega^2, \qquad \zeta = \frac{\omega h}{2\pi}$$

The last equations specify a circle lying in the plane $\zeta = \omega h/(2\pi)$ with centre on the axis $O\zeta$ and radius $R\omega$ (Fig. 7.14). As is seen in the figure, to the point M_0 there corresponds the point G_0 of the velocity hodograph.

If the acceleration vector $\boldsymbol{w}$ is set off from a fixed point, the terminus H of the vector $\boldsymbol{w}$ describes a curve which is called the *acceleration hodograph*. Its equations can be obtained by eliminating time t from the equations

$$\xi = w_x = -R\omega^2 \cos \omega t$$

$$\eta = w_y = -R\omega^2 \sin \omega t, \quad \zeta = w_z = 0$$

This elimination yields

$$\xi^2 + \eta^2 = R^2\omega^4, \quad \zeta = 0$$

The last two equations specify a circle in the plane $O\xi\eta$ with centre at the point O and radius $R\omega^2$ (see Fig. 7.14). To the point M_0 there corresponds the point H_0 of the acceleration hodograph.

3.5. Rectilinear Motion of a Particle. To conclude the chapter we shall consider the rectilinear motion of a particle as a special case of curvilinear motion.

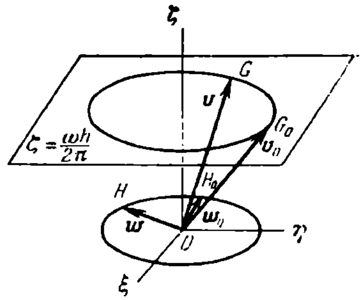

Fig. 7.14

Let us take the trajectory of motion (which is a straight line) as the axis Ox; to this end we choose the origin (the point O), the positive direction along the line and, of course, a length scale. As was mentioned at the end of Sec. 1.5, in this case the natural method and the coordinate method of specifying motion coincide. Thus, let a function $x = x\ (t)$ be given. Since we have $y = z \equiv 0$ for the rectilinear motion under consideration, there must be

$$\boldsymbol{v} = \frac{dx}{dt}\,\boldsymbol{i}, \qquad \boldsymbol{w} = \frac{d^2x}{dt^2}\,\boldsymbol{i}$$

Therefore we shall use the algebraic values of the velocity and of the acceleration because their signs determine the directions of the corresponding vectors. Then (and only then!) we can write

$$v = \frac{dx}{dt}, \qquad w = \frac{dv}{dt}$$

As an example, we shall consider the case of uniformly variable rectilinear motion; in this case $w \equiv a$, where a is a positive or a negative constant. Thus, $dv/dt = a$, whence $dv = a\ dt$. Integrating with respect to t from 0 to t and with respect to v from v_0 to v we obtain

$$\int_{v_0}^{v} dv = \int_{0}^{t} a\,dt, \quad v = v_0 + at \tag{7.29}$$

Further, we have

$$\frac{dx}{dt} = v = v_0 + at, \quad dx = (v_0 + at)\,dt$$

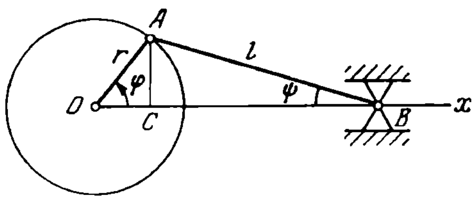

Fig. 7.15

Integrating the last relation with respect to t within the same limits and with respect to x from x_0 to x we derive the equality

$$\int_{x_0}^{x} dx = \int_{0}^{t} (v_0 + at)\,dt, \quad \text{whence } x = x_0 + v_0 t + \frac{1}{2} at^2 \qquad (7.30)$$

For given x_0 and v_0 formulas (7.29) and (7.30) can be regarded as two equalities connecting four quantities: t, x, v and a. When two of them are known the other two can be found. The following formula derived from formulas (7.29) and (7.30) is of use:

$$x = x_0 + \frac{1}{2} v_0 t + \frac{1}{2} (v_0 + at)\,t = x_0 + \frac{1}{2} (v_0 + v)\,t \qquad (7.31)$$

If $x_0 = v_0 = 0$ there holds the formula

$$v^2 = a^2t^2 = 2a\,\frac{at^2}{2} = 2ax \qquad (7.32)$$

EXAMPLE 7.6. The crank $OA = r$ is in uniform rotation about the fixed point O (Fig. 7.15); this means that the angle of rotation φ is proportional to time: $\varphi = \omega t$. To the point A of the crank the connecting rod $AB = l$ is hinged; the connecting rod imparts motion to the slide block B with which it is connected by a hinge; the slide block moves between two parallel guides. Determine the velocity and the acceleration of the slide block B.

Solution. Let us place the origin at the point O and draw the axis Ox in the direction of OB. From Fig. 7.15 we derive

$$x = OB = OC + CB = r\cos\varphi + l\cos\psi$$

According to the law of sines, for the triangle OAB we have

$$\frac{\sin\psi}{r} = \frac{\sin\varphi}{l}, \quad \text{that is } \sin\psi = \frac{r}{l}\sin\varphi$$

Let us denote the ratio r/l by λ and compute $\cos\psi$:

$$\cos\psi = \sqrt{1 - \sin^2\psi} = \sqrt{1 - \lambda^2\sin^2\varphi}$$

Substituting this expression of $\cos\psi$ into the equation of x and taking into account that $\varphi = \omega t$ we obtain the equation of motion of the slide block B in the form

$$x = r\cos\omega t + l\sqrt{1 - \lambda^2\sin^2\omega t}$$

For any instant t the velocity of the slide block is

$$v=\frac{dx}{dt}=-r\omega\sin\omega t-\frac{1}{2}\lambda^2 l\omega\frac{\sin 2\omega t}{\sqrt{1-\lambda^2\sin^2\omega t}}$$

The acceleration of the slide block can be found as function of t by performing the repeated differentiation.

Usually the quantity λ^2 is small (of the order of several hundredths). Let us use the expansion of $\sqrt{1-\alpha}$ into the power series:

$$\sqrt{1-\alpha}=1-\frac{1}{2}\alpha+\dots$$

(see [1]). The application of this expansion provides an approximate expression

$$x\approx\xi=r\cos\omega t+l\left(1-\frac{1}{2}\lambda^2\sin^2\omega t\right)$$

for the abscissa of the slide block. Differentiating with respect to time we find approximate expressions for the velocity and the acceleration of the slide block;

$$v\approx\frac{d\xi}{dt}=-r\omega\sin\omega t-\frac{1}{2}\lambda^2 l\omega\sin 2\omega t$$

and

$$w\approx\frac{d^2\xi}{dt^2}=-r\omega^2\cos\omega t-\lambda^2 l\omega^2\cos 2\omega t$$

We see that the approximate expression for the velocity of the slide block is obtained from the exact expression if we discard the term of the radicand involving λ^2 taking into account that λ^2 is small in comparison with unity.

Problems

PROBLEM 7.1. Two bodies A and B connected by an inextensible rod AB of length $2l$ slide along the coordinate axes (Fig. 7.16). The law of motion of the body A is expressed by the equation $x = 2l\sin\omega t$. Find the velocities of both the bodies and also the trajectory and the velocity of the point C lying at a distance $2l/3$ from the body A.

Answer. The point C describes an ellipse with centre at O and semiaxes $4l/3$ and $2l/3$;

$$v_A=2l\omega\cos\omega t,\quad v_B=-2l\omega\sin\omega t,\quad v_C=\frac{2}{3}l\omega\sqrt{1+3\cos^2\omega t}$$

PROBLEM 7.2. A particle M is in the motion described by the equations $x=\beta t$, $y=\gamma t-(gt^2/2)$. Determine the trajectory of the particle, find the velocity, the total, the tangential and the normal accelerations of M and also the radius of curvature of the particle for an arbitrary instant t. Explain the physical significance of the constants β and γ and construct the velocity hodograph.

Answer. The trajectory is a part of the parabola

$$y=\frac{\gamma}{\beta}x-\frac{g}{2\beta^2}x^2$$

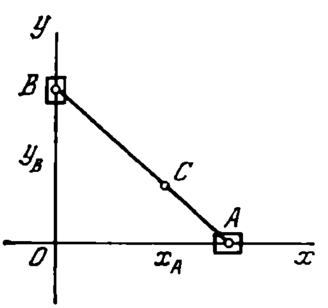

Fig. 7.16

passing through the origin, with vertex at the point $\left(\beta\gamma/g,\ \frac{1}{2}\gamma^2/g\right)$ and branches going downward;

$$v=\sqrt{\beta^2+(\gamma-gt)^2},\quad \cos(\widehat{v,\ x})=\frac{\beta}{v},\quad w=g,\quad \angle(w,\ y)=\pi$$

$$w_\tau=-\frac{g(\gamma-gt)}{v},\quad w_n=\frac{\beta g}{v},\quad \rho=\frac{v^3}{\beta g}$$

$\beta = v_0 \cos\alpha$, $\gamma = v_0 \sin\alpha$, where α is the angle between the vector of the initial velocity $\boldsymbol{v}_0 = \boldsymbol{v}(0)$ and the axis Ox. The equation of the velocity hodograph is $x = \beta$.

PROBLEM 7.3. Let M be a point on the rim of a wheel of radius R rolling without sliding along a rectilinear rail, the velocity v_O of the centre of the wheel being constant. The point M describes a cycloid determined by the parametric equations $x = R(\omega t - \sin\omega t)$, $y = R(1 - \cos\omega t)$ (where $\omega = v_0/R$) which are nothing other than equations of motion of M. Find the velocity and the acceleration of M and the radius of curvature of the trajectory for an arbitrary instant t.

Answer. $v=2R\omega\left|\sin\frac{\omega t}{2}\right|$, $\cos(\widehat{v,\ y})=R\omega\sin\omega t/v$, $w=R\omega^2$, $\angle(w,\ y)=\omega t$, $\rho=4R\left|\sin\frac{\omega t}{2}\right|$.

PROBLEM 7.4. A particle moves along an ellipse described by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

The particle starts moving from the vertex $x = 0$, $y = b$ of the ellipse with initial velocity $\boldsymbol{v}_0$, its total acceleration permanently remaining parallel to the axis Oy. Find the algebraic value of the total acceleration as function of the ordinate y (this is known as Newton's problem).

Hint. Differentiate identity (1) with respect to time, put $\dot{x} = v_0$ in accordance with the conditions of the problem and then express $\dot{y}$. After this differentiate identity (1) with respect to time once again and make use of the expression of $\dot{y}$ and the expression of x^2 determined by (1).

Answer. $w=\ddot{y}=-\frac{b^4v_0^2}{a^2y^3}$.

PROBLEM 7.5. A bridge has the shape of the parabola $y = -0.005x^2$, the coordinates x and y being expressed in metres. An automobile moves on the bridge with a velocity of constant modulus 72 km/h. Find the acceleration of the automobile at the instant when it is at the vertex of the parabola.

Hint. Use the formula for the radius of curvature established in differential calculus.

Answer. $w = w_n = 4$ m/s^2.

PROBLEM 7.6. A point moves in a circle of radius 8 m, the law of motion being $s = 2t^3/3$ m. Find the velocity of the point at the instant t_1 when the modulus of its normal acceleration is equal to that of the tangential acceleration.

Answer. $t_1 = 2$ s, $v(2) = 8$ m/s.

Chapter 8 Simplest Forms of Motion of a Rigid Body

Let us choose a rectangular Cartesian coordinate system $Oxyz$ in space which we conditionally regard as being fixed. The position of a rigid body in space can be specified by the positions of its three points not lying in one straight line. Before proceeding to the study of the general case of motion of a rigid body we shall consider the simplest types of motion.

§ 1. Translatory Motion of a Rigid Body

1.1. Definition of Translatory Motion. Let us consider a motion of a rigid body such that the vector $\boldsymbol{MN}$ connecting any two of the points of the rigid body remains constant during the whole time of motion. Thus, the vector $\boldsymbol{MN}$ retains not only its length (which is always the case when a rigid body is in arbitrary motion) but also its direction. Such a motion of a rigid body is called *translatory*. Hence, by a translatory motion of a rigid body is meant a *motion during which any straight line chosen arbitrarily within the body remains parallel to itself*. Members of many mechanisms are in translatory motion, for instance, the parallel rods of the wheels of a locomotive, the pedals of a bicycle, etc.

In Fig. 8.1 the positions of the line segment MN are shown for instant t and also for instants t' and t''. We see that if the body is in a translatory motion then, by definition,

$$\boldsymbol{MN} = \boldsymbol{M'N'} \tag{8.1}$$

It is evident that the quadrilateral $MM'N'N$ is a parallelogram and consequently $MM' = NN'$. By virtue of the definition, equality (8.1) must hold for all the particles of the rigid body (it is in fact sufficient to require that equality (8.1) should hold for any three points of the body not lying in one straight line) and for any instants t and t' during the time of motion.

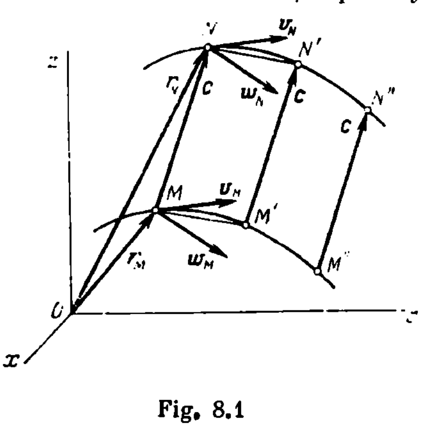

Fig. 8.1

1.2. Theorem on the Trajectories, Velocities and Accelerations of the Particles of a Body in Translatory Motion. *In a translatory motion of a rigid body the trajectories of all its particles*

are congruent (that is they can be made to coincide by means of parallel translation), *the laws of motion of the particles along their trajectories being the same, and the particles having geometrically equal velocities and accelerations.*

Proof. Let $\boldsymbol{r}_M$ and $\boldsymbol{r}_N$ denote the radius vectors $\boldsymbol{OM}$ and $\boldsymbol{ON}$; then the identity

$$\boldsymbol{MN} = \boldsymbol{r}_N - \boldsymbol{r}_M = \boldsymbol{c}$$

where $\boldsymbol{c}$ is a constant free vector, holds for any two points M and N of the body during the whole time of the translatory motion. Consequently the position of the point N at any instant t can be determined from the identity

$$\boldsymbol{r}_N(t) = \boldsymbol{r}_M(t) + \boldsymbol{c} \tag{8.2}$$

It follows that in the translatory motion the trajectories of the particles of the rigid body are all of the same shape and can be made to coincide by means of parallel translation.

Differentiating identity (8.2) and taking into account formula (7.9) we obtain

$$\boldsymbol{v}_N(t) \equiv \boldsymbol{v}_M(t) \tag{8.3}$$

This means that the velocities of all the particles of the rigid body have equal moduli and the same directions at any instant during the translatory motion. Conversely, if identity (8.3) holds for any two points of the rigid body then, integrating the identity, we arrive at identity (8.2), and hence the motion is translatory. Thus, any translatory motion is characterized completely by one vector which depends on time solely and expresses at each instant the velocity of the translatory motion of the rigid body, the modulus and the direction of the velocity remaining one and the same during the motion.

Let us write identity (8.3) for the algebraic velocities (see Sec. 2.3 of Chap. 7), multiply it by dt and integrate with respect to time from 0 to t:

$$\int_0^t v_{N_\tau}(t)\,dt = \int_0^t v_{M_\tau}(t)\,dt$$

By virtue of formula (7.13), the last relation means that

$$\int_0^t ds_N(t) = \int_0^t ds_M(t), \text{ whence } s_N(t) = s_M(t)$$

It follows that the particles of the body describe their trajectories (which are congruent) according to one and the same law of motion.

The differentiation of identity (8.3) with respect to time results in

$$\boldsymbol{w}_N(t) \equiv \boldsymbol{w}_M(t) \tag{8.4}$$

Hence, we come to the conclusion that the accelerations of all the particles of the rigid body have the same moduli and directions at any instant during the translatory motion. Thus, the acceleration of a translatory motion of a rigid body is completely characterized by the acceleration of an arbitrarily chosen point of the body. The theorem is proved.

When the modulus and the direction of the velocity vector of a translatory motion are invariable during the whole time of motion the acceleration is equal to zero and all the particles of the rigid body are in uniform rectilinear motion. In this case the congruence of the trajectories implies that the rectilinear trajectories along which the particles of the body move with the same velocity are parallel. Such a motion is called a *uniform rectilinear translatory motion of the body.*

To conclude the section we stress that a translatory motion of a rigid body is completely specified by the motion of any of its points. The kinematic description of a translatory motion of a rigid body reduces to the description of the motion of a particle (Chaps. 7 and 11).

§ 2. Rotation of a Rigid Body about a Fixed Axis

2.1. Equation of Rotational Motion. The second simple type of motion of a rigid body is rotation about a fixed axis, that is a motion during which two points O and O' of the body remain fixed (Fig. 8.2). From the definition of a rigid body it follows that all the points of the straight line OO' also remain fixed; the line OO' is called the *axis of rotation**. Let us place the origin of a fixed coordinate system at the point O and let the axis Oz coincide with the axis of rotation OO', the axes Ox and Oy being directed along two arbitrary perpendicular directions so that $Oxyz$ is a right-handed coordinate system. Since the position of a rigid body in space is completely specified by the positions of its three points not lying in one straight line, for the motion under consideration it is sufficient to specify the position of a point M of the body not lying on the axis of rotation. This can be performed in the following way.

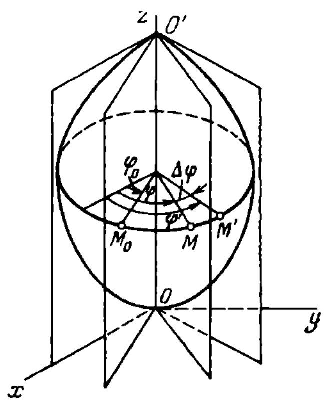

Fig. 8.2

* More generally, the axis of rotation must not necessarily lie within the rigid body itself.

Let M_0 be the position of the point M at the initial instant $t = 0$. We shall consider the half-planes $OO'M_0$ and $OO'M$: by φ_0 and φ we shall denote the dihedral angles between $OO'M_0$ and $OO'M$ and the plane Ozx respectively. Let us choose the direction of rotation from the axis Ox to the axis Oy as the positive direction of reckoning the angle φ. The position of the point M and thus the position of the whole body are completely specified if the value of the angle φ is known for any instant t. That is why the relation

$$\varphi = \varphi(t)$$

is called the *equation of rotational motion of the rigid body* about the fixed axis. As usual, the function $\varphi(t)$ is supposed to be one-valued, continuous and twice-differentiable.

2.2. Angular Velocity and Angular Acceleration of a Body. Let M' be the position of the point M at instant $t' = t + \Delta t$, the corresponding value of the angle φ being $\varphi' = \varphi + \Delta\varphi$. The ratio $\Delta\varphi/\Delta t$ is called the *average angular velocity of the body* during the time interval $(t, t + \Delta t)$. The limit of this ratio for Δt tending to zero is called the *angular velocity of the body* at instant t; we shall denote the angular velocity by ω. Thus, by definition, the angular velocity of the body is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \frac{d\varphi}{dt} \tag{8.5}$$

In the International System of Units (SI) the unit for measuring the angular velocity is rad/s (one radian per second). In engineering rotational motion is also characterized by another quantity, namely by the number n of revolutions per minute (denoted by rev/min or rpm). If we divide n by 60 the quotient is equal to the number of revolutions per second (denoted by rev/s or rps). The product of this quotient by 2π is equal to the number of radians per second, that is the angular velocity is

$$\omega = \frac{n}{60} 2\pi = \frac{\pi n}{30} \text{ rad/s}$$

The angular velocity itself is a function of time: $\omega = \omega(t)$. The derivative of this function with respect to time is called the *angular acceleration of the body*; we shall denote the angular acceleration by ε:

$$\varepsilon = \frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2} \tag{8.6}$$

In the International System of Units (SI) the unit for measuring the angular acceleration is rad/s^2. It should be noted that depending on the direction of rotation the angular velocity ω may be positive or negative. At those instants when $\omega > 0$ the body rotates in the

positive direction. If the angular acceleration is positive ($\varepsilon > 0$) for $\omega > 0$ the rotation of the body is accelerated, and if $\varepsilon < 0$ and $\omega > 0$ the rotation is decelerated.

Now let us suppose that $\omega < 0$. Then for $\varepsilon > 0$ the absolute value of the angular velocity decreases, and the body is in a decelerated rotation; conversely, for $\varepsilon < 0$ the angular velocity increases in its absolute value, that is the rotation is accelerated.

When the product of the angular velocity by the angular acceleration is positive the body is in an accelerated rotation. Thus, the rotation of the body at a given instant is accelerated if at this instant the absolute value of its angular velocity tends to increase.

2.3. Uniform Rotation and Uniformly Variable Rotation.

(a) In the case when the angular velocity is constant ($\omega = \Omega = \text{const}$) equality (8.5) yields $d\varphi = \Omega\, dt$. Integrating the last equality with respect to time t from 0 to t and with respect to φ from φ_0 to φ, we obtain

$$\int_{\varphi_0}^{\varphi} d\varphi = \int_0^t \Omega\, dt$$

whence

$$\varphi - \varphi_0 = \Omega t \tag{8.7}$$

Formula (8.7) implies that in a *uniform rotation* the angle of rotation $\varphi - \varphi_0$ is proportional to time, and the constant angular velocity Ω is expressed as

$$\Omega = \frac{\varphi - \varphi_0}{t}$$

(b) In the case when the angular acceleration is constant ($\varepsilon = \mathrm{E} = \text{const}$) equality (8.6) yields $d\omega = \mathrm{E}\, dt$; the integration of the last relation results in

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \mathrm{E}\, dt$$

whence we find

$$\omega = \omega_0 + \mathrm{E}t \tag{8.8}$$

Now formula (8.5) gives

$$d\varphi = \omega\, dt = (\omega_0 + \mathrm{E}t)\, dt$$

Next we compute the integral:

$$\int_{\varphi_0}^{\varphi} d\varphi = \int_0^t (\omega_0 + \mathrm{E}t)\, dt$$

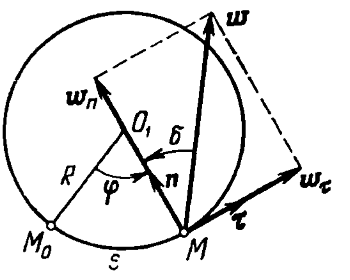

Fig. 8.3

This finally yields

$$\varphi = \varphi_0 + \omega_0 t + \frac{1}{2} \mathrm{E} t^2 \tag{8.9}$$

Formulas (8.8) and (8.9) describe a *uniformly variable* (that is uniformly accelerated or uniformly decelerated) *motion* and are analogous to formulas (7.29) and (7.30) for a uniformly variable rectilinear motion of a particle. For given φ_0 and ω_0 these equations can be regarded as a system of two simultaneous equations connecting the three variables t, φ, ω and one constant quantity E. If two of these four quantities are known the other two quantities can be found from the equations.

2.4. Velocity and Acceleration of a Particle of a Body in Rotational Motion. In this section we shall find the velocity and the acceleration of an arbitrary particle M of a rigid body rotating about a fixed axis. Let us denote by O_1 the point of intersection of the plane perpendicular to the axis of rotation OO' and passing through the particle M with that axis (Fig. 8.3). The trajectory of the particle M is a circle (or its part) of radius $O_1M = R$ lying in the indicated plane. Let us reckon the coordinate s (the arc length) from the initial position M_0 of the particle M; for s we retain the same positive direction in which the angle φ is reckoned. Then

$$s = R\varphi$$

Consequently, the so-called algebraic value of the velocity of the particle M is determined by formula (7.13):

$$v_\tau = \frac{ds}{dt} = R\frac{d\varphi}{dt} = R\omega \tag{8.10}$$

The unit vector $\boldsymbol{\tau}$ along the tangent goes in the direction of increase of the coordinate s, that is in the direction of increase of the angle φ, while the unit vector $\boldsymbol{n}$ of the normal is directed, as usual, toward the centre of curvature of the trajectory, that is to the point O_1. Then for the velocity vector $\boldsymbol{v}$ of the particle M we have the expression

$$\boldsymbol{v} = R\omega\boldsymbol{\tau}$$

and the modulus of the velocity is given by the formula

$$v = R\,|\,\omega\,| \tag{8.11}$$

Formulas (7.25a) and (7.26) yield the following expressions for the projections of the acceleration vector of the particle M on the

natural axes:

$$w_\tau = \frac{d^2s}{dt^2} = R\frac{d^2\varphi}{dt^2} = R\varepsilon \tag{8.12}$$

and

$$w_n = \frac{v^2}{\rho} = \frac{R^2\omega^2}{R} = R\omega^2 \tag{8.13}$$

Consequently, the acceleration vector $\boldsymbol{w}$ is written as

$$\boldsymbol{w} = w_\tau \boldsymbol{\tau} + w_n \boldsymbol{n} = R\varepsilon\boldsymbol{\tau} + R\omega^2\boldsymbol{n} \tag{8.14}$$

The tangential acceleration $R\varepsilon\boldsymbol{\tau}$ goes in the positive or in the negative direction of reckoning the angle φ depending on whether the angular acceleration ε is positive or negative. The normal acceleration $R\omega^2\boldsymbol{n}$ always goes along the vector $\boldsymbol{n}$ or is equal to zero (because $R\omega^2 \geqslant 0$); it is also called the *centripetal acceleration*. The modulus of the acceleration vector of the particle M is

$$w = \sqrt{w_\tau^2 + w_n^2} = R\sqrt{\varepsilon^2 + \omega^4} \tag{8.15}$$

The direction of the acceleration vector of the particle M can be specified by the acute angle δ reckoned from the acceleration vector to the unit normal vector $\boldsymbol{n}$. From the right triangle shown in Fig. 8.3 we find

$$\tan\delta = \frac{w_\tau}{w_n} = \frac{\varepsilon}{\omega^2} \tag{8.16}$$

The sign of $\tan\delta$ is determined by that of the angular acceleration ε. In the case of a uniform rotation with constant angular velocity we have $\varepsilon \equiv 0$, and hence $w_\tau \equiv 0$ and $\delta \equiv 0$. Consequently, in this case $\boldsymbol{w} = R\omega^2\boldsymbol{n}$, and the acceleration vector is directed along the normal during the whole time of motion.

2.5. Formulas for the Velocity Vector and the Acceleration Vector of a Particle of a Rotating Body. The expressions for the velocity vector and the acceleration vector of an arbitrary particle M of a rigid body rotating about a fixed axis can also be found in a different way. To this end we shall consider the angular velocity of the rotating body *as a vector* $\boldsymbol{\omega}$ whose modulus is equal to the absolute value of the angular velocity (that is to $|d\varphi/dt|$) and which is directed along the axis of rotation according to the right-hand screw rule. For instance, if the rotation is in the positive direction about the axis of rotation (coinciding with the axis Oz; see Fig. 8.4) then the vector $\boldsymbol{\omega}$ is directed along the positive z-axis. The origin of the vector $\boldsymbol{\omega}$ is placed arbitrarily on the axis of rotation. Thus, this is a sliding vector.

Similarly, the angular acceleration of the body will also be regarded as a sliding vector $\boldsymbol{\varepsilon}$ lying on the axis of rotation. If $\boldsymbol{k}$ denotes the

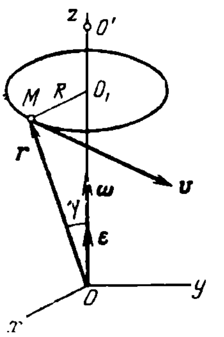

Fig. 8.4

unit vector along the axis of rotation then

$$\boldsymbol{\omega} = \omega \boldsymbol{k} = \frac{d\varphi}{dt}\boldsymbol{k}, \quad \boldsymbol{\varepsilon} = \varepsilon \boldsymbol{k} = \frac{d\omega}{dt}\boldsymbol{k}$$

Since both the length and the direction of the unit vector $\boldsymbol{k}$ are constant there holds the vector equality

$$\boldsymbol{\varepsilon} = \frac{d\boldsymbol{\omega}}{dt}$$

Let us set off the vector $\boldsymbol{\omega}$ from its point of application placed at the origin O (see Fig. 8.4) and construct the radius vector $\boldsymbol{r} = \boldsymbol{OM}$. The point of intersection of the plane passing through the point M perpendicularly to the axis of rotation with that axis (Oz) will be denoted by O_1; $O_1M = R$. Further, let the velocity vector $\boldsymbol{v}$ of the particle M be drawn as is shown in Fig. 8.4. Now we consider the vector product $[\boldsymbol{\omega}, \boldsymbol{r}]$. This vector product is a vector perpendicular to the plane O_1OM, its direction being determined by the right-hand screw rule. Hence, the direction of the vector product coincides with that of the vector $\boldsymbol{v}$. The modulus of the vector product is equal to $|\omega| r \sin\gamma = |\omega| R$, that is to the modulus of the vector $\boldsymbol{v}$. We have thus proved the vector equality

$$\boldsymbol{v} = [\boldsymbol{\omega}, \boldsymbol{r}] \tag{8.17}$$

whose meaning can be expressed thus:

The velocity of any particle of a rigid body rotating about a fixed axis is geometrically equal to the vector product of the vector of angular velocity of the body by the radius vector of that particle drawn from an arbitrary point of the axis of rotation. This means that as the radius vector of the particle M we can also take the vector $\boldsymbol{O_1M} = \boldsymbol{R}$ and write formula (8.17) in the form $\boldsymbol{v} = [\boldsymbol{\omega}, \boldsymbol{R}]$.

The expression for the acceleration vector $\boldsymbol{w}$ of the particle M is obtained by differentiating identity (8.17) with respect to t according to the differentiation rule for the vector product (see formula (3) in Sec. 4 of Introduction to Kinematics):

$$\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt} = \left[\frac{d\boldsymbol{\omega}}{dt}, \boldsymbol{r}\right] + \left[\boldsymbol{\omega}, \frac{d\boldsymbol{r}}{dt}\right]$$

Since we have

$$\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\varepsilon}, \quad \frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}$$

the formula for the acceleration can be written as

$$\boldsymbol{w} = [\boldsymbol{\varepsilon}, \boldsymbol{r}] + [\boldsymbol{\omega}, \boldsymbol{v}] \tag{8.18}$$

It can be shown (see Sec. 2.3 of Chap. 9) that the vector products on the right-hand side of formula (8.18) are equal to the first and the second summands in formula (8.14) respectively. Consequently, formula (8.18), like formula (8.14), expresses the resolution of the acceleration vector of a particle of a rotating body into components along the tangential and normal directions. We shall restrict ourselves to formula (8.14) because it is more visual.

In conclusion we consider some examples.

EXAMPLE 8.1. A wheel of radius R is in uniformly decelerated motion about a fixed axis perpendicular to the plane of the wheel; making N revolutions it stops moving. The initial angular velocity is $\omega_0 > 0$. Find the angular acceleration of the wheel and the acceleration of a point on the rim of the wheel.

Solution. Let us choose the direction of rotation as the positive direction and assume that $\varphi = 0$ at the initial instant $t = 0$. Then at the instant $t = T$ at which the wheel stops rotating the angle of rotation of the wheel is $\varphi = 2\pi N$. From formulas (8.8) and (8.9) we obtain the equations

$$0=\omega_0+\varepsilon T, \quad 2\pi N=\omega_0 T+\frac{1}{2}\varepsilon T^2$$

From the first equation we find $T = -\omega_0/\varepsilon$; the substitution of this expression into the second equation results in

$$2\pi N=-\frac{\omega_0^2}{\varepsilon}+\frac{\omega_0^2}{2\varepsilon}=-\frac{\omega_0^2}{2\varepsilon}$$

whence we find

$$\varepsilon=-\frac{\omega_0^2}{4\pi N}$$

Using formula (8.12) we find that the projection of the acceleration of the point on the rim of the wheel on the tangent is expressed by the formula

$$w_\tau=R\varepsilon=-\frac{R\omega_0^2}{4\pi N}$$

It is constant and negative during the whole time of motion. Since $w_\tau < 0$, this means that the direction of the tangential acceleration is opposite to that of the motion.

The angular velocity ω of the wheel at instant $t \leqslant T$ is determined by formula (8.8):

$$\omega=\omega_0+\varepsilon t=\omega_0-\frac{\omega_0^2}{4\pi N}t$$

Using formula (8.3) we compute the projection of the acceleration on the normal:

$$w_n=R\omega^2=R\left(\omega_0-\frac{\omega_0^2}{4\pi N}t\right)^2 \quad (t \leqslant T)$$

We see that in the example under consideration the modulus of the centripetal acceleration (which is proportional to the square of the angular velocity of rotation) varies with time. The acceleration vector of the point on the rim of the wheel is expressed by formula (8.14):

$$\boldsymbol{w}=w_\tau\boldsymbol{\tau}+w_n\boldsymbol{n}=-\frac{R\omega_0^2}{4\pi N}\boldsymbol{\tau}+R\left(\omega_0-\frac{\omega_0^2}{4\pi N}t\right)^2\boldsymbol{n} \quad (t \leqslant T)$$

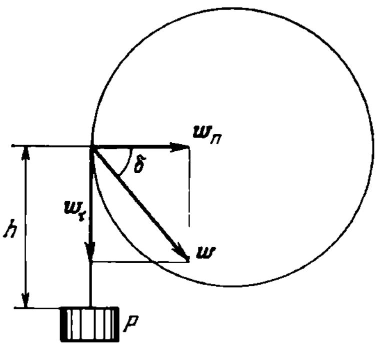

Fig. 8.5

The modulus of the acceleration vector of the point on the rim of the wheel is

$$w = \sqrt{w_\tau^2 + w_n^2}$$

$$= R \sqrt{\frac{\omega_0^4}{16\pi^2 N^2} + \left(\omega_0 - \frac{\omega_0^2}{4\pi N} t\right)^4}$$

$$(t \leqslant T)$$

and its direction is specified by formula (8.15):

$$\tan \delta = \frac{\varepsilon}{\omega^2}$$

$$= -\frac{1}{4\pi N \left(1 - \frac{\omega_0}{4\pi N} t\right)^2} \quad (t \leqslant T)$$

The minus sign indicates that the acute angle δ reckoned from the acceleration vector to the unit vector along the principal normal is negative. This means that the acceleration vector of the point on the rim of the wheel goes at an angle δ reckoned from the normal in the clockwise direction.

EXAMPLE 8.2. A weight P attached to a thread wound on a wheel of radius R moves vertically downward and rotates the wheel (Fig. 8.5), the initial velocity of the weight being equal to zero. During the first t_1 seconds the weight P passes a distance of h metres. Find the angular velocity and the angular acceleration of the wheel and also the velocity and the acceleration of the points on the rim of the wheel at instant t_2.

Solution. Let us denote the acceleration of the weight P by a. Then, by formula (7.30), we have

$$h = \frac{1}{2} a t_1^2, \quad \text{that is} \quad a = \frac{2h}{t_1^2} \text{ m/s}^2$$

The modulus of the tangential acceleration of the points lying on the rim of the wheel is obviously equal to the acceleration a and hence

$$w_\tau = R\varepsilon = a = \frac{2h}{t_1^2} \text{ m/s}^2$$

whence we find the angular acceleration of the wheel:

$$\varepsilon = \frac{2h}{R t_1^2} \text{ rad/s}^2$$

From formula (8.8), for $\omega_0 = 0$, we find

$$\omega(t_2) = \varepsilon t_2 = \frac{2h t_2}{R t_1^2} \text{ rad/s}$$

The modulus of the normal acceleration of the points lying on the rim of the wheel is

$$w_n(t_2) = R\omega(t_2)^2 = \frac{4h^2 t_2^2}{R t_1^4} \text{ m/s}^2$$

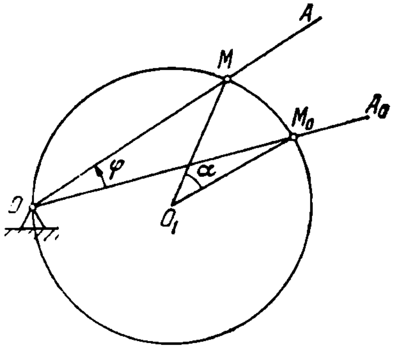

Fig. 8.6

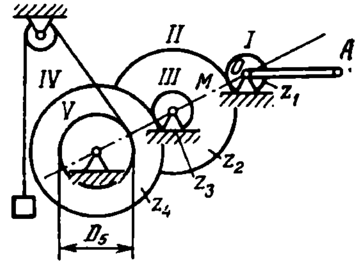

Fig. 8.7

The acceleration of the points on the rim of the wheel is determined by formulas (8.15) and (8.16):

$$w(t_2)=R\sqrt{\left(\frac{2h}{Rt_1^2}\right)^2+\left(\frac{2ht_2}{Rt_1^2}\right)^4}=\frac{2h}{t_1^2}\sqrt{1+\frac{4h^2t_2^4}{R^2t_1^4}}\ \text{m/s}^2$$

$$\tan\delta=\frac{2h}{Rt_1^2}\left(\frac{Rt_1^2}{2ht_2}\right)^2=\frac{Rt_1^2}{2ht_2^2}$$

EXAMPLE 8.3. A small ring M is put on a fixed wire circle of radius R with centre at O_1 (Fig. 8.6); the rod OA passing through the ring rotates uniformly about a fixed axis perpendicular to the plane of the circle and passing through the point O of the circle. The angular velocity of the rod OA is equal to ω. Find the velocity and the acceleration of the ring M.

Solution. When the rod turns through the angle $\varphi=\omega t$ the ring M traverses the arc $\cup M_0M = R\alpha$. From geometry we know that $\alpha = 2\varphi$ and therefore

$$s = \cup M_0M = 2R\omega t$$

Using formula (7.10) we find

$$v=\frac{ds}{dt}=2R\omega$$

and then, by formulas (7.25) and (7.26), we obtain

$$w_\tau=\frac{dv}{dt}=0,\quad w=w_n=\frac{v^2}{R}=4R\omega^2$$

EXAMPLE 8.4. The handle of the winch OA shown in Fig. 8.7 is rotated with constant acceleration $\varepsilon_1 = \pi$ rad/s². The numbers of the teeth of the gears of the winch are $z_1 = 8$, $z_2 = 32$, $z_3 = 12$ and $z_4 = 36$, and the diameter of the drum is $D_5 = 400$ mm. Find the velocity and the acceleration of the weight and also the height to which the weight is lifted in 1/2 minute after the beginning of the motion.

Solution. The velocity of the point M belonging to both the gears I and II has the modulus equal to

$$v_M = R_1\omega_1 = R_2\omega_2$$

Therefore the relation

$$\omega_2=\frac{R_1}{R_2}\,\omega_1=\frac{z_1}{z_2}\,\omega_1$$

holds identically (for any instant t). Similarly, we have

$$\omega_4 = \frac{z_3}{z_4}\omega_3$$

Since $\omega_3 = \omega_2$ (because gears *III* and *II* are attached to one shaft) we obtain the identity

$$\omega_4 = \frac{z_3 z_1}{z_4 z_2}\omega_1$$

Differentiating the last identity we find

$$\frac{d\omega_4}{dt} = \frac{z_3 z_1}{z_4 z_2}\frac{d\omega_1}{dt}, \text{ that is } \varepsilon_4 = \frac{z_3 z_1}{z_4 z_2}\varepsilon_1$$

The substitution of the numerical values results in

$$\varepsilon_5 = \varepsilon_4 = \frac{12\cdot 8}{36\cdot 32}\pi = \frac{1}{12}\pi \text{ rad/s}^2$$

The angular velocity and the angle of rotation of gear *IV* and drum *V* for $t = 30$ s are determined by formulas (8.8) and (8.9):

$$\omega_5(30) = \varepsilon_5\cdot 30 = \frac{5}{2}\pi \text{ rad/s}, \quad \varphi_5(30) = \frac{1}{2}\varepsilon_5\cdot 30^2 = \frac{75}{2}\pi \text{ rad}$$

Multiplying the angle of rotation of the drum by its radius R_5 we obtain the height to which the weight is lifted:

$$h(30) = R_5\varphi_5(30) = 0.2\cdot\frac{75}{2}\pi = 23.6 \text{ m}$$

Since for any instant t we have

$$h(t) = R_5\varphi_5(t)$$

the weight is in rectilinear motion, its velocity and acceleration are

$$v(t) = \frac{dh}{dt} = R_5\frac{d\varphi_5}{dt} = R_5\omega_5(t), \quad w(t) = \frac{dv(t)}{dt} = R_5\frac{d\omega_5}{dt} = R_5\varepsilon_5$$

Finally, we substitute the numerical values and obtain

$$v(30) = 0.2\cdot\frac{5}{2}\pi = 1.57 \text{ m/s}, \quad w(30) = 0.2\cdot\frac{\pi}{12} = 0.0524 \text{ m/s}^2$$

Problems

PROBLEM 8.1. A wheel of radius $r = 2$ m rotates about a fixed horizontal axis O so that the modulus of the velocity of the point A on the rim of the wheel is constant: $v_A = 3.6$ m/s. The rod triangle ABC (Fig. 8.8) is hinged to the rim of the wheel at its vertex A; the triangle is in a motion during which the base BC always remains parallel to the horizon. Determine the trajectory, the velocity and the acceleration of the vertex C of the triangle ABC.

Hint. By the condition of the problem, the motion of the triangle ABC is translatory and consequently $\mathbf{v}_C = \mathbf{v}_A$ and $\mathbf{w}_C = \mathbf{w}_A$.

Answer. The trajectory of the point C is a circle of radius r with centre on the perpendicular to $\mathbf{v}_C$ erected at the point C; $v_C = 3.6$ m/s, $w_C = w_n = \frac{1}{r}v_C^2 = 6.48$ m/s^2.

PROBLEM 8.2. After an electric motor is switched off its rotor makes 675 revolutions and stops in 30 seconds. Assuming that the motion is uniformly

decelerated find the initial velocity and the law of rotation of the rotor of the electric motor.

Answer. $\omega_0 = 90\pi$ rad/s, $\varphi = \frac{3}{2}\pi t\,(60-t)$ rad.

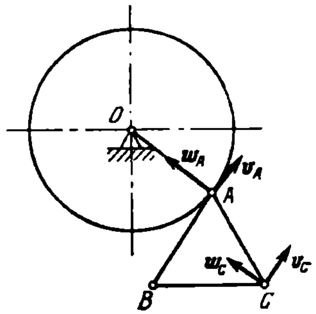

Fig. 8.8

PROBLEM 8.3. For a point M_1 on the equator of the Earth and for a point M_2 with latitude 60°, which are fixed relative to the Earth (Fig. 8.9), find the velocity and the acceleration due to the rotation of the Earth. The radius of the Earth is $R = 6370$ km.

Answer. $v_1 = 464$ m/s; $w_1 = w_n = 0.0337$ m/s²; $v_2 = 232$ m/s; $w_2 = w_n = 0.0169$ m/s².

PROBLEM 8.4. In the mechanism of the lifting jack shown in Fig. 8.10 gears z_2 and z_3 are rigidly attached to shaft *II*, and gears z_4 and z_5 to shaft *III*.

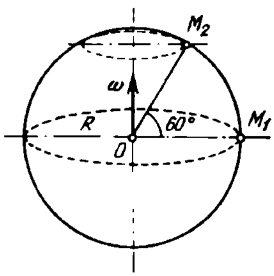

Fig. 8.9

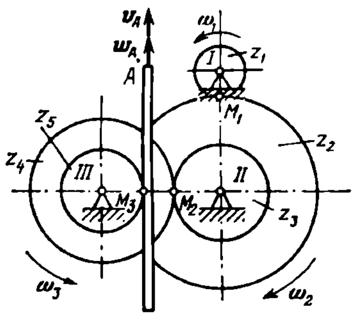

Fig. 8.10

Drive shaft *I* imparts rotation with the aid of gears z_1 and z_2 to shaft *II*; shaft *II* imparts motion with the aid of gears z_3 and z_4 to shaft *III*, and the latter drives the tooth rack *A* with the aid of gear z_5. The numbers of the gear teeth are $z_1 = 8$, $z_2 = 32$, $z_3 = 16$ and $z_4 = 24$; the radius of gear z_5 is $r_5 = 5$ cm.

Find the velocity and the acceleration of the tooth rack *A* at the instant $t = 3$ s after the beginning of motion on condition that drive shaft *I* is in

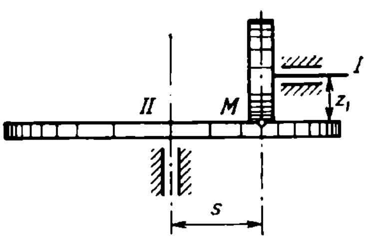

Fig. 8.11

uniformly accelerated motion with angular acceleration $\varepsilon_1 = 6\pi$ rad/s², the members of the mechanism being at rest at the initial instant.

Answer. $v_A = 0.471$ m/s, $w_A = 0.157$ m/s².

PROBLEM 8.5. Drive shaft I of a friction gear rotates with constant angular velocity corresponding to $n = 360$ rpm (Fig. 8.11). Simultaneously with the rotation, shaft I moves along its axis from right to left so that the point of contact M of wheels I and II moves according to the law $s = 16 - 2t$ cm, where t is measured in seconds and the distance s is reckoned from the axis of wheel II. For the instant $t = 5$ s find the angular velocity and the angular acceleration of driven wheel II, the radius r_1 of drive wheel I being equal to 8 cm.

Answer. $\omega(5) = 50.2$ rad/s, $\varepsilon(5) = 16.7$ rad/s².

Chapter 9 General Case of Motion of a Free Rigid Body. Motion of a Rigid Body about a Fixed Point

§ 1. General Case of Motion of a Free Rigid Body

1.1. Instantaneous Motion. The kinematic state of any material body at a given instant is specified by the configuration in space of all its particles and by their velocities at that instant. A motion of a body is thought of as a continuous passage from one kinematic state to another. Besides the specification of the location of the particles of the moving body, there arises an independent problem on the distribution of the velocities of the particles of the body at the given instant.

In the foregoing chapter we considered two simplest forms of motion of a rigid body: a translatory motion and a rotational motion. Now we proceed to establishing the law of distribution of the velocities of the particles of the body at a given instant (of the instantaneous velocities) for the general case of motion of a free rigid body.

In what follows by *instantaneous motion of a body* we shall only mean the *distribution of the velocities of the particles of the moving body at the given instant*. For example, an *instantaneous translatory motion* of a body is understood as a distribution of velocities in which the velocity vectors of all the particles of the moving body are geometrically equal at the given instant. As to the trajectories of the particles of the body and their accelerations, they can be quite arbitrary.

Now let us suppose that the distribution of the velocities of the particles of a rigid body is given by formula (8.17), that is the velocity v_M of any particle M of the body at the given instant t is expressed by

$$v_M = [\omega, OM]$$

where O is a particle of the body and ω is a vector set off from that point. In such a case we shall speak of an *instantaneous rotational motion of the body* about an instantaneous axis of rotation (which passes through the point O and coincides with the line of action of the vector ω) with instantaneous angular velocity specified by the vector ω. As to the trajectories of the particles of the body and their accelerations, they can again be quite arbitrary.

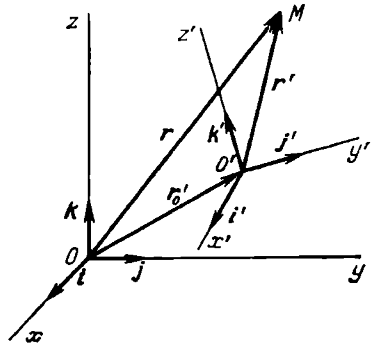

Fig. 9.1

1.2. Euler's Theorem. *An arbitrary instantaneous motion of a rigid body can be represented as the result of composition of an instantaneous translatory motion whose velocity coincides with that of an arbitrarily chosen particle of the body and an instantaneous rotational motion about an instantaneous axis of rotation passing through that point.*

Proof. Let us take a fixed coordinate system $Oxyz$ in space. The motion of the rigid body can be specified by the motion of its three points not lying in one straight line. To this end, let us take a coordinate system $O'x'y'z'$ thought of as being rigidly connected with the rigid body. The position of the coordinate system $O'x'y'z'$ determines that of the body itself relative to the fixed coordinate system $Oxyz$ (Fig. 9.1). By $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and $\boldsymbol{i}'$, $\boldsymbol{j}'$, $\boldsymbol{k}'$ we shall denote the unit vectors along the axes Ox, Oy, Oz and $O'x'$, $O'y'$, $O'z'$ respectively. From Fig. 9.1 we see that for the point M there holds the equality

$$\boldsymbol{r} = \boldsymbol{r}_{O'} + \boldsymbol{r}' \tag{9.1}$$

where $\boldsymbol{r} = \boldsymbol{OM}$, $\boldsymbol{r}_{O'} = \boldsymbol{OO'}$ and $\boldsymbol{r}' = \boldsymbol{O'M}$. Let the coordinates of the points M and O' relative to the fixed coordinate system $Oxyz$ be (x, y, z) and $(x_{O'}, y_{O'}, z_{O'})$ respectively. Then the expressions for the vectors $\boldsymbol{r}$ and $\boldsymbol{r}_{O'}$ can be written in full thus:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \quad \boldsymbol{r}_{O'} = x_{O'}\boldsymbol{i} + y_{O'}\boldsymbol{j} + z_{O'}\boldsymbol{k}$$

We shall denote the coordinates of the point M in the coordinate system $O'x'y'z'$ as (x', y', z'); then

$$\boldsymbol{r}' = x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}' \tag{9.2}$$

On the basis of identity (9.1), for the velocity vector of the point M we can write

$$\boldsymbol{v}_M = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}_{O'}}{dt} + \frac{d\boldsymbol{r}'}{dt} = \boldsymbol{v}_{O'} + \frac{d\boldsymbol{r}'}{dt} \tag{9.3}$$

The second summand on the right-hand side of formula (9.3) is found by differentiating identity (9.2):

$$\frac{dr'}{dt}=x'\frac{di'}{dt}+y'\frac{dj'}{dt}+z'\frac{dk'}{dt} \tag{9.4}$$

It should be stressed that since the quantities x', y' and z' are the coordinates of a particle of the body relative to the coordinate system $O'x'y'z'$ rigidly connected with the body they do not change with time. Therefore in each term on the right-hand side of identity (9.2) only the second factor is differentiated.

Let us elucidate the mechanical meaning of formula (9.4). Multiplying scalarly both the sides of that formula by the unit vectors $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ we obtain

$$\begin{aligned}
\left(\frac{dr'}{dt},\ i'\right)&=x'\left(\frac{di'}{dt},\ i'\right)+y'\left(\frac{dj'}{dt},\ i'\right)+z'\left(\frac{dk'}{dt},\ i'\right)\\
\left(\frac{dr'}{dt},\ j'\right)&=x'\left(\frac{di'}{dt},\ j'\right)+y'\left(\frac{dj'}{dt},\ j'\right)+z'\left(\frac{dk'}{dt},\ j'\right)\\
\left(\frac{dr'}{dt},\ k'\right)&=x'\left(\frac{di'}{dt},\ k'\right)+y'\left(\frac{dj'}{dt},\ k'\right)+z'\left(\frac{dk'}{dt},\ k'\right)
\end{aligned} \tag{9.5}$$

The scalar squares of the unit vectors are equal to unity, and the scalar products of two different unit vectors $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ are equal to zero since these vectors are mutually perpendicular. Thus,

$$i'^2=j'^2=k'^2=1, \quad (j',\ k')=(k',\ i')=(i',\ j')=0$$

Differentiating the last identities according to formulas (2) in Sec. 5 of Introduction to Kinematics and cancelling the first three identities by two we obtain the relations

$$\begin{gathered}
\left(i',\frac{di'}{dt}\right)=\left(j',\frac{dj'}{dt}\right)=\left(k',\frac{dk'}{dt}\right)=0\\
\left(\frac{dj'}{dt},\ k'\right)+\left(j',\frac{dk'}{dt}\right)=\left(\frac{dk'}{dt},\ i'\right)+\left(k',\frac{di'}{dt}\right)\\
=\left(\frac{di'}{dt},\ j'\right)+\left(i',\frac{dj'}{dt}\right)=0
\end{gathered} \tag{9.6}$$

Let us introduce the notation

$$\left(\frac{dj'}{dt},\ k'\right)=p,\ \left(\frac{dk'}{dt},\ i'\right)=q,\ \left(\frac{di'}{dt},\ j'\right)=\tilde{r}$$

and rewrite the last three equalities (9.6) thus:

$$\left(j',\frac{dk'}{dt}\right)=-\left(\frac{dj'}{dt},\ k'\right)=-p,\quad \left(k',\frac{di'}{dt}\right)=-q,\quad \left(i',\frac{dj'}{dt}\right)=-\tilde{r}$$

Then, taking into account (9.6), we can write equalities (9.5) in the form

$$\left(\frac{dr'}{dt}, i'\right) = qz' - \tilde{r}y', \quad \left(\frac{dr'}{dt}, j'\right) = \tilde{r}x' - pz', \\ \left(\frac{dr'}{dt}, k'\right) = py' - qx' \tag{9.7}$$

Let us construct the following sliding vector ω whose line of action passes through the point O':

$$\omega = pi' + qj' + \tilde{r}k' \tag{9.8}$$

Equalities (9.7) now reduce to one vector equality (see formula (1.16) in Sec. 1.2 of Chap. 1):

$$\frac{dr'}{dt} = [\omega, r'] = \begin{vmatrix} i' & j' & k' \\ p & q & \tilde{r} \\ x' & y' & z' \end{vmatrix} \\ = (qz' - \tilde{r}y')\, i' + (\tilde{r}x' - pz')\, j' + (py' - qx')\, k' \tag{9.9}$$

Indeed, the scalar multiplication of the last equality by the unit vectors i', j' and k' results in the corresponding equalities (9.7). Now we substitute (9.9) into (9.3), which results in the *formula for the velocity vector v_M of an arbitrary particle M of the rigid body*:

$$v_M = v_{O'} + [\omega, r'] \tag{9.10}$$

It should be stressed that the vectors $v_{O'}$ and ω do not depend on the choice of the point M; as to the vector r', it is the radius vector of the chosen point M relative to the point O':

$$r' = O'M = x'i' + y'j' + z'k'$$

Taking into account formula (8.17) we note that the second summand in formula (9.10) is the velocity which the particle M would have if the body rotated about a fixed axis passing through the point O' with angular velocity vector equal to ω. Thus, the motion of the rigid body can be regarded as the result of composition of two motions: a motion in which all the particles of the body have one and the same velocity $v_{O'}$ at the given instant (this corresponds to an instantaneous translatory motion) and an instantaneous rotation with angular velocity ω about an axis passing through the point O'.

In the general case, at the following instant of time all the vectors on the right-hand side of formula (9.10) change. As is seen from the proof we have presented, the rigid body has no other instantaneous motions. Euler's theorem is proved.

1.3. Independence of the Angular Velocity Vector of a Body of the Choice of the Centre. We shall prove that the vector ω representing the instantaneous angular velocity of the body is independent of the

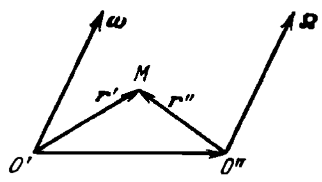

Fig. 9.2

choice of the centre O'. Besides the point O', let us take some other point O'' of the body. Let the angular velocity of the rotation of the body about the instantaneous axis passing through the point O'' be equal to $\mathbf{\Omega}$ (Fig. 9.2). If the point O'' is taken as a centre then, by analogy with (9.10), we have

$$\mathbf{v}_M = \mathbf{v}_{O''} + [\mathbf{\Omega}, \mathbf{r}''] \tag{9.11}$$

The comparison of formulas (9.10) and (9.11) shows that

$$\mathbf{v}_{O'} + [\boldsymbol{\omega}, \mathbf{r}'] = \mathbf{v}_{O''} + [\mathbf{\Omega}, \mathbf{r}'']$$

The velocity of the point O'' can be found with the aid of formula (9.10) in which we put $\mathbf{r}' = \mathbf{O'O''}$:

$$\mathbf{v}_{O''} = \mathbf{v}_{O'} + [\boldsymbol{\omega}, \mathbf{O'O''}]$$

The substitution of $\mathbf{v}_{O''}$ into the foregoing formula yields

$$[\boldsymbol{\omega}, \mathbf{r}'] = [\boldsymbol{\omega}, \mathbf{O'O''}] + [\mathbf{\Omega}, \mathbf{r}'']$$

whence we obtain

$$[\boldsymbol{\omega}, \mathbf{r}' - \mathbf{O'O''}] = [\mathbf{\Omega}, \mathbf{r}'']$$

Now, from Fig. 9.2 we see that $\mathbf{r}' - \mathbf{O'O''} = \mathbf{r}''$, and consequently $[\boldsymbol{\omega}, \mathbf{r}''] = [\mathbf{\Omega}, \mathbf{r}'']$, whence we finally obtain

$$[\boldsymbol{\omega} - \mathbf{\Omega}, \mathbf{r}''] = 0$$

Since the last equality must hold for any point M, that is for any value of the vector $\mathbf{r}''$, the vector $\boldsymbol{\omega} - \mathbf{\Omega}$ must be identically equal to zero and hence

$$\boldsymbol{\omega} = \mathbf{\Omega}$$

which is what we had to prove.

The formula for the acceleration of the particles of a free rigid body in the general case of motion will be given at the end of Sec. 2.3.

§ 2. Motion of a Rigid Body about a Fixed Point

2.1. Instantaneous Axis of Rotation and Instantaneous Angular Velocity of a Body. Let us consider the motion of a rigid body with one fixed point about which the body can rotate in an arbitrary way. Examples of this kind are the motion of a rigid body whose only constraint is a ball-and-socket joint and the motion of a top whose point of support remains fixed. Such a motion is called *spherical*.

Let us place the origin of a *fixed coordinate system Oxyz* at the fixed point O of the rigid body. We shall also consider a *moving coordinate system Ox′y′z′* whose origin is at the same point O and whose axes are rigidly connected with the body (Fig. 9.3).

For the motion under consideration Euler's theorem can be stated thus:

Any instantaneous motion of a rigid body having one fixed point is an instantaneous rotational motion about an instantaneous axis of rotation passing through that point.

From formula (9.10) it follows that the velocity vector of an arbitrary particle M of a rigid body moving about a fixed point O is expressed by the formula

$$v_M = [\omega, r] \tag{9.12}$$

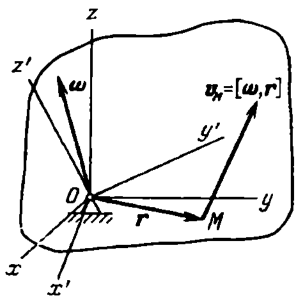

Fig. 9.3

where $r = OM$ and ω is the vector representing the instantaneous angular velocity of the body (see Fig. 9.3).

In contrast to the case of rotation of a rigid body about a fixed axis (see § 2 of Chap. 8), not only the modulus but also the line of action of the vector ω may vary with time when a body moves about a fixed point; however, the line of action of the vector ω passes through the fixed point O during the whole time of motion.

It should be stressed that the modulus of the second vector factor in formula (9.12), that is of $r = OM$, is constant because the distance between any two points of a rigid body is invariable. By definition, we have $v = dr/dt$ (see (7.9)) and therefore formula (9.12) can be written in the form

$$\frac{dr}{dt} = [\omega, r] \tag{9.13}$$

2.2. Velocities of the Particles of a Rigid Body Moving about a Fixed Point. Euler's Formulas. Let us denote by ω_x, ω_y and ω_z the projections of the instantaneous angular velocity vector ω on the fixed coordinate axes Ox, Oy and Oz, and by x, y and z the coordinates of the particle M relative to the same coordinate system; then

$$\omega = \omega_x i + \omega_y j + \omega_z k, \quad r = xi + yj + zk$$

where i, j and k are unit vectors along the axes Ox, Oy and Oz respectively. Formula (9.12) can be written in full in the form

$$v_M = \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = (\omega_y z - \omega_z y)\, i + (\omega_z x - \omega_x z)\, j + (\omega_x y - \omega_y x)\, k$$

It follows that the *projections of the velocity vector of the particle M on the fixed coordinate axes* are expressed by the formulas

$$v_x = \omega_y z - \omega_z y, \quad v_y = \omega_z x - \omega_x z, \quad v_z = \omega_x y - \omega_y x \tag{9.14}$$

On the right-hand sides of formula (9.14) not only the quantities ω_x, ω_y and ω_z change with time but also the coordinates x, y and z of the particle M relative to the fixed coordinate system $Oxyz$.

In order to deal with constant coordinates we can project vector equality (9.12) on the coordinate axes $Ox'y'z'$ rigidly connected with the body (we call them moving axes; see Fig. 9.3). Let p, q and $\tilde{r}$ denote the projections of the instantaneous angular velocity on the moving axes Ox', Oy' and Oz', and let x', y' and z' be the coordinates of the particle M relative to the same coordinate system. Then formula (9.12) can be written in full thus:

$$v_M = \begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ p & q & \tilde{r} \\ x' & y' & z' \end{vmatrix} = (qz' - \tilde{r}y')\,\mathbf{i}' + (\tilde{r}x' - pz')\,\mathbf{j}' + (py' - qx')\,\mathbf{k}'$$

where $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$ are unit vectors along the axes Ox', Oy' and Oz' respectively.

It follows that the projections $v_{x'}$, $v_{y'}$ and $v_{z'}$ *of the velocity vector of the particle M on the moving coordinate axes* are expressed by the formulas

$$v_{x'} = qz' - \tilde{r}y', \quad v_{y'} = \tilde{r}x' - pz', \quad v_{z'} = py' - qx' \qquad (9.15)$$

On the right-hand sides of formulas (9.15) only the quantities p, q and $\tilde{r}$ may vary with time while the coordinates x', y' and z' of the particle M relative to the moving coordinate system $Ox'y'z'$ connected with the body remain invariable. Formulas (9.14) and (9.15) were established by L. Euler.

The modulus of the velocity of an arbitrary particle of the rigid body can be found by means of the known projections using the formulas

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}, \quad v = \sqrt{v_{x'}^2 + v_{y'}^2 + v_{z'}^2}$$

On the other hand, since the velocity of any particle of the body can be regarded as the velocity of rotational motion about the instantaneous axis the modulus of the velocity is determined by formula (8.11):

$$v_M = R\,|\omega| \qquad (9.16)$$

Here $R = MO_1$ is the length of the perpendicular dropped from the point M to the instantaneous axis of rotation, and $|\omega|$ is the modulus of the instantaneous angular velocity vector of the body.

2.3. Accelerations of the Particles of a Rigid Body Moving about a Fixed Point. Differentiating identity (9.12) with respect to time and using formula (3) of Sec. 5 in Introduction to Kinematics we

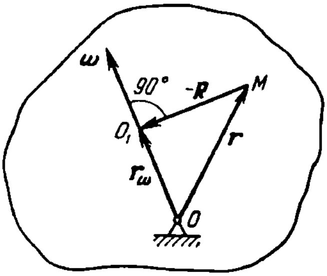

Fig. 9.4

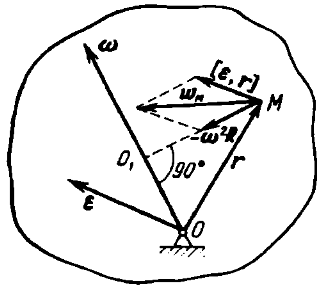

Fig. 9.5

obtain the acceleration of an arbitrary particle of the body:

$$w = \frac{dv}{dt} = \left[\frac{d\omega}{dt},\ r\right] + \left[\omega,\ \frac{dr}{dt}\right]$$

However, here we have

$$\frac{dr}{dt} = [\omega, r]$$

(see formula (9.13)) and therefore

$$w = \left[\frac{d\omega}{dt},\ r\right] + [\omega, [\omega, r]]$$

Let us write in full the expression for the second summand on the right-hand side of the last equality using formula (1.20) of Chap. 1 for the triple vector product:

$$w = \left[\frac{d\omega}{dt},\ r\right] + (\omega, r)\,\omega - \omega^2 r$$

Let us introduce the unit vector ω^0 along the instantaneous axis of rotation (the direction of ω^0 coinciding with that of ω) and write the instantaneous angular velocity vector ω in the form

$$\omega = |\omega|\,\omega^0$$

Then we obtain

$$w = \left[\frac{d\omega}{dt},\ r\right] + \omega^2 \{(\omega^0, r)\,\omega^0 - r\}$$

Now we note that (ω^0, r) is the projection of the radius vector r of the point M on the instantaneous axis of rotation, and $(\omega^0, r)\omega^0$ is equal to the component r_ω of the radius vector r along ω (Fig. 9.4). The difference of the vectors in braces is equal to the vector $-R$:

$$r_\omega - r = -R$$

The vector $-\boldsymbol{R} = \boldsymbol{MO}_1$ is shown in Fig. 9.4. Thus, for the acceleration of the particle we obtain

$$\boldsymbol{w} = \left[\frac{d\boldsymbol{\omega}}{dt}, \boldsymbol{r}\right] - \omega^2 \boldsymbol{R} \tag{9.17}$$

Let us discuss the mechanical meaning of both the components on the right-hand side of formula (9.17). The vector $d\boldsymbol{\omega}/dt$ is called the *instantaneous angular acceleration of the body*; we denote it by $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = \frac{d\boldsymbol{\omega}}{dt}$$

If the instantaneous angular velocity vector $\boldsymbol{\omega}$ is set off from a fixed point O then its terminus, the point K, describes a curve which is the hodograph of the angular velocity. The derivative of the vector $\boldsymbol{\omega}$ with respect to time goes along the tangent to the hodograph in the direction of motion of the point K. Let us set off this derivative, that is the angular acceleration vector $\boldsymbol{\varepsilon}$, from the point O and then construct the vector $[\boldsymbol{\varepsilon}, \boldsymbol{r}]$ at the point M (Fig. 9.5). The vector $\left[\frac{d\boldsymbol{\omega}}{dt}, \boldsymbol{r}\right] = [\boldsymbol{\varepsilon}, \boldsymbol{r}]$ is the rotational component of the acceleration $\boldsymbol{w}$, and the vector $-\omega^2\boldsymbol{R}$ an analogue of the centripetal component $R\omega^2\boldsymbol{n}$ in formula (8.14).

In Fig. 9.5 the acceleration vector of a particle M of a body moving about a fixed point O is constructed.

2.4. Accelerations of the Particles of a Free Rigid Body. In this section we present some supplementary material to § 1 concerning the acceleration of a particle of a free rigid body for the general case of motion. We obtain this acceleration by differentiating with respect to time formula (9.10) expressing the velocity vector $\boldsymbol{v}_M$ of an arbitrary particle M of the rigid body:

$$\boldsymbol{w}_M = \frac{d\boldsymbol{v}_M}{dt} = \frac{d\boldsymbol{v}_{O'}}{dt} + \left[\frac{d\boldsymbol{\omega}}{dt}, \boldsymbol{r}'\right] + \left[\boldsymbol{\omega}, \frac{d\boldsymbol{r}'}{dt}\right]$$

The first vector on the right-hand side of this formula is the acceleration $\boldsymbol{w}_{O'}$ of the centre O'. Transforming the other two vectors with the aid of the same method as the one used in the derivation of formula (9.17) we obtain

$$\boldsymbol{w}_M = \boldsymbol{w}_{O'} + [\boldsymbol{\varepsilon}, \boldsymbol{r}'] - \omega^2\boldsymbol{R}' \tag{9.18}$$

Here $\boldsymbol{\varepsilon}$ is the instantaneous angular acceleration vector of the body:

$$\boldsymbol{\varepsilon} = \frac{d\boldsymbol{\omega}}{dt}$$

and $\boldsymbol{r}' = \boldsymbol{O'M}$ (see Fig. 9.1); the vector $\boldsymbol{R}'$ corresponds to the perpendicular dropped from the point M to the instantaneous axis of

rotation, that is to the line of action of the instantaneous angular velocity vector constructed at the centre O'.

The corresponding examples are given in Sec. 1.2 of Chap. 12 (Examples 12.1 and 12.2).

Problems

PROBLEM 9.1. Prove that for any motion of a line segment AB of constant length the projections of the velocities of the ends of the line segment on the straight line AB are equal.

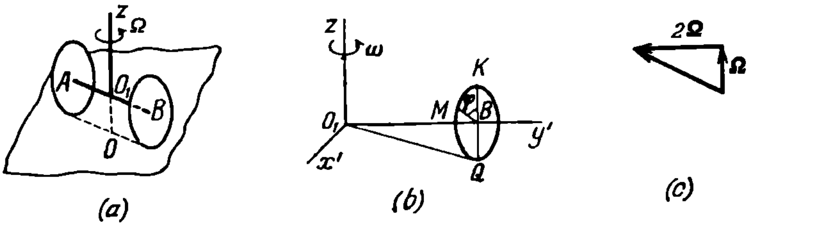

Fig. 9.6

Hint. Take the point A as the centre, make use of formula (9.10) and show that the projection of the second summand on $\boldsymbol{r}' = \boldsymbol{AB}$ is equal to zero.

PROBLEM 9.2. Two wheels (thin discs) of radius R are put on a horizontal axis AB ($AB = 4R$); through the midpoint O_1 of the axis AB a fixed vertical axis Oz passes (Fig. 9.6a). The axis AB rotates about Oz with constant angular velocity Ω. The discs touch the horizontal plane and roll without sliding on it. Determine the velocity $\boldsymbol{v}$ of the point M lying on the circumference of one of the discs.

Hint. Let us introduce the fixed moving coordinate system $O_1x'y'z$ shown in Fig. 9.6b. As is seen from Fig. 9.6c, the angular velocity $\boldsymbol{\omega}$ is expressed by $\boldsymbol{\omega} = -2\Omega \boldsymbol{j}' + \Omega \boldsymbol{k}'$, that is $p = 0$, $q = -2\Omega$ and $\tilde{r} = \Omega$ (see Sec. 2.2). Let the position of the point M under consideration be specified by an angle φ (see Fig. 9.6b); then its coordinates are $x' = R \sin \varphi$, $y' = 2R$, $z = R \cos \varphi$. The projections of the velocity $\boldsymbol{v}$ are determined by formulas (9.15).

Answer. $v = R\omega \sqrt{5 \sin^2 \varphi + 4 (1 + \cos \varphi)^2}$.

Chapter 10 Plane Motion of a Rigid Body

By a *plane (-parallel) motion* is meant a motion of a rigid body in which *all its particles move only in the planes parallel to a given fixed plane* Π. Thus, in such a motion the distance from every particle of the body to the given fixed plane remains constant (Fig. 10.1).

Let us take the plane Π or some other plane Π' parallel to the former as the coordinate plane Oxy. Let a coordinate system $O'x'y'$ be rigidly connected with the moving body, the initial position (at the beginning of the motion) of the plane $O'x'y'$ coinciding with that of the plane Oxy (Fig. 10.2). The moving plane $O'x'y'$ remains coin-

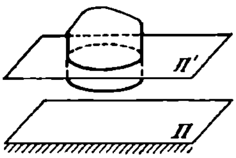

Fig. 10.1

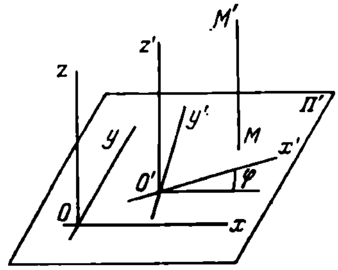

Fig. 10.2

cident with the fixed plane during the whole time of plane motion Indeed, by definition, the displacements of all the particles of the moving plane $O'x'y'$ are parallel to the fixed plane Oxy, and since the initial positions of the particles are in the latter plane they lie in that plane during the whole time of motion. Therefore the moving axis $O'z'$ and any straight line MM' rigidly connected with the body and perpendicular to the plane Oxy are parallel to the fixed axis Oz during the whole time of plane motion (see Fig. 10.2).

§ 1. Velocity of Plane Motion

1.1. Equations of Plane Motion of a Figure. From what was said it follows that a *plane motion of a rigid body* is completely determined by the *motion of a plane figure* (or, which is the same, of a moving plane) *in the plane where it lies*. Therefore, for the sake of brevity we can simply speak of a plane motion of the figure. To specify the position of a plane figure in its plane, that is to specify the position of a movable coordinate system $O'x'y'$ rigidly connected with the figure relative to the fixed coordinate system Oxy, it is sufficient to specify the coordinates of the point O' $(x_{O'}, y_{O'})$ (called the *centre*) and the angle φ between the moving axis $O'x'$ and the fixed axis Ox. This means that the *equations of plane motion* can be written in the form

$$x_{O'} = f(t), \quad y_{O'} = g(t), \quad \varphi = \varphi(t)$$

The derivative of the angle of rotation φ with respect to time is called the *angular velocity of the plane motion of the figure*; we denote it by ω:

$$\omega = \frac{d\varphi}{dt}$$

The angular velocity of the figure is independent of the choice of the centre and the moving coordinate axes (this was proved in Sec. 1.3 of Chap. 9 for the general case of motion of a body). Here we shall

demonstrate this property for the special case of plane motion of a figure. In Fig. 10.3 a plane figure is shown as a rectangle, the centre O' is taken at the centre of the rectangle, and the moving axes $O'x'$ and $O'y'$ rigidly connected with the figure are drawn along the symmetry axes of the rectangle; the angle of rotation is $\varphi = \angle x'O'x$. Now let us choose another centre O'' at one of the vertices of the rectangle, and let the axis $O''x''$ go along the diagonal and the axis $O''y''$ be perpendicular to the diagonal. The new angle of rotation is $\Phi = \angle x''O'x$ and we have

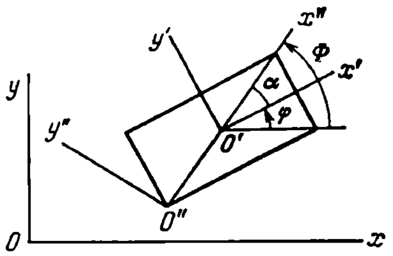

Fig. 10.3

$$\Phi = \varphi + \alpha$$

Since the angle α is constant the differentiation of the last identity results in the equality

$$\frac{d\Phi}{dt} = \frac{d\varphi}{dt} = \omega$$

which holds during the whole time of motion; this is what we intended to prove.

Let us consider two important *special cases*. If $\varphi = \text{const}$ then only the coordinates of the centre O' change with time, the moving axes $O'x'y'$ undergoing displacement in which they remain parallel to the fixed axes Oxy. This means that the plane figure and, consequently, the rigid body are in *plane translatory motion*.

If $x_{O'} = \text{const}$ and $y_{O'} = \text{const}$ then the centre does not move and only the angle φ changes with time. In this case the plane figure rotates in its plane about the axis $O'z'$ which is fixed in the present case. Hence, a *rotation of a rigid body about a fixed axis* is a special case of plane motion. This of course does not apply to the general case of translatory motion because it may turn out to be plane only in a special case.

1.2. Velocities of the Particles of a Figure in Plane Motion. Let us state Euler's theorem (see Sec. 1.2 of Chap. 9) for the special case of plane motion:

Every instantaneous motion of a plane figure in its plane (an instantaneous plane motion) reduces to the composition of an instantaneous translatory motion in the plane of the figure and an instantaneous rotational motion about the axis passing through the centre O' and perpendicular to the plane of the figure.

Proof. The assertion we have stated can be proved by repeating the proof of Euler's theorem for the special case under consideration in which the axes Oz and $O'z'$ are parallel during the whole time of

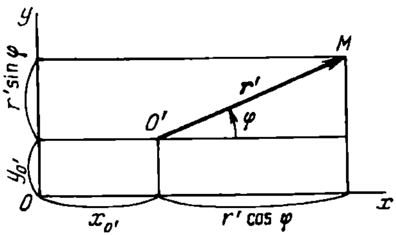

Fig. 10.4

motion. However, it is useful to prove it directly, that is to derive the formula for the velocity vector of an arbitrary point M of the figure. Let us choose a centre O' in the figure and let the vector $O'M$ be denoted by $\boldsymbol{r}'$. The angle between the vector $\boldsymbol{r}'$ and the positive direction of the fixed axis Ox, reckoned counterclockwise, will be taken as the angle of rotation φ of the figure. From Fig. 10.4 we see that the coordinates of the point M of the figure are

$$x_M = x_{O'} + r' \cos \varphi, \quad y_M = y_{O'} + r' \sin \varphi$$

The projections of the velocity of the point M of the figure on the axes Ox and Oy are found by differentiating the last equalities with respect to time:

$$v_x^M \equiv \dot{x}_M = \dot{x}_{O'} - r'\dot{\varphi} \sin \varphi, \quad v_y^M \equiv \dot{y}_M = \dot{y}_{O'} + r'\dot{\varphi} \cos \varphi \qquad (10.1)$$

Since $\dot{\varphi} = \omega$ is the angular velocity of the plane motion of the figure, and $r' \cos \varphi$ and $r' \sin \varphi$ are equal to the projections r'_x and r'_y of the vector $\boldsymbol{r}'$ respectively, formulas (10.1) can be rewritten in the form

$$v_x^M = v_x^{O'} - \omega r'_y, \quad v_y^M = v_y^{O'} + \omega r'_x$$

Now, using the vector notation we can write

$$\boldsymbol{v}_M = (v_x^{O'} - \omega r'_y)\, \boldsymbol{i} + (v_y^{O'} + \omega r'_x)\, \boldsymbol{j} = \boldsymbol{v}_{O'} + \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \omega \\ r'_x & r'_y & 0 \end{vmatrix}$$

that is

$$\boldsymbol{v}_M = \boldsymbol{v}_{O'} + [\boldsymbol{\omega}, \boldsymbol{r}'] \quad (\boldsymbol{r}' = \boldsymbol{O'M}) \qquad (10.2)$$

which is what we had to prove.

The vector $\boldsymbol{v}_M$ is equal to the vector sum of the velocity vector of an arbitrarily chosen centre O' in the plane figure and the velocity vector of the point M in the rotational motion of the figure about the axis $O'z'$ (perpendicular to the plane of the figure). Knowing the velocity $\boldsymbol{v}_{O'}$ of the centre O' and the angular velocity ω of the figure we can construct the velocity vector for any point M of the figure as shown in Fig. 10.5.

Since the vectors $\boldsymbol{\omega}$ and $\boldsymbol{r}'$ form a right angle the modulus of the second summand in formula (10.2) is equal to $r' |\omega|$.

We once again stress that, as was proved in Sec. 1.1, the angular velocity ω of the figure does not depend on the choice of the centre.

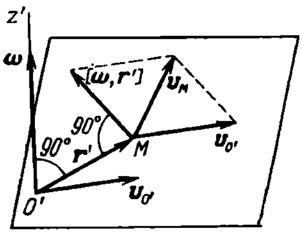

Fig. 10.5

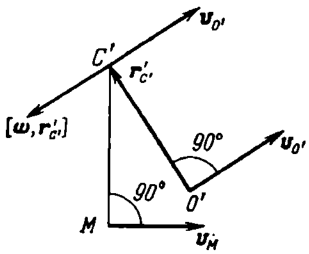

Fig. 10.6

1.3. Instantaneous Centre of Zero Velocity. There arises the question whether a figure in a plane motion possesses a point whose velocity at a given instant is equal to zero. Let us denote by $r'_{C'}$ the radius vector of such a point C' drawn from the centre O'; then equating to zero the right-hand side of formula (10.2) we obtain for $r'_{C'}$ the vector equation

$$[\omega,\ r'_{C'}] = -v_{O'}$$

First let us find the modulus of the vector $r'_{C'}$; to this end we equate the moduli of both the members of the last vector equation:

$$r'_{C'}\ |\ \omega\ | = v_{O'}$$

From the equality we have obtained it follows that

$$r'_{C'} = \frac{1}{|\omega|} v_{O'} \tag{10.3}$$

provided that $\omega \neq 0$.

It now remains to determine the direction of the vector $r'_{C'}$; to this end we shall consider Fig. 10.6. Let us turn the vector $v_{O'}$ through a right angle about the point O' in the direction of rotation; further, along this direction we set off from the point O' a vector whose modulus is determined by formula (10.3). The vector thus constructed is the sought-for vector $r'_{C'}$. The velocity vector of the point C' is found from formula (10.2) and turns out to be equal to zero:

$$v_{C'} = v_{O'} + [\omega,\ r'_{C'}] = 0$$

Indeed, from Fig. 10.6 it is seen that the vectors $v_{O'}$ and $[\omega,\ r'_{C'}]$ have opposite directions while, as formula (10.3) shows, their moduli are equal. Hence, the velocity of the point C' of the figure *at the given instant* is equal to zero. This point of the plane figure (of the moving plane $O'x'y'$) is called the *instantaneous centre of zero velocity* (or, simply, the *instantaneous centre*).

If at the given instant we take the centre not at the point O' but at the point C', formula (10.2) takes the form

$$v_M = [\omega, C'M] \tag{10.4}$$

(see Fig. 10.6) because the first summand, that is $v_{C'}$, is equal to zero. It should be stressed that, since the point C' moves, the position of C' in formula (10.4) is different for different instants. Formula (10.4) shows that the velocity of an arbitrary point M of a figure in a plane motion at a given instant t coincides with the velocity of a rotational motion in the fixed plane about the point C which occupies the position of the point C' at time t (or, more precisely, about the axis Cz') with angular velocity ω. That is why the point C of the fixed plane Oxy is called the *instantaneous centre* (*of zero velocity*).

If the position of the instantaneous centre of zero velocity C' is found and the angular velocity ω of the figure at that instant is known the velocity vector of an arbitrary point can be determined as shown in Fig. 10.7. In the figure the velocity vectors of two points A and B are constructed, and we have

$$v_A = C'A\,|\omega| \text{ and } v_A \perp C'A; \quad v_B = C'B\,|\omega| \text{ and } v_B \perp C'B \tag{10.5}$$

This construction may also be regarded as an example of determining the instantaneous centre for the case when the lines of action of the vectors representing the velocities of two points of the figure are known. As is seen, *the instantaneous centre C' is at the point of intersection of the perpendiculars erected to the lines of action of the velocity vectors at these points.* The direction of one of the velocity vectors determines that of instantaneous rotation of the figure.

The construction of the point C' cannot be performed when the indicated perpendiculars do not intersect; however, in this case the directions of the velocity vectors are parallel. Below we shall show that in the case we have indicated the instantaneous distribution of

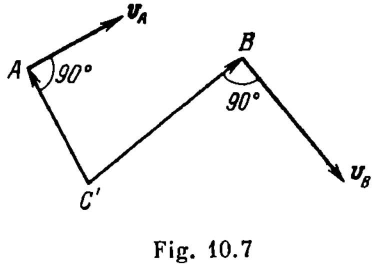

Fig. 10.7

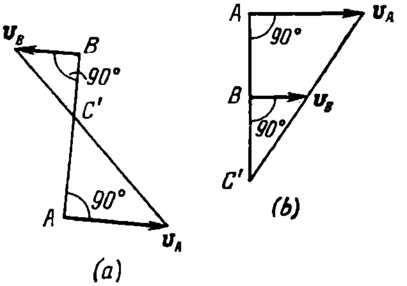

Fig. 10.8

the velocities of the points of the figure is the same as in a translatory motion. At the instant when the velocities become parallel we have $\omega = 0$ and the instantaneous centre of zero velocity recedes to infinity; therefore this situation can be regarded as a limiting case described by formula (10.3).

From equalities (10.5) it follows that $\frac{v_A}{C'A} = |\omega|$ and $\frac{v_B}{C'B} = |\omega|$, whence

$$\frac{v_A}{v_B} = \frac{C'A}{C'B} \tag{10.6}$$

Thus, *the moduli of the velocities of the points of the figure in a plane motion are proportional to the distances from these points to the instantaneous centre of zero velocity.*

The position of the point C' in the above geometrical construction may be indeterminate when the perpendiculars we constructed coincide (Fig. 10.8). In this case the instantaneous centre C' can be found as the point of intersection of the straight line passing through the tips of the vectors $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ with the perpendicular AB (or with its extension) to the velocities. Indeed, for both the cases shown in Fig. 10.8*a* and *b* the similarity of the triangles implies the proportion

$$\frac{v_A}{v_B} = \frac{C'A}{C'B}$$

In the cases shown in Fig. 10.8 the moduli and the directions of the velocity vectors of two points of the figure in a plane motion can be set quite arbitrarily; in all the other cases this is impossible because these vectors are connected by the relationship established in the following theorem:

The projections of the velocity vectors of any two points of the figure on the straight line connecting these points are equal.

Proof. See Fig. 10.9 where the vectors $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are constructed. Let us take the point A as the centre and write formula (10.2) in the form

$$\boldsymbol{v}_B = \boldsymbol{v}_A + [\boldsymbol{\omega}, \boldsymbol{AB}]$$

In order to find the indicated projections let us multiply scalarly both the sides of the last vector equality by the unit vector $\boldsymbol{e}$ along the straight line AB:

$$(\boldsymbol{v}_B, \boldsymbol{e}) = (\boldsymbol{v}_A, \boldsymbol{e}) + ([\boldsymbol{\omega}, \boldsymbol{AB}], \boldsymbol{e})$$

Further, $[\boldsymbol{\omega}, \boldsymbol{AB}]$ is a vector perpendicular to AB, and its scalar product by the vector $\boldsymbol{e}$ (which is directed along AB) is equal to zero.

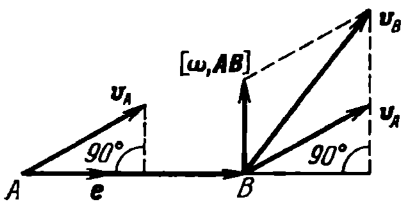

Fig. 10.9

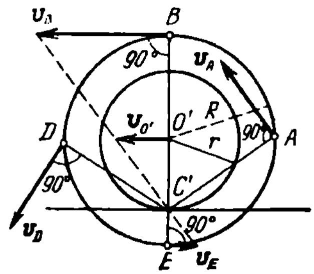

Fig. 10.10

Therefore the last scalar equality yields

$$\text{proj}_{AB}\, v_B = \text{proj}_{AB}\, v_A \tag{10.7}$$

which completes the proof of the theorem*.

By the way, the validity of the theorem can be seen directly from Fig. 10.9: the projections of the vectors v_B and v_A (transferred to the point B) on the straight line AB are equal because the tips of these vectors lie on one perpendicular to AB.

Now let us suppose that $v_A \parallel v_B$. If $v_A \perp AB$ (and consequently $v_B \perp AB$) in this case, then v_A and v_B may assume any values (Fig. 10.8*a* and *b*). This does not contradict the theorem because the projections of the vectors v_A and v_B on the line segment AB are equal to zero.

In the other case when the parallel vectors v_A and v_B are not perpendicular to the line segment AB the value of v_A must be equal to that of v_B, which follows from the theorem proved above. To show this it is sufficient to construct the vectors v_A and v_B and their projections on the line segment AB. Finally, the equality $v_A = v_B$ implies that the instantaneous distribution of the velocities of the points of the plane figure is the same as in a translatory motion.

EXAMPLE 10.1. A wheel of a railway car rolls without sliding along a rail (Fig. 10.10). The radii of the rims of the wheel are r and R, and the velocity of its centre O' is $v_{O'}$. Determine the velocities of the points A, B, D and E of the wheel.

Solution. Since the wheel rolls without sliding, the instantaneous velocity of the point of tangency C' of the wheel and the rail is equal to zero. Hence, this point is the instantaneous centre of zero velocity. Let us take the point C' as the centre; then the modulus of the velocity $v_{O'}$ of the point O' can be written as

$$v_{O'} = r \mid \omega \mid$$

From this equality we find the modulus of the angular velocity ω of rotation of the wheel (we remind the reader that the angular velocity is independent of

* In the proof of the theorem we have not used the fact that $\omega \perp AB$. Therefore the theorem also remains valid for the general case of motion of a rigid body.

the choice of the centre):

$$|\omega| = \frac{1}{r} v_{O'}$$

Now the moduli of the sought-for velocities can be found as the products of $|\omega|$ by the distances from the points of the wheel to the instantaneous centre C':

$$v_A = C'A\,|\omega| = \frac{1}{r}\sqrt{R^2+r^2}\,v_{O'},$$

$$v_B = C'B\,|\omega| = \frac{1}{r}(R+r)\,v_{O'},$$

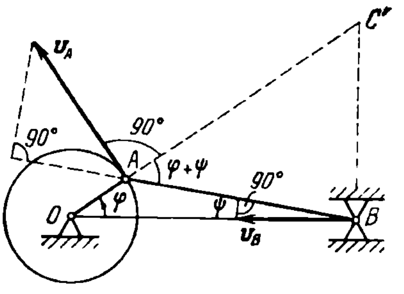

Fig. 10.11

$$v_D = C'D\,|\omega| = \frac{1}{r}\sqrt{R^2+r^2}\,v_{O'}, \qquad v_E = C'E\,|\omega| = \frac{1}{r}(R-r)\,v_{O'}$$

The velocity vectors are perpendicular to $C'A$, $C'B$, $C'D$ and $C'E$ respectively, and the directions of the vectors are determined by the direction of rotation about the instantaneous centre C'.

EXAMPLE 10.2. The crank $OA = r$ of the slider-crank mechanism shown in Fig. 10.11 rotates uniformly about the fixed point O with angular velocity Ω. The connecting rod $AB = l$ is hinged to the crank at the point A; the connecting rod imparts motion to the slide block B, the guides of the slide block being parallel to the axis OB. Find the velocity of the slide block B and the angular velocity ω of the connecting rod AB at the instant when the angle AOB is equal to φ.

Solution. Constructing the configuration of the mechanism for the given angle φ we can determine the angle ψ (it can also be found analytically; see Example 7.6). Therefore we shall consider the angle ψ as known. The velocity vector $\boldsymbol{v}_A$ of the point A of the crank is perpendicular to the radius OA and its modulus is equal to $r\Omega$. The velocity vector of the point B is directed along the straight line BO. The instantaneous centre of zero velocity C' of the crank AB can be found as the point of intersection of the perpendiculars erected at the points A and B to the velocities of these points.

Since the moduli of the velocities of the points are proportional to their distances from the instantaneous centre (see formula (10.6)) we have

$$v_B = \frac{C'B}{C'A} v_A = \frac{C'B}{C'A} r\Omega$$

Another method of solution is based on the application of the theorem on the equality of the projections of the velocity vectors of the ends of a line segment on the direction of that line segment. Let us take the crank AB as that line segment; then

$$v_A \cos\left[\frac{\pi}{2} - (\varphi+\psi)\right] = v_B \cos\psi$$

whence

$$v_B = \frac{\sin(\varphi+\psi)}{\cos\psi} v_A = \frac{\sin(\varphi+\psi)}{\cos\psi} r\Omega$$

Using the law of sines for the triangle ABC' we can readily show that this expression of the modulus of the velocity coincides identically with the one obtained earlier.

The angular velocity ω of the crank is found by formula (10.6):

$$\omega = \frac{v_A}{C'A} = \frac{r}{C'A}\,\Omega$$

§ 2. Acceleration of Plane Motion

2.1. Accelerations of the Particles of a Figure in Plane Motion. To find the projections of the acceleration of a point M of a figure in its plane motion on the fixed coordinate axes Ox and Oy we differentiate expressions (10.1) with respect to time:

$$w_x^M = \ddot{x}_{O'} - \ddot{\varphi} r' \sin\varphi - \dot{\varphi}^2 r' \cos\varphi, \quad w_y^M = \ddot{y}_{O'} + \ddot{\varphi} r' \cos\varphi - \dot{\varphi}^2 r' \sin\varphi$$

Using the notation introduced in Sec. 1.2 and denoting the acceleration of the centre by $w_{O'}$ and the angular acceleration of the plane figure ($\varepsilon = \dot{\omega} = \ddot{\varphi}$) by ε we obtain

$$w_x^M = w_x^{O'} - \varepsilon r_y' - \omega^2 r_x', \quad w_y^M = w_y^{O'} + \varepsilon r_x' - \omega^2 r_y'$$

In the vector form this can be written as

$$w_M = w_x^M \boldsymbol{i} + w_y^M \boldsymbol{j} = w_x^{O'} \boldsymbol{i} + w_y^{O'} \boldsymbol{j} + \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \varepsilon \\ r_x' & r_y' & 0 \end{vmatrix} - \omega^2 (r_x' \boldsymbol{i} + r_y' \boldsymbol{j})$$

The same expression can also be derived from formula (9.18) for the acceleration of an arbitrary point M in the general case of motion of a rigid body. When the plane figure moves in its plane (see Fig. 9.5) the perpendicular dropped from the point M to the instantaneous axis of rotation is $\boldsymbol{MO'}$, and therefore the vector $\boldsymbol{R}'$ in formula (9.18) is equal to the vector $\boldsymbol{r}' = \boldsymbol{O'M}$. Hence, we can write the above expression for the acceleration in the form

$$\boldsymbol{w}_M = \boldsymbol{w}_{O'} + [\boldsymbol{\varepsilon}, \boldsymbol{r}'] - \omega^2 \boldsymbol{r}' \tag{10.8}$$

Thus, *the acceleration vector $\boldsymbol{w}_M$ of the point M in a plane motion is equal to the vector sum of the vector $\boldsymbol{w}_{O'}$ (the acceleration of an arbitrarily chosen centre O' in the plane figure), the vector $\boldsymbol{w}_\tau^{\mathrm{rel}} = [\boldsymbol{\varepsilon}, \boldsymbol{r}']$ (the rotational acceleration) and the vector $\boldsymbol{w}_n^{\mathrm{rel}} = -\omega^2 \boldsymbol{r}'$ (the centripetal acceleration).* All these vectors are shown in Fig. 10.12. Here $\boldsymbol{\varepsilon}$ is the vector representing the instantaneous angular acceleration of the plane figure:

$$\boldsymbol{\varepsilon} = \frac{d\boldsymbol{\omega}}{dt}$$

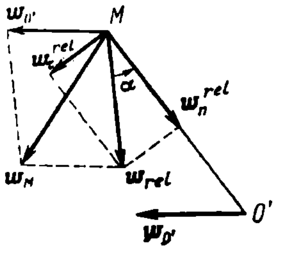

Fig. 10.12

(see Sec. 2.4 of Chap. 9). Since the vector $\boldsymbol{\omega}$ representing the angular velocity of the plane figure is perpendicular to the plane of the figure during the whole time of plane motion, the vector $\boldsymbol{\varepsilon}$ also goes along the instantaneous axis of rotation in the same direction as $\boldsymbol{\omega}$ when the rotation is accelerated and in the opposite direction when the rotation is decelerated. In what follows we shall also use the algebraic values of the angular velocity and of the angular acceleration of the plane figure:

$$\omega = \frac{d\varphi}{dt}, \qquad \varepsilon = \frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2}$$

Figure 10.12 is drawn under the assumption that $\varepsilon > 0$ and therefore the rotational acceleration w_τ^{rel} goes in the positive direction (it corresponds to counterclockwise angular acceleration relative to the centre O'). The same result can be obtained if we set off from the point O' the vector $\boldsymbol{\varepsilon}$ perpendicular to the plane of the figure in the direction to the reader and then construct, according to the right-hand screw rule, the vector w_τ^{rel} equal to the vector product of the vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{r}' = \boldsymbol{O'M}$ and place the origin of the vector $\boldsymbol{w}_\tau^{\mathrm{rel}}$ at the point M. As to the last summand, that is the centripetal acceleration $\boldsymbol{w}_n^{\mathrm{rel}}$, it is always directed from the point M toward the centre O' or is equal to zero (at those instants when the angular velocity ω of the plane figure is equal to zero). The moduli of the last two summands in formula (10.8) are

$$w_\tau^{\mathrm{rel}} = r' \,|\varepsilon|, \qquad w_n^{\mathrm{rel}} = r'\omega^2 \tag{10.9}$$

These expressions coincide with the ones given by formulas (8.12) and (8.13). Now, from formula (10.8) it follows that the acceleration of the point M in a plane motion can be represented as the vector sum of the acceleration $w_{O'}$ of the centre O' and the acceleration w_{rel} of the point M in its rotation about the centre O' (regarded as a fixed centre):

$$\boldsymbol{w}_M = \boldsymbol{w}_{O'} + \boldsymbol{w}_{\mathrm{rel}} \qquad (\boldsymbol{w}_{\mathrm{rel}} = \boldsymbol{w}_\tau^{\mathrm{rel}} + \boldsymbol{w}_n^{\mathrm{rel}}) \tag{10.8a}$$

The vector $\boldsymbol{w}_{\mathrm{rel}}$ is regarded as the acceleration of the point M relative to the centre O', and the subscript rel means relative. As to the components $\boldsymbol{w}_\tau^{\mathrm{rel}}$ and $\boldsymbol{w}_n^{\mathrm{rel}}$, although they must not necessarily be directed along the tangent and the normal to the trajectory of the point M in the general case under consideration, we retain for the rotational and the centripetal accelerations the same subscripts as in formulas (8.12) and (8.13). The components of the vector $\boldsymbol{w}_{\mathrm{rel}}$ are mutually perpendicular and therefore its modulus is

$$w_{\mathrm{rel}} = \sqrt{(w_\tau^{\mathrm{rel}})^2 + (w_n^{\mathrm{rel}})^2} = r'\sqrt{\varepsilon^2 + \omega^4}$$

(see formula (8.15)). To determine the direction of the vector $\boldsymbol{w}_{\mathrm{rel}}$ we must take into account that the rotational acceleration $\boldsymbol{w}_\tau^{\mathrm{rel}}$

goes in the positive or in the negative direction of reckoning the angle φ depending on whether ε is positive or negative, and the centripetal acceleration w_n^{rel} always goes in the direction from the point M to the centre O' (see Fig. 10.12). Denoting by α the acute angle between the vector $w_{\text{rel}} = w_\tau^{\text{rel}} + w_n^{\text{rel}}$ and the direction of MO' we can write

$$\tan\alpha = \frac{w_\tau^{\text{rel}}}{w_n^{\text{rel}}} = \frac{|\varepsilon|}{\omega^2} \tag{10.10}$$

Hence, at the given instant the angle α is one and the same for all the points of the figure and does not depend on the choice of the centre O'.

2.2. **Instantaneous Acceleration Centre.** The point S' of the plane figure whose acceleration at the given instant is equal to zero is called the *instantaneous acceleration centre*. If this point exists at the given instant we can take it as the centre for that instant and then, by formula (10.8a), we have

$$w_M = w_\tau^{\text{rel}} + w_n^{\text{rel}}$$

Thus, the acceleration of an arbitrary point of the figure at the given instant can be regarded as the vector sum of the rotational acceleration w_τ^{rel} and the centripetal acceleration w_n^{rel} *with respect to the instantaneous acceleration centre* S'. In this case the angle α determined by formula (10.10) is the acute angle between the vector w_M representing the acceleration of the point M and the direction of MS'.

This property of the instantaneous acceleration centre is analogous to that of the instantaneous centre of zero velocity C': the velocity of any point M forms a constant angle (namely, a right angle) with the direction of MC'. The indicated property makes it possible to determine the instantaneous acceleration centre S' when the acceleration vectors w_A and w_B of two points A and B are known (Fig. 10.13). To this end let us construct at one of the points, say at B, the vector $w_{\text{rel}} = w_B - w_A = w_B + (-w_A)$: this vector is the acceleration of the point B relative to the point A because if the point A is taken as the centre then, by formula (10.8a), we have

$$w_B = w_A + w_{\text{rel}}$$

Next we determine from the figure the angle α between the vector $w_B - w_A$ and the direction of BA, and draw two straight lines through the points A and B forming angles equal to α (and reckoned in the same direction as α) with the accelerations w_A and w_B respectively; it is the point of intersection S' of these straight lines that gives us the position of the instantaneous acceleration centre.

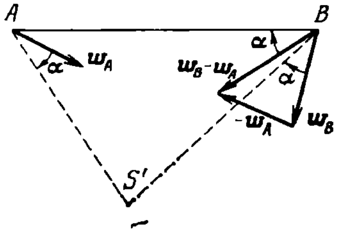

Fig. 10.13

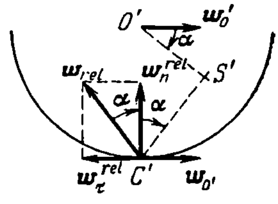

Fig. 10.14

EXAMPLE 10.3. Let us consider the conditions of Example 10.1 and, in addition, assume that the algebraic value of the acceleration of the centre O' of the wheel at the given instant, which is equal to $-w_{O'}$, is known ($w_{O'}$ denotes the modulus of the acceleration vector). It is required to find the acceleration of the instantaneous centre of zero velocity C'.

Solution. Let us construct in Fig. 10.14 the vector $w_{O'}$ directed opposite to that of the motion of the centre O'. The angular velocity of rotation of the wheel is $\omega = v_{O'}/r$; consequently, the angular acceleration is

$$\varepsilon = \frac{d\omega}{dt} = \frac{1}{r}\frac{dv_{O'}}{dt} = -\frac{1}{r} w_O$$

Let us take the centre at the point O' and construct for the point C' the vectors $w_{O'}$, w_τ^{rel} and w_n^{rel}. The moduli of the last two vectors are

$$w_\tau^{\text{rel}} = O'C'\,|\varepsilon| = w_{O'}, \qquad w_n^{\text{rel}} = O'C'\omega^2 = \frac{1}{r} v_{O'}^2$$

As usual, the centripetal acceleration w_n^{rel} is directed toward the centre O', and the rotational acceleration w_τ^{rel} is perpendicular to w_n^{rel} and is directed, in the case under consideration, opposite to that of the rotation because ε is negative. By formula (10.8a), we have

$$w_{C'} = w_{O'} + w_\tau^{\text{rel}} + w_n^{\text{rel}} = w_n^{\text{rel}}$$

For the point C' the equality $w_{O'} + w_\tau^{\text{rel}} = 0$ holds and therefore the acceleration of the point C' is equal to w_n^{rel}.

Let us determine the instantaneous acceleration centre. To this end we construct at the point C' the vector

$$w_{C'} - w_{O'} = w_\tau^{\text{rel}} + w_n^{\text{rel}} = w_{\text{rel}}$$

Next we find from the figure the angle α between the vector w_{rel} and the direction of $C'O'$ and draw through the points O' and C' two straight lines forming angles equal to α (and reckoned in negative direction, that is clockwise, because $\varepsilon < 0$) with the vectors $w_{O'}$ and $w_{C'}$. The instantaneous acceleration centre S' is at the point of intersection of these straight lines (see Fig. 10.14).

EXAMPLE 10.4. Determine the acceleration of the slide block B for the conditions of Example 10.2 (Fig. 10.15).

Solution. Let us take the centre at the point A of the connecting rod AB and write for the acceleration vector of the point B the expression

$$w_B = w_A + w_\tau^{\text{rel}} + w_n^{\text{rel}}$$

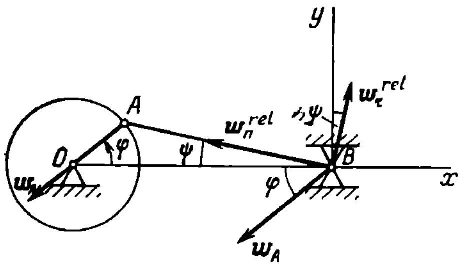

Fig. 10.15

using formula (10.9). Each of the three vectors on the right-hand side of this expression is constructed in Fig. 10.15; the unknown angular acceleration ε of the connecting rod AB is at present assumed to be positive, which corresponds to the rotation of the vector w_τ^{rel} about the point A in counterclockwise direction. The moduli of these vectors are

$$w_A = r\Omega^2, \quad w_\tau^{\text{rel}} = AB\varepsilon, \quad w_n^{\text{rel}} = AB\omega^2 = AB\,\frac{r^2\Omega^2}{C'A^2}$$

where the angular velocity ω of the connecting rod AB is written using the expression given at the end of Example 10.2. Let us project the vector formula expressing w_B on the axes Bx and By:

$$w_x^B = -w_A\cos\varphi + w_\tau^{\text{rel}}\sin\psi - w_n^{\text{rel}}\cos\psi = -r\Omega^2\cos\varphi + AB\varepsilon\sin\psi - AB\,\frac{r^2\Omega^2}{C'A^2}\cos\psi$$

$$0 = w_y^B = -w_A\sin\varphi + w_\tau^{\text{rel}}\cos\psi + w_n^{\text{rel}}\sin\psi = -r\Omega^2\sin\varphi + AB\varepsilon\cos\psi + AB\,\frac{r^2\Omega^2}{C'A^2}\sin\psi$$

Here we have $w_y^B = 0$ since the slide block B is in a rectilinear motion along the axis Ox. From the second equation we find the angular acceleration ε of the connecting rod AB in its plane motion:

$$\varepsilon = \frac{r\Omega^2}{AB\cos\psi}\left(\sin\varphi - \frac{rAB}{C'A^2}\sin\psi\right)$$

The substitution of this value of ε into w_x^B yields

$$w_B = w_x^B = -r\Omega^2\left[\cos\varphi + \frac{rAB}{C'A^2}\cos\psi - \tan\psi\left(\sin\varphi - \frac{rAB}{C'A^2}\sin\psi\right)\right] = -\frac{r\Omega^2}{\cos\psi}\left[\cos(\varphi+\psi) + \frac{rAB}{C'A^2}\right]$$

For the configuration shown in Fig. 10.15 we have $w_B < 0$, that is the acceleration of the slide block is directed toward the point O.

EXAMPLE 10.5. A line segment AB is in a plane motion. The velocities and the accelerations of its end points are known. Determine the velocity and the acceleration of the midpoint D of the line segment and also the angular velocity and the angular acceleration of AB (Fig. 10.16).

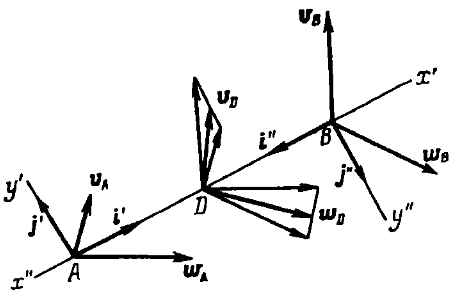

Fig. 10.16

Solution. Let us choose the point A as the centre and draw the axis Ax' (of the moving coordinate system rigidly connected with the moving line segment) along AB, the axis Ay' being perpendicular to Ax' and drawn so that the coordinate system $Ax'y'$ is right-handed (the rotation from the axis Ax' to the axis Ay' is counterclockwise). The unit vectors along the coordinate axes are denoted by $\boldsymbol{i}'$ and $\boldsymbol{j}'$. Taking into account the equality $\boldsymbol{r}' = \boldsymbol{AD}$ and using formulas (10.2) and (10.8) we obtain

$$\boldsymbol{v}_D = \boldsymbol{v}_A + [\boldsymbol{\omega}, \boldsymbol{AD}] \tag{1}$$

and

$$\boldsymbol{w}_D = \boldsymbol{w}_A + \varepsilon AD\boldsymbol{j}' - \omega^2 AD\boldsymbol{i}' \tag{2}$$

Now let us take the centre at the point B and draw the axis Bx'' of the moving coordinate system in the direction of BA, the axis By'' being again drawn perpendicularly to the moving line segment so that the coordinate system $Bx''y''$ is right-handed. For the unit vectors $\boldsymbol{i}''$ and $\boldsymbol{j}''$ along the axes of the new coordinate system we have

$$\boldsymbol{i}'' = -\boldsymbol{i}', \quad \boldsymbol{j}'' = -\boldsymbol{j}'$$

Applying formulas (10.2) and (10.8) we obtain for $\boldsymbol{r}' = \boldsymbol{BD} = -\boldsymbol{AD}$ the formulas

$$\boldsymbol{v}_D = \boldsymbol{v}_B + [\boldsymbol{\omega}, \boldsymbol{BD}] = \boldsymbol{v}_B - [\boldsymbol{\omega}, \boldsymbol{AD}] \tag{3}$$

and

$$\boldsymbol{w}_D = \boldsymbol{w}_B + \varepsilon BD\boldsymbol{j}'' - \omega^2 BD\boldsymbol{i}'' = \boldsymbol{w}_B - \varepsilon AD\boldsymbol{j}' + \omega^2 AD\boldsymbol{i}' \tag{4}$$

Adding together vector equalities (1) and (3) and then (2) and (4) we obtain

$$\boldsymbol{v}_D = \frac{1}{2}(\boldsymbol{v}_A + \boldsymbol{v}_B) \tag{5}$$

and

$$\boldsymbol{w}_D = \frac{1}{2}(\boldsymbol{w}_A + \boldsymbol{w}_B) \tag{6}$$

Thus, the velocity (the acceleration) of the midpoint of the line segment in its plane motion is equal to half the vector sum of the velocities (of the accelerations) of its end points.

The substitution of expression (5) into formula (1) results in

$$\frac{1}{2}(\boldsymbol{v}_A + \boldsymbol{v}_B) = \boldsymbol{v}_A + [\boldsymbol{\omega}, \boldsymbol{AD}], \quad \text{that is} \quad \frac{1}{2}(\boldsymbol{v}_B - \boldsymbol{v}_A) = [\boldsymbol{\omega}, \boldsymbol{AD}] \tag{7}$$

In a plane motion the vector $\boldsymbol{\omega}$ representing the angular velocity of a figure (or of a line segment) is perpendicular to the plane of motion. Therefore equating

the moduli of the vectors in the last equality we obtain

$$\frac{1}{2}|v_B - v_A| = |\omega| AD$$

From this relation we find the modulus of the instantaneous angular velocity of the figure (or of the line segment) in its plane motion:

$$|\omega| = \frac{|v_B - v_A|}{2AD} = \frac{|v_B - v_A|}{AB} \tag{8}$$

To determine the direction of rotation we must resort to vector equality (7). Now we substitute expression (6) into formula (2), which yields

$$\frac{1}{2}(w_B - w_A) + \omega^2 ADi' = \varepsilon ADj' \tag{9}$$

Let us multiply scalarly both the sides of vector equality (9) by the unit vector j'; then, since $(i', j') = 0$ and $2AD = AB$, we obtain

$$(w_B, j') - (w_A, j') = \varepsilon AB$$

Noting that the left-hand side of the last relation is the difference of the projections of the acceleration vectors of the end points of the line segment on the axis Ay' we can write the algebraic value ε of the instantaneous angular acceleration of the line segment in the form

$$\varepsilon = \frac{1}{AB}(\text{proj}_{Ay'}\, w_B - \text{proj}_{Ay'}\, w_A) \tag{10}$$

Remark. The velocity vectors v_A and v_B of the end points of the line segment cannot be set arbitrarily because, according to the theorem proved in Sec. 1.3, they are connected by a definite relation. For given v_A and v_B the acceleration vectors w_A and w_B of the end points of the line segment cannot be arbitrary either. Indeed, multiplying scalarly both the sides of vector equality (9) by the unit vector i' we obtain

$$(w_B, i') - (w_A, i') + 2\omega^2 AD = 0$$

From the last relation, using formula (8), we find

$$\text{proj}_{AB}\, w_B = \text{proj}_{AB}\, w_A - \frac{|v_B - v_A|^2}{AB}$$

Problems

PROBLEM 10.1. The crank OA of the slider-crank mechanism shown in Fig. 10.17 rotates uniformly about the axis O with constant angular velocity ω_{OA} corresponding to 240 rpm. The rocker GD is hinged to the midpoint G of the connecting rod AB; the other end D of the rocker GD is hinged to the crank DE which can rotate about the point E. The points B and E lie in one vertical; the other data are: $OA = 0.15$ m, $AB = 0.3$ m, $DE = 0.4$ m, $\angle GDE = 90°$ and $\angle BED = 30°$. Determine the angular velocity of the crank DE.

Answer. $\omega_{DE} = \frac{3}{4}\pi = 2.36$ rad/s.

PROBLEM 10.2. The end A of a rod AB lying in the vertical plane slides along the horizontal plane with velocity $v_A = 1.2$ m/s while the upper part of the rod reclines on a brickwork of height $h = 2$ m (Fig. 10.18). For the instant when the angle φ is equal to 60° determine the velocity of the point D of the rod AB at which it touches the brickwork; also find the angular velocity ω of the rod.

Answer. $v_D = 0.6$ m/s, $\omega = 0.45$ rad/s.

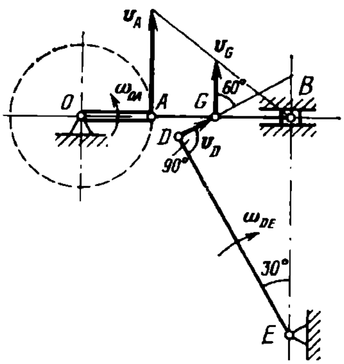

Fig. 10.17

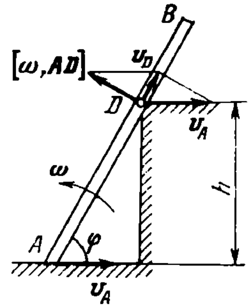

Fig. 10.18

PROBLEM 10.3. Consider the general plane motion of a plane figure. For the instant when the instantaneous angular velocity of the figure is equal to zero prove that the projections of the acceleration vectors of any two points of the figure on the straight line connecting the points are equal.

Hint. The proof is analogous to that of the theorem on the projections of the velocity vectors of two points of the figure on the straight line connecting the points.

PROBLEM 10.4. For a plane motion of a rigid line segment prove that the tips of the acceleration vectors of its points (see the points a, d and b in Fig. 10.19) lie in one straight line and divide that line into parts proportional to the distances between these points.

Hint. Prove that

$$\frac{|w_D - w_A|}{|w_B - w_A|} = \frac{AD}{AB}$$

This is equivalent to

$$\frac{a\delta}{a\beta} = \frac{AD}{AB}$$

Finally, from the similarity of the triangles derive the proportion

$$\frac{ad}{ab} = \frac{a\delta}{a\beta}$$

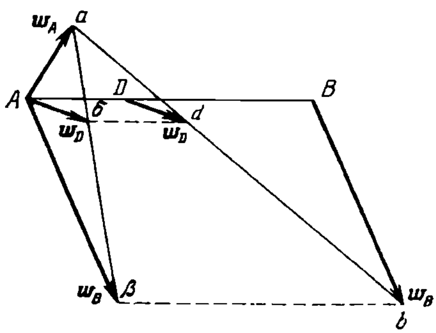

Fig. 10.19

Chapter 11 Compound Motion of a Particle

§ 1. Velocity of Compound Motion of a Particle

1.1. Absolute, Relative and Transportation Motion. In Chap. 7 we studied the motion of a particle relative to a coordinate system regarded as fixed. Now let us suppose that we consider the motion of a point (particle) M relative to a body S which in its turn moves

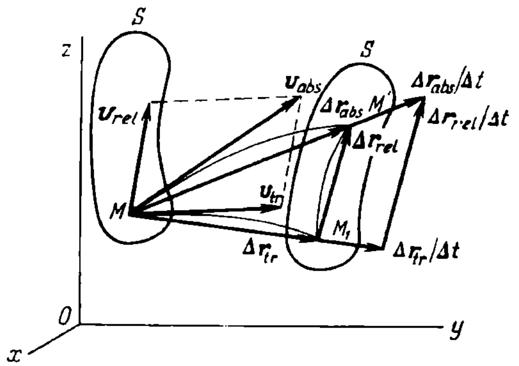

Fig. 11.1

relative to a fixed coordinate system $Oxyz$. To begin with, let us discuss the terminology and introduce the notation.

The motion (or the trajectory or the velocity or the acceleration) of the point M relative to the fixed coordinate system is called *absolute*. The motion of the same point M with respect to the body S is called *relative*. Now suppose that the position of the point M is fixed within the body S; then the motion of the point M (which it would have if it were connected with the body S) relative to the fixed coordinate system is called the *transportation* (motion) at the given instant.

Thus, the absolute motion of the point M is the motion which is seen by an observer connected with the fixed coordinate system; as to the relative motion, it is seen by an observer moving together with the body S. Finally, the transportation of the point M is the motion of the point of the body S, seen by an observer connected with the fixed coordinate system, with which at the given instant the moving point M coincides.

The scalar and the vector quantities related to the absolute motion will be marked by the subscript abs; the quantities connected with the transportation will be supplied by the subscript tr; finally, the subscript rel will indicate the quantities connected with the relative motion.

1.2. Theorem on Composition of Velocities. Shown in Fig. 11.1 are the positions of the body S and of the point (particle) M at instants t and $t' = t + \Delta t$, the point M' indicating the position of M at instant $t + \Delta t$. Let M_1 be the position of the point M which it would have at instant $t + \Delta t$ if it were fixed within the body S at instant t. The vectors $\boldsymbol{MM'}$, $\boldsymbol{MM_1}$ and $\boldsymbol{M_1M'}$ represent the displacements of the point M in its absolute, transportation and relative motions respectively. The vectors $\boldsymbol{MM'}$, $\boldsymbol{MM_1}$ and $\boldsymbol{M_1M'}$ are chords of the arcs of the absolute, the transportation and the relative

trajectories shown in Fig. 11.1. The obvious vector equality $\boldsymbol{MM'} = \boldsymbol{MM_1} + \boldsymbol{M_1M'}$ can be written in the form

$$\Delta \boldsymbol{r}_{abs} = \Delta \boldsymbol{r}_{tr} + \Delta \boldsymbol{r}_{rel}$$

Here and henceforth (and in the foregoing sections) the symbol $\Delta \boldsymbol{r}$ designates a displacement vector. It should be noted that the relative displacement vector $\boldsymbol{M_1M'} = \Delta \boldsymbol{r}_{rel}$ is shown in the figure for time $t + \Delta t$. Dividing termwise the last vector equality by the scalar Δt we obtain

$$\frac{\Delta \boldsymbol{r}_{abs}}{\Delta t} = \frac{\Delta \boldsymbol{r}_{tr}}{\Delta t} + \frac{\Delta \boldsymbol{r}_{rel}}{\Delta t}$$

Remembering the definition of the vector of average velocity (see Sec. 1.1 of Chap. 7) we rewrite this vector equality in the form

$$\boldsymbol{v}_{av}^{abs} = \boldsymbol{v}_{av}^{tr} + \boldsymbol{v}_{av}^{rel}$$

Finally, passing to the limit for $\Delta t \to 0$ in this equality we obtain

$$\boldsymbol{v}_{abs} = \boldsymbol{v}_{tr} + \boldsymbol{v}_{rel} \tag{11.1}$$

We have thus proved the *theorem on composition of velocities for a compound motion* (or *composite* or *resultant motion*) *of a particle*: *In a compound motion the absolute velocity of a particle is equal to the vector sum of the transportation and the relative velocities of that particle.* In other words, in order to find the absolute velocity vector of the particle we must add together the vectors of the transportation and the relative velocities of the particle using the parallelogram law (or, which is the same, the triangle law). Marked in Fig. 11.1 are the vectors $\boldsymbol{v}_{abs}$, $\boldsymbol{v}_{tr}$ and $\boldsymbol{v}_{rel}$ directed along the tangents to the corresponding trajectories, the vector $\boldsymbol{v}_{rel}$ being shown, as it must be, at instant t.

The modulus of the absolute velocity of the particle is found by the law of cosines:

$$v_{abs} = \sqrt{v_{tr}^2 + v_{rel}^2 + 2v_{tr}v_{rel}\cos(\widehat{\boldsymbol{v}_{tr}, \boldsymbol{v}_{rel}})} \tag{11.2}$$

where $(\widehat{\boldsymbol{v}_{tr}, \boldsymbol{v}_{rel}})$ is the angle between the vectors of transportation and relative velocities (see Fig. 11.1).

In particular, when the last two vectors are perpendicular, the velocity parallelogram turns into a rectangle, and we have

$$v_{abs} = \sqrt{v_{tr}^2 + v_{rel}^2} \quad (\text{if } \boldsymbol{v}_{tr} \perp \boldsymbol{v}_{rel}) \tag{11.2a}$$

When the vectors representing the transportation and the relative velocities of the point go in one direction along one straight line we have

$$v_{abs} = v_{tr} + v_{rel} \quad (\text{if } \boldsymbol{v}_{tr} \uparrow\uparrow \boldsymbol{v}_{rel}) \tag{11.2b}$$

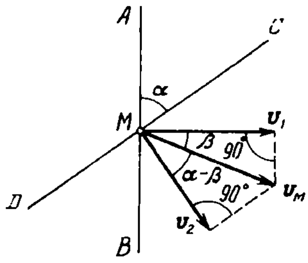

Fig. 11.2

(the symbol $\uparrow\uparrow$ means "parallel"). Finally, if they have opposite directions along one line then

$$v_{abs} = |v_{tr} - v_{rel}| \qquad (\text{if } v_{tr} \uparrow\downarrow v_{rel}) \tag{11.2c}$$

(the symbol $\uparrow\downarrow$ means "antiparallel").

EXAMPLE 11.1. Two rods AB and CD are in a translatory motion in one plane, their velocities v_1 and v_2 being perpendicular to AB and CD respectively. The angle between the rods is equal to α. Find the velocity of the small ring M put on both the rods (Fig. 11.2).

Solution. The motion of the ring M can be regarded as a compound motion in the following two ways: as the result of composition of the transportation with velocity v_1 (equal to the velocity of the translatory motion of the rod AB) and the relative motion of the ring M along the rod CD or as the result of composition of the transportation with velocity v_2 (together with the rod CD) and the relative motion of the ring M along the rod CD. Since the absolute velocity v_M of the ring is equal to the vector sum of the transportation and the relative velocities, the terminus of the vector v_M must lie on a straight line parallel to the rod AB and passing through the terminus of the vector v_1; on the other hand, it must also lie on a straight line parallel to the rod CD and passing through the terminus of the vector v_2. It is the point of intersection of these straight lines that serves as the terminus of the absolute velocity vector v_M of the point M.

From the right triangles in Fig. 11.2 we find

$$v_M = \frac{v_1}{\cos\beta}, \quad v_M = \frac{v_2}{\cos(\alpha-\beta)}$$

To determine the angle β we equate the right-hand sides of the last equalities, whence

$$v_1 \cos(\alpha - \beta) = v_2 \cos\beta$$

This can be written in full as

$$v_1 \cos\alpha \cos\beta + v_1 \sin\alpha \sin\beta = v_2 \cos\beta$$

whence

$$\tan\beta = \frac{v_2 - v_1\cos\alpha}{v_1 \sin\alpha}$$

From this formula we find

$$\sec\beta = \sqrt{1 + \frac{(v_2 - v_1\cos\alpha)^2}{v_1^2 \sin^2\alpha}} = \frac{1}{v_1 \sin\alpha}\sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\alpha}$$

Finally, using the first formula expressing v_M we obtain

$$v_M = v_1 \sec\beta = \frac{1}{\sin\alpha}\sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\alpha}$$

As to the direction of the vector v_M, it is specified by the angle β.

1.3. Expression for the Velocity of a Particle in Polar Coordinates. Let a particle move in a fixed plane, its equations of motion being written in polar coordinates:

$$r = r(t), \qquad \varphi = \varphi(t)$$

Here we shall determine the velocity of that particle considering this problem as an example of compound motion.

Let us interpret the polar radius r as a rod S and the moving particle M as a small ring moving along the rotating rod. The equation of rotation of the rod is $\varphi = \varphi(t)$. The transportation velocity v_{tr} can be found if we fix the position of the ring M on the rotating rod whose angular velocity of rotation is $\omega = d\varphi/dt$.

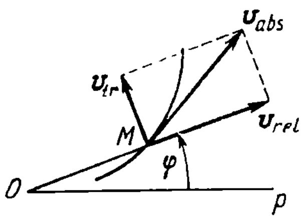

Fig. 11.3

Then for the modulus of the transportation velocity we obtain

$$v_{tr} = r\,|\omega| = r\left|\frac{d\varphi}{dt}\right|$$

The relative motion of the particle M is the rectilinear motion of the ring along the rod, the modulus of the relative velocity being

$$v_{rel} = \left|\frac{dr}{dt}\right|$$

Since the vectors representing the transportation and the relative velocities of the particle M are perpendicular (Fig. 11.3), the modulus of the absolute velocity can be found as the length of the diagonal of the rectangle constructed on the vectors v_{tr} and v_{rel} (see formula (11.2a)):

$$v_{abs} = \sqrt{v_{tr}^2 + v_{rel}^2} = \sqrt{r^2\left(\frac{d\varphi}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2} \qquad (11.3)$$

The path length travelled by the particle M (in its absolute motion, that is with respect to the fixed plane) is found by formula (7.14):

$$S = \int_0^t \sqrt{r^2\left(\frac{d\varphi}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2}\,dt$$

Let us transform the integrand expression by inserting dt under the radical sign and then taking $d\varphi$ out of the radical sign. Replacing in the definite integral the variable of integration t by φ we find

$$S = \int_{\varphi_0}^{\varphi} \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2}\,d\varphi$$

It is this formula that expresses the arc length of a plane curve in polar coordinates (see [1]).

1.4. Analytical Proof of the Theorem on Composition of Velocities. To carry out the analytical proof let us consider, besides the fixed coordinate system $Oxyz$, a moving coordinate system $O'x'y'z'$ rigidly connected with the body S. Then the motion of the system $O'x'y'z'$

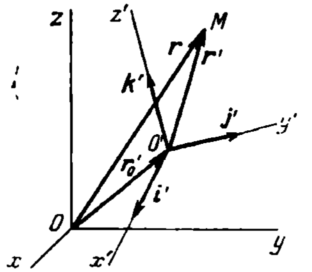

Fig. 11.4

represents that of the body. By i', j' and k' we shall denote the unit vectors along the axes $O'x'$, $O'y'$ and $O'z'$. From Fig. 11.4 we conclude that for the point M there holds the equality

$$r = r_{O'} + r'$$

where $r_{O'} = OO'$; $r = OM$ and $r' = O'M$ are the absolute and the relative radius vectors of the point M respectively. Let us denote by x', y' and z' the coordinates of the point M relative to the moving coordinate system $O'x'y'z'$; then

$$r' = x'i' + y'j' + z'k'$$

and consequently

$$r = r_{O'} + (x'i' + y'j' + z'k')$$

The velocity of absolute motion of the point is the vector

$$v_{\text{abs}} = \frac{dr}{dt}$$

and therefore in order to find the velocity we differentiate the expression for r with respect to time:

$$v_{\text{abs}} = \left(\frac{dr_{O'}}{dt} + x'\frac{di'}{dt} + y'\frac{dj'}{dt} + z'\frac{dk'}{dt}\right) + \left(\frac{dx'}{dt}i' + \frac{dy'}{dt}j' + \frac{dz'}{dt}k'\right) \qquad (11.4)$$

Note that, generally speaking, since the coordinate system $O'x'y'z'$ moves, the directions of the unit vectors i', j' and k' change with time. That is why first parentheses in (11.4) involve the derivatives of the unit vectors with respect to time.

When the point M does not move with respect to the body S, that is when x', y' and z' do not vary with time, the *transportation velocity of the point M coincides with its absolute velocity*. In this case the last three summands in formula (11.4) vanish and we obtain the vector representing the velocity of the transportation of the point:

$$v_{\text{tr}} = \frac{dr_{O'}}{dt} + x'\frac{di'}{dt} + y'\frac{dj'}{dt} + z'\frac{dk'}{dt} \qquad (11.5)$$

The *relative velocity of the point* is the velocity of that point with respect to the moving coordinate system $O'x'y'z'$ and consequently is expressed by the vector determined by a formula analogous to (7.8):

$$v_{\text{rel}} = \frac{dx'}{dt}i' + \frac{dy'}{dt}j' + \frac{dz'}{dt}k' \qquad (11.6)$$

Now formula (11.4) can be written in form (11.1):

$$v_{\text{abs}} = v_{\text{tr}} + v_{\text{rel}}$$

The last relation expresses the theorem on composition of velocities for a compound motion of a particle.

If the vectors entering into formula (11.1) are projected on the fixed coordinate axes Ox, Oy and Oz this yields the formulas

$$v_x^{\text{abs}} = v_x^{\text{tr}} + v_x^{\text{rel}}, \quad v_y^{\text{abs}} = v_y^{\text{tr}} + v_y^{\text{rel}}, \quad v_z^{\text{abs}} = v_z^{\text{tr}} + v_z^{\text{rel}} \qquad (11.7)$$

The projection of the absolute velocity of the point M on an axis is equal to the algebraic sum of the projections of the transportation and the relative velocities of the point on that axis.

Let us write formula (11.5) for the following two *special cases* of motion of the body S.

1. If the body S is in *translatory motion* the coordinate system $O'x'y'z'$ remains parallel to itself during the motion, and consequently the unit vectors $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ are constant. In this case the last three summands in formula (11.5) vanish and the velocity vector of the transportation of the point M for the translatory motion of the body S is expressed by the formula

$$v_{\text{tr}} = \frac{d\boldsymbol{r}_{O'}}{dt} = v_{O'}$$

Since the velocity of any point of the body can be taken as the velocity of the translatory motion of the body, we can say that in the case under consideration the transportation velocity of the point M is equal to the velocity of the translatory motion of the body S.

2. Let the body S move so that the *point O' remains fixed*. For instance, this can be the case when the body rotates about a fixed axis passing through the point O'. In this case the first summand in formula (11.5) vanishes and for the vector representing the velocity of the transportation of the point M we obtain the formula

$$v_{\text{tr}} = x' \frac{d\boldsymbol{i}'}{dt} + y' \frac{d\boldsymbol{j}'}{dt} + z' \frac{d\boldsymbol{k}'}{dt} \qquad (11.8)$$

§ 2. Acceleration of Compound Motion of a Particle

2.1. The Coriolis Theorem on Composition of Accelerations. The absolute acceleration vector w_{abs} of the point M is expressed by the formula

$$w_{\text{abs}} = \frac{dv_{\text{abs}}}{dt}$$

In order to find this vector we differentiate formula (11.4) with respect to time:

$$w_{\text{abs}} = \left(\frac{d^2\boldsymbol{r}_{O'}}{dt^2} + x' \frac{d^2\boldsymbol{i}'}{dt^2} + y' \frac{d^2\boldsymbol{j}'}{dt^2} + z' \frac{d^2\boldsymbol{k}'}{dt^2} \right)$$

$$+\left(\frac{d^2x'}{dt^2}i'+\frac{d^2y'}{dt^2}j'+\frac{d^2z'}{dt^2}k'\right)$$
$$+2\left(\frac{dx'}{dt}\frac{di'}{dt}+\frac{dy'}{dt}\frac{dj'}{dt}+\frac{dz'}{dt}\frac{dk'}{dt}\right) \qquad (11.9)$$

Let us discuss the meaning of the formula we have derived. If the point M is rigidly connected with the coordinate system $O'x'y'z'$, that is if x', y' and z' have constant values, then only the first four summands in formula (11.9) may remain nonzero; the sum of these four summands expresses the vector w_{tr} representing the *transportation acceleration of the point*:

$$w_{\mathrm{tr}}=\frac{d^2r_{O'}}{dt^2}+x'\frac{d^2i'}{dt^2}+y'\frac{d^2j'}{dt^2}+z'\frac{d^2k'}{dt^2}$$

The *relative acceleration of the point* is the acceleration of that point with respect to the moving coordinate system $O'x'y'z'$ and consequently is represented by the vector determined by a formula analogous to (7.18):

$$w_{\mathrm{rel}}=\frac{d^2x'}{dt^2}i'+\frac{d^2y'}{dt^2}j'+\frac{d^2z'}{dt^2}k'$$

We have thus elucidated the mechanical meaning of the first two brackets in formula (11.9). However, the formula expressing the vector w_{abs} involves one more vector summand (we denote it by w_{C}) which is called the *Coriolis* acceleration of the point*:

$$w_{\mathrm{C}}=2\left(\frac{dx'}{dt}\frac{di'}{dt}+\frac{dy'}{dt}\frac{dj'}{dt}+\frac{dz'}{dt}\frac{dk'}{dt}\right) \qquad (11.10)$$

We have thus proved the *Coriolis theorem on composition of accelerations: the acceleration of the absolute motion of a particle is equal to the vector sum of three accelerations, namely the transportation, the relative and the Coriolis accelerations*, that is

$$w_{\mathrm{abs}}=w_{\mathrm{tr}}+w_{\mathrm{rel}}+w_{\mathrm{C}} \qquad (11.11)$$

Projecting the vectors involved in formula (11.11) on the axes of the fixed coordinate system $Oxyz$ we find

$$\begin{aligned} w_x^{\mathrm{abs}}&=w_x^{\mathrm{tr}}+w_x^{\mathrm{rel}}+w_x^{\mathrm{C}}\\ w_y^{\mathrm{abs}}&=w_y^{\mathrm{tr}}+w_y^{\mathrm{rel}}+w_y^{\mathrm{C}}\\ w_z^{\mathrm{abs}}&=w_z^{\mathrm{tr}}+w_z^{\mathrm{rel}}+w_z^{\mathrm{C}} \end{aligned} \qquad (11.12)$$

Thus, the projection of the absolute acceleration of a particle on an axis is equal to the algebraic sum of the projections of the transportation, the relative and the Coriolis accelerations of the particle on that axis.

2.2. Vector of Coriolis Acceleration of a Particle. Now let us analyse formula (11.10) expressing the vector of Coriolis acceleration.

* Coriolis, G.G. (1792-1843), a French physicist.

Formula (11.8) expresses the transportation velocity of the point M, that is of the terminus of the vector

$$\boldsymbol{O'M} = x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}'$$

when the origin of this vector, the point O', is fixed. If we replace x', y' and z' in formula (11.8) by dx'/dt, dy'/dt and dz'/dt respectively, we obtain

$$\frac{dx'}{dt}\frac{d\boldsymbol{i}'}{dt}+\frac{dy'}{dt}\frac{d\boldsymbol{j}'}{dt}+\frac{dz'}{dt}\frac{d\boldsymbol{k}'}{dt}$$

This expression represents the transportation velocity of the terminus of the vector $\boldsymbol{v}_{\text{rel}}$ of relative velocity of the point M:

$$\boldsymbol{v}_{\text{rel}}=\frac{dx'}{dt}\boldsymbol{i}'+\frac{dy'}{dt}\boldsymbol{j}'+\frac{dz'}{dt}\boldsymbol{k}'$$

on condition that the origin of that vector is fixed (is transferred to the point O).

Now, formula (11.10) can be interpreted as follows: *the vector representing the Coriolis acceleration of a particle is geometrically equal to twice the vector of transportation velocity of the terminus of the vector* $\boldsymbol{v}_{\text{rel}}$ *provided that this vector* $\boldsymbol{v}_{\text{rel}}$ *is transferred to the fixed point* O.

The same result can be obtained by transforming formula (11.10). To this end let us find the derivatives with respect to time of the unit vectors along the axes of the moving coordinate system. The vector $d\boldsymbol{i}'/dt$ can be interpreted as the velocity of the terminus of the unit vector $\boldsymbol{i}'$ when the origin of $\boldsymbol{i}'$ is transferred to a fixed point, for instance, to O. Therefore, by formula (9.13) of Chap. 9, we have

$$\frac{d\boldsymbol{i}'}{dt}=[\boldsymbol{\omega}_{\text{tr}},\ \boldsymbol{i}']$$

Here $\boldsymbol{\omega}_{\text{tr}}$ is the vector representing the instantaneous angular velocity of the coordinate system $O'x'y'z'$ (of the body S). Similarly, we have

$$\frac{d\boldsymbol{j}'}{dt}=[\boldsymbol{\omega}_{\text{tr}},\ \boldsymbol{j}'],\qquad \frac{d\boldsymbol{k}'}{dt}=[\boldsymbol{\omega}_{\text{tr}},\ \boldsymbol{k}']$$

The substitution of these values of the derivatives into formula (11.10) results in

$$\boldsymbol{w}_{\text{C}}=2\left\{\frac{dx'}{dt}[\boldsymbol{\omega}_{\text{tr}},\ \boldsymbol{i}']+\frac{dy'}{dt}[\boldsymbol{\omega}_{\text{tr}},\ \boldsymbol{j}']+\frac{dz'}{dt}[\boldsymbol{\omega}_{\text{tr}},\ \boldsymbol{k}']\right\}$$
$$=2\left[\boldsymbol{\omega}_{\text{tr}},\ \left(\frac{dx'}{dt}\boldsymbol{i}'+\frac{dy'}{dt}\boldsymbol{j}'+\frac{dz'}{dt}\boldsymbol{k}'\right)\right]$$

Further, the expression in the parentheses is the relative velocity $\boldsymbol{v}_{\text{rel}}$ of the point M (see formula (11.6)) and we finally obtain

$$\boldsymbol{w}_{\text{C}} = 2\,[\boldsymbol{\omega}_{\text{tr}},\ \boldsymbol{v}_{\text{rel}}] \tag{11.13}$$

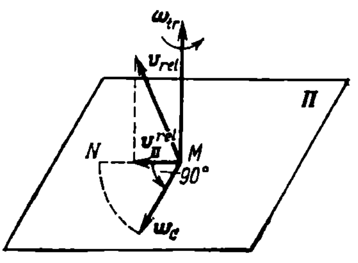

Fig. 11.5

The Coriolis acceleration of a particle in its compound motion is geometrically equal to twice the vector product of the instantaneous angular velocity of the coordinate system $O'x'y'z'$ (of the body S) by the relative velocity of the particle.

The Joukowski rule for constructing the Coriolis acceleration vector. According to this rule, in order to construct the Coriolis acceleration vector we draw through the point M a plane Π perpendicular to the vector $\boldsymbol{\omega}_{tr}$ of instantaneous angular velocity of the moving coordinate system $O'x'y'z'$ and project the relative velocity vector v_{rel} on that plane Π (Fig. 11.5). The component v_{Π}^{rel} is then increased $2\omega_{tr}$ times (where ω_{tr} is the modulus of the angular velocity of the coordinate system $O'x'y'z'$) and the line segment MN thus obtained is turned through a right angle in the plane Π in the direction of the transportation rotation. The resultant vector thus constructed is the vector $\boldsymbol{w}_C$ of the Coriolis acceleration.

By virtue of formula (11.13), for the modulus of the Coriolis acceleration we have

$$w_C = 2\omega_{tr} v_{rel} \sin\,(\widehat{\boldsymbol{\omega}_{tr}, \boldsymbol{v}_{rel}}) \tag{11.14}$$

From formula (11.13) it follows that the Coriolis acceleration of a particle vanishes at those instants when either $\omega_{tr} = 0$ or $v_{rel} = 0$ or the vectors $\boldsymbol{\omega}_{tr}$ and $\boldsymbol{v}_{rel}$ are collinear (in the latter case the angle $(\widehat{\boldsymbol{\omega}_{tr}, \boldsymbol{v}_{rel}})$ is equal to 0 or 180°).

Let us consider the expression for the Coriolis acceleration for the following two *special cases* of motion of the coordinate system $O'x'y'z'$ rigidly connected with the body S.

1. Suppose that the coordinate system $O'x'y'z'$ is in a *translatory motion*. This means that the instantaneous angular velocity ω_{tr} of $O'x'y'z'$ is identically equal to zero. From formula (11.13) it follows that the Coriolis acceleration w_C is identically equal to zero and formula (11.11) takes the form

$$\boldsymbol{w}_{abs} = \boldsymbol{w}_{tr} + \boldsymbol{w}_{rel} \tag{11.15}$$

In this case the absolute acceleration of the point M is equal to the vector sum of its transportation and relative accelerations. Thus if the transportation of the point is determined by translatory motion of the body S then not only the velocities are added together according to the parallelogram law (this is valid for any transportation of the point) but the accelerations as well.

2. Suppose that the coordinate system $O'x'y'z'$ *rotates* about a fixed axis passing through the point O' with angular velocity ω_{tr}. In this case the transportation of the point M is determined by the rotational motion of the body S. Therefore the vector w_C of the Coriolis acceleration is determined by general formula (11.13) in which the vector $\boldsymbol{\omega}_{tr}$ retains an invariable line of action.

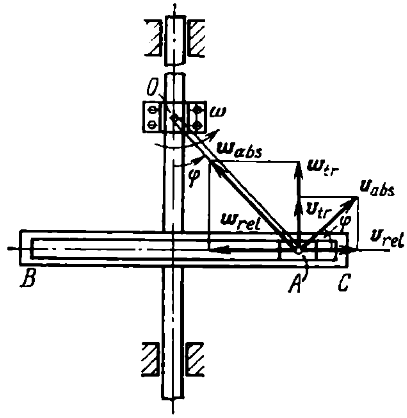

Fig. 11.6

For instance, if the direction of the relative velocity vector v_{rel} of the particle is parallel to the line of action of the vector $\boldsymbol{\omega}_{tr}$, that is to the axis of rotation of the body S, the angle $(\widehat{\boldsymbol{\omega}_{tr}, v_{rel}})$ is equal to 0 or 180°, and the Coriolis acceleration w_C is equal to zero. Hence, the Coriolis acceleration may also be equal to zero when the transportation of the point is determined by a rotational motion of the body.

EXAMPLE 11.2. In the oscillating crank gear with a reciprocating link BC the crank OA of length l rotates with constant angular velocity $\omega > 0$ (Fig. 11.6). The link BC can only be in a translatory motion; it moves under the action of the link bolt A at the end of the crank which slides in the slot of the link. Find the velocity and the acceleration of the link and of the bolt A.

Solution. In its absolute motion the bolt A of the crank OA moves in a circle of radius l with centre at the point O. Its absolute velocity v_A is perpendicular to the crank OA, and its modulus is

$$v_A = l\omega$$

On the other hand, the motion of the bolt A can be regarded as a resultant motion composed of two motions: the relative motion along the slot in the link with velocity v_{rel} and the transportation determined by the translatory motion of the link with velocity v_{tr}. From the velocity parallelogram (in the case under consideration it is a rectangle) we find

$$v_{rel} = v_{abs} \cos \varphi = l\omega \cos \omega t$$

and

$$v_{tr} = v_{abs} \sin \varphi = l\omega \sin \omega t$$

(here we assume that $\varphi(0) = 0$). The last quantity is equal to the algebraic velocity of the translatory motion of the link.

In its absolute motion the bolt A moves in a circle of radius l with velocity whose modulus is constant: $v_{abs} = l\omega$. Consequently the tangential acceleration w_τ^{abs} is equal to zero, and the normal acceleration w_n^{abs} coincides with the total acceleration:

$$w_{abs} = w_n^{abs} = \frac{v_{abs}^2}{l} = l\omega^2$$

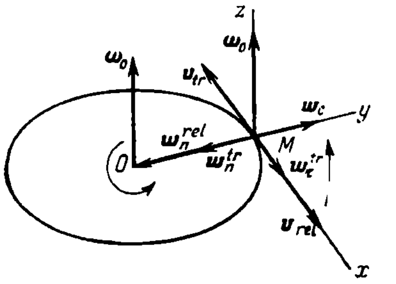

Fig. 11.7

The same result can be obtained if the motion of the bolt A is considered as compound. Since the transportation of the bolt is determined by the translatory motion of the link, the Coriolis acceleration is equal to zero, and the acceleration of the absolute motion is the result of composition of the relative acceleration whose modulus is

$$w_{\mathrm{rel}} = \left| \frac{dv_{\mathrm{rel}}}{dt} \right| = l\omega^2 \, |\sin \omega t|$$

and the transportation acceleration equal to that of the link in its reciprocating motion:

$$w_{\mathrm{tr}} = \frac{dv_{\mathrm{tr}}}{dt} = l\omega^2 \cos \omega t$$

Forming the vector sum of both the accelerations we find the modulus of the absolute acceleration of the bolt:

$$w_{\mathrm{abs}} = \sqrt{w_{\mathrm{tr}}^2 + w_{\mathrm{rel}}^2} = l\omega^2$$

It can readily be verified that the vector $\boldsymbol{w}_{\mathrm{abs}}$ thus found is directed along the crank OA.

EXAMPLE 11.3. Shown in Fig. 11.7 is a platform of radius R which is in a uniformly decelerated rotation about a fixed axis perpendicular to the plane of the platform, the initial angular velocity being $\omega_0 > 0$ and the constant angular acceleration being $\varepsilon < 0$. Along the rim of the platform a point M moves with a velocity of constant modulus u relative to the platform in the direction opposite to that of the rotation of the platform at the initial instant. Find the absolute velocity and the absolute acceleration of the point M at the initial instant.

Solution. Let us set off along the axis of rotation the vector $\boldsymbol{\omega}_0$ and then construct the velocity vector $\boldsymbol{v}_{\mathrm{tr}}$ of the transportation of the point M:

$$\boldsymbol{v}_{\mathrm{tr}} = [\boldsymbol{\omega}_0, \boldsymbol{OM}]$$

The direction of the vector $\boldsymbol{v}_{\mathrm{tr}}$ is determined by the right-hand screw rule specifying the direction of the vector product; this direction is also determined by that of the rotation of the platform. Since the vectors $\boldsymbol{\omega}_0$ and $\boldsymbol{OM}$ form a right angle we have $v_{\mathrm{tr}} = R\omega_0$; this also follows from the well-known formula for the linear velocity of a particle in a rotational motion. The relative velocity vector $\boldsymbol{v}_{\mathrm{re}}$ of the point M is collinear to the vector $\boldsymbol{v}_{\mathrm{tr}}$, has opposite direction, and its modulus is equal to u. The absolute velocity vector $\boldsymbol{v}_{\mathrm{abs}}$ of the point M is expressed by formula (11.1), and its modulus by formula (11.2c):

$$v_{\mathrm{abs}} = |R\omega_0 - u|$$

The direction of $\boldsymbol{v}_{\mathrm{abs}}$ coincides with that of the greatest of the two vectors $\boldsymbol{v}_{\mathrm{tr}}$ and $\boldsymbol{v}_{\mathrm{rel}}$.

In the example under consideration the transportation of the point M is the motion of that point of the platform with which the point M coincides at the given instant. In this sense the transportation of the point M is determined by the rotation of the platform about the fixed axis (see § 2 of Chap. 8). The transportation acceleration vector $\boldsymbol{w}_{\mathrm{tr}}$ of the point M is equal to the vector sum of its tangential and normal accelerations:

$$\boldsymbol{w}_{\mathrm{tr}} = \boldsymbol{w}_{\tau}^{\mathrm{tr}} + \boldsymbol{w}_{n}^{\mathrm{tr}}$$

Both the vectors are shown in Fig. 11.7. The angular acceleration being negative, the direction of the vector w_τ^{tr} is as shown in the figure. The moduli of these vectors are expressed by formulas (8.12) and (8.13). The relative velocity vector v_{rel} of the point M has a constant modulus, and according to the conditions of this example we have $v_{rel} = u = \text{const}$.

This however does not imply that the relative acceleration vector w_{rel} of the point M is equal to zero. Indeed, when determining the kinematic characteristics of the relative motion we must consider the motion of the particle with respect to the moving coordinate system rigidly connected with the body S. The relative motion of the point M with respect to the platform is along the rim of the platform; the linear velocity v_{rel} of this motion has a constant modulus (but varies in its direction). Hence, we again have

$$w_{rel} = w_\tau^{rel} + w_n^{rel}$$

and formulas (7.25) and (7.26) imply

$$w_\tau^{rel} = \frac{du}{dt} = 0, \quad w_n^{rel} = \frac{u^2}{R}$$

The vector w_n^{rel} is shown in Fig. 11.7. It now remains to compute the Coriolis acceleration vector w_C of the point M. By formula (11.13), we have

$$w_C = 2\,[\omega_0, \ v_{rel}]$$

and hence the modulus of the Coriolis acceleration is found from formula (11.14) (it should be taken into account that $v_{rel} \perp \omega_0$):

$$w_C = 2\omega_0 u$$

To construct the Coriolis acceleration vector let us set off from the point M a vector geometrically equal to ω_0 and then apply the right-hand screw rule. We can also apply the Joukowski rule. Since $v_{rel} \perp \omega_0$, the vector v_{rel} lies in the plane Π, and it only remains to increase it $2\omega_0$ times and to turn it through a right angle in the direction of the transportation rotation.

The absolute acceleration vector w_{abs} is found by formula (11.11). Since in the example under consideration the vector w_{tr} is equal to the vector sum of w_τ^{tr} and w_n^{tr} we have

$$w_{abs} = w_\tau^{tr} + w_n^{tr} + w_n^{rel} + w_C$$

To simplify Fig. 11.7 we do not show the construction of the vector sum of these vectors. Let us use formulas (11.12) to compute the projections of the absolute acceleration vector on the axes Mx, My and Mz:

$$w_x^{abs} = w_\tau^{tr} = R\varepsilon$$

$$w_y^{abs} = -w_n^{tr} - w_n^{rel} + w_C = -R\omega_0^2 - \frac{u^2}{R} + 2\omega_0 u, \quad w_z^{abs} = 0$$

Finally, we compute the modulus of the absolute acceleration:

$$w_{abs} = \sqrt{(w_x^{abs})^2 + (w_y^{abs})^2 + (w_z^{abs})^2} = \sqrt{R^2\varepsilon^2 + \left(R\omega_0^2 + \frac{u^2}{R} - 2\omega_0 u\right)^2}$$

EXAMPLE 11.4. A train M goes to the south along the meridian with a velocity of 20 m/s. Determine the Coriolis acceleration of the train when it is at the latitude of 60° (Fig. 11.8).

Solution. The transportation of the train M is determined by the rotation of the Earth. Let us construct at the point M the angular velocity vector ω_{tr}

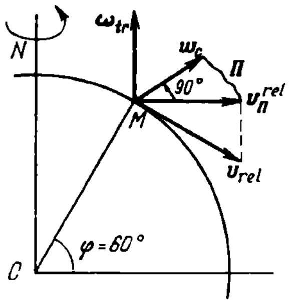

Fig. 11.8

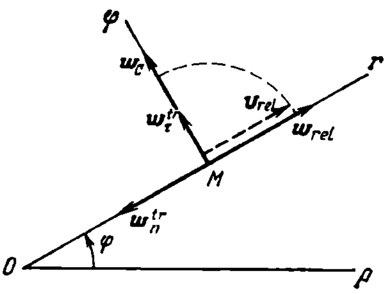

Fig. 11.9

of the rotation of the Earth (this vector is directed toward the North Star). The Earth makes one revolution during 24 hours and therefore

$$\omega_{tr} = \frac{2\pi}{24 \cdot 60 \cdot 60} = \frac{\pi}{43\,200} \text{ rad/s}$$

The modulus of the Coriolis acceleration is found by formula (11.14):

$$w_C = 2\omega_{tr} v_{rel} \sin 120° = 2\frac{\pi}{43\,200}\, 20\frac{\sqrt{3}}{2} = 0.00252 \text{ m/s}^2$$

The direction of the Coriolis acceleration can be found with the aid of the Joukowski rule. To this end we project the relative velocity vector v_{rel} on the plane П perpendicular to the vector ω_{tr} and turn it through a right angle in the direction of rotation of the Earth. The result we obtain shows that the vector w_C goes to the east, that is toward the left rail (see Fig. 11.8).

It should be noted that if the train went to the north in the northern hemisphere the Coriolis acceleration would go to the west, that is again toward the left rail. Let the reader verify this by applying the Joukowski rule or the right-hand screw rule for constructing the vector product. The dynamic effect connected with the Coriolis acceleration appearing due to the rotation of the Earth will be considered in Example 16.3 of Chap. 16.

2.3. Expression for the Acceleration of a Particle in Polar Coordinates. To conclude the present chapter we shall derive formulas expressing the acceleration of a particle in polar coordinates. We shall obtain these formulas by solving a problem concerning a compound motion of a particle (see Sec. 1.3). Namely, let us consider the motion of a small ring M sliding along a rotating rod (Fig. 11.9). The transportation acceleration w_{tr} can be found if we fix the ring M on the rotating rod whose angular velocity and angular acceleration are

$$\omega = \frac{d\varphi}{dt}, \quad \varepsilon = \frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2}$$

The modulus of the tangential component w_τ^{tr} of transportation acceleration is

$$w_\tau^{tr} = r\,|\varepsilon| = r\left|\frac{d^2\varphi}{dt^2}\right|$$

This component is perpendicular to the polar radius and its direction is specified by the sign of $d^2\varphi/dt^2$. The modulus of the normal component w_n^{tr} is

$$w_n^{\text{tr}} = r\omega^2 = r\left(\frac{d\varphi}{dt}\right)^2$$

This component is directed along the polar radius toward the centre O.

The relative motion of the point M is the rectilinear motion of the ring along the rod, and the modulus of the relative acceleration is equal to

$$w_{\text{rel}} = \left|\frac{d^2r}{dt^2}\right|$$

By formula (11.14), the modulus of the Coriolis acceleration is

$$w_{\text{C}} = 2\,|\,\omega\,|\,|\,v_{\text{rel}}\,| = 2\left|\frac{d\varphi}{dt}\right|\left|\frac{dr}{dt}\right|$$

because $\boldsymbol{\omega} \perp \boldsymbol{v}_{\text{rel}}$. The direction of the Coriolis acceleration vector can be found using the Joukowski rule; to this end the vector $\boldsymbol{v}_{\text{rel}}$ (lying in the plane Π) should be turned through a right angle in the direction of rotation of the rod, that is counterclockwise if $d\varphi/dt > 0$ and clockwise if $d\varphi/dt < 0$.

The vector representing the absolute acceleration of the point M is determined by formula (11.11):

$$\boldsymbol{w}_{\text{abs}} = \boldsymbol{w}_\tau^{\text{tr}} + \boldsymbol{w}_n^{\text{tr}} + \boldsymbol{w}_{\text{rel}} + \boldsymbol{w}_{\text{C}}$$

Let us compute the projections of $\boldsymbol{w}_{\text{abs}}$ on the polar radius OM and on the transversal $M\varphi$ (see Fig. 11.9):

$$\begin{aligned} w_{\text{rel}}^{\text{abs}} &= \frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 \\ w_\varphi^{\text{abs}} &= r\,\frac{d^2\varphi}{dt^2} + 2\,\frac{d\varphi}{dt}\,\frac{dr}{dt} = \frac{1}{r}\,\frac{d}{dt}\left(r^2\,\frac{d\varphi}{dt}\right) \end{aligned} \tag{11.16}$$

The modulus of the absolute acceleration is equal to the length of the diagonal of the rectangle constructed on the radial and transversal components:

$$w_{\text{abs}} = \sqrt{(w_{\text{rel}}^{\text{abs}})^2 + (w_\varphi^{\text{abs}})^2}$$

$$= \sqrt{\left[\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2\right]^2 + \left(r\,\frac{d^2\varphi}{dt^2} + 2\,\frac{d\varphi}{dt}\,\frac{dr}{dt}\right)^2}$$

Problems

PROBLEM 11.1. The crank CD (Fig. 11.10) rotates about the horizontal axis C with constant angular velocity $\omega_{CD} = 2\pi$ rad/s; the crank imparts rotation to the link AB (the link rotates about the horizontal axis A) with the aid of the slide block D hinged to the crank. Determine the angular velocity

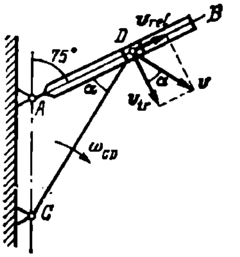

Fig. 11.10

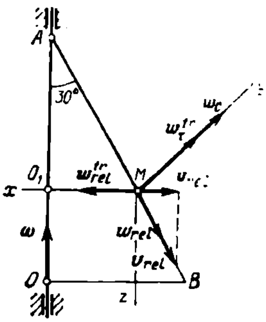

Fig. 11.11

of the link AB at the instance when the link forms an angle of 75° with the vertical CA; the other data are: $AC = 0.3$ m and $CD = 0.5$ m.

Answer. $\omega_{AB} = 2.5\pi = 7.85$ rad/s.

PROBLEM 11.2. The triangle OAB shown in Fig. 11.11 rotates about the axis OA lying in the plane of the figure, the angular velocity being $\omega = 6t^2$ rad/s. A point M moves along the hypotenuse of the triangle in the direction from the vertex A to the vertex B according to the law of motion $s = AM = (12t^2 + 4)$ cm; the angle BAO is equal to 30°. Assuming that at instant $t = 1$ s the plane of the triangle coincides with that of the figure find the absolute acceleration w of the point M at that instant.

Answer. $w = 3.66$ m/s², $\cos(\widehat{w, x}) = 0.754$, $\cos(\widehat{w, y}) = 0.656$, $\cos(\widehat{w, z}) = 0.057$.

PROBLEM 11.3. An airplane flies along the Earth's meridian from the equator to the North Pole, the velocity of the airplane having a constant modulus of 300 m/s. Find the components of the absolute acceleration vector of the airplane for the equator and for the North Pole and also compute the modulus of the total acceleration of the airplane (the radius of the Earth is equal to 6370 km).

Answer. For the equator we have $w_{tr} = 0.0337$ m/s², $w_{rel} = 0.0141$ m/s², $w_C = 0$, $w_{abs} = 0.0478$ m/s², and for the North Pole we have $w_{tr} = 0$, $w_{rel} = 0.0141$ m/s², $w_C = 0.0436$ m/s², $w_{abs} = 0.0458$ m/s².

Chapter 12 Compound Motion of a Rigid Body

Let a rigid body be in motion relative to a coordinate system $O'x'y'z'$ which, in its turn, moves relative to a fixed coordinate system $Oxyz$. We shall denote by v_M^{rel} the relative velocity of a particle M of the body in its motion with respect to the coordinate system $O'x'y'z'$ and by v_M^{tr} the transportation velocity of that particle. The absolute velocity v_M^{abs} of the particle M in its compound (resultant) motion can be found using the theorem on composition of velocities (see Sec. 1.2 of Chap. 11); the absolute velocity is found

as a vector sum:

$$v_M^{\text{abs}} = v_M^{\text{tr}} + v_M^{\text{rel}}$$

In this chapter we shall find the instantaneous distribution of the velocities of the particles of a rigid body corresponding to the resultant (compound) motion of the body for various special assumptions concerning the character of the transportation and the relative motion of the body at the given instant.

§ 1. Composition of Simplest Motions

1.1. Composition of Two Translatory Motions. The simplest case is the one when both the relative motion of a rigid body and its transportation (that is the motion of the moving coordinate system $O'x'y'z'$) are translatory (see § 1 of Chap. 8). If the body is in a translatory motion with velocity $\boldsymbol{v}_2$ relative to the coordinate system $O'x'y'z'$ which, in its turn, is in a translatory motion with velocity $\boldsymbol{v}_1$ relative to the fixed coordinate system $Oxyz$ then the absolute velocity of every point of the body is equal to the vector sum of the transportation velocity $\boldsymbol{v}_1$ and the relative velocity $\boldsymbol{v}_2$:

$$\boldsymbol{v}_M^{\text{abs}} = \boldsymbol{v}_1 + \boldsymbol{v}_2$$

The absolute velocity vector being one and the same for all the particles of the body at each instant, the *absolute motion of the body is also translatory* and its velocity is

$$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$$

1.2. Composition of Rotations about Two Intersecting Axes. Let us consider the composition of two motions which are, at the given instant, rotations about instantaneous axes with instantaneous angular velocities ω_1 and ω_2 respectively. In this section we shall consider the case when the instantaneous axes of rotation *intersect* at a point O. Let us apply the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ at the point O and add them together using the parallelogram law (Fig. 12.1). Next we compute the velocity of the end point O_1 of the diagonal of the parallelogram; to this end we use formula (8.17) for each of the rotations:

$$\boldsymbol{v}_{O_1} = [\boldsymbol{\omega}_1, \boldsymbol{OO}_1] + [\boldsymbol{\omega}_2, \boldsymbol{OO}_1]$$

The moduli of the two vector summands on the right-hand side of this equality are equal to twice the areas of the triangles constructed

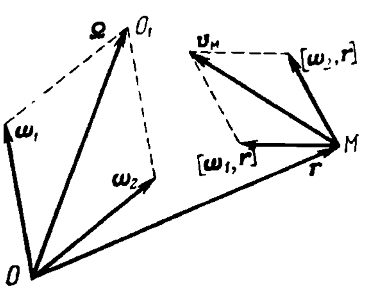

Fig. 12.1

on the vectors ω_1 and ω_2 respectively, that is they are equal to the area of the parallelogram $O\omega_2O_1\omega_1$. Both these vectors are perpendicular to the plane of the parallelogram and, obviously, have opposite directions. Therefore the velocities of both the point O_1 and the point O are equal to zero; the straight line OO_1 is the *instantaneous axis of rotation of the resultant motion.*

Let us apply the theorem on composition of velocities (see Sec. 1.2 of Chap. 11) to compute the velocity of an arbitrary point M of the body:

$$v_M = [\omega_1, r] + [\omega_2, r] = [(\omega_1 + \omega_2), r] = [\Omega, r]$$

where

$$r = OM, \qquad \Omega = \omega_1 + \omega_2$$

The comparison of the expression for the velocity v_M with Euler's formula (9.10) shows that the instantaneous resultant motion is a rotation about an instantaneous axis passing through the point O, the instantaneous angular velocity of the resultant motion being equal to the vector sum of the given instantaneous angular velocities ω_1 and ω_2. Thus, *if the vectors representing the instantaneous angular velocities of two given motions intersect then they can be added together according to the parallelogram law.*

EXAMPLE 12.1. A top (Fig. 12.2) rotates about its axis with angular velocity ω_1, and the axis of the top rotates about the vertical passing through the point of contact of the top and the horizontal plane with angular velocity ω_2 (this rotation of the axis of the top is called precession*). The resultant (compound) motion of the top is a rotation about the instantaneous axis passing along the diagonal of the parallelogram constructed on the vectors ω_1 and ω_2, the resultant instantaneous angular velocity being

$$\Omega = \omega_1 + \omega_2$$

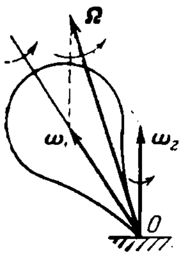

Fig. 12.2

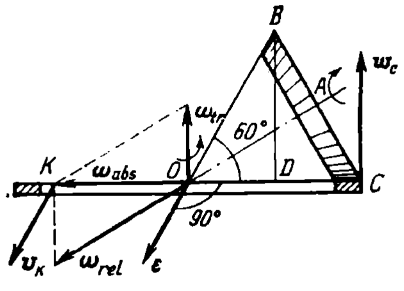

Fig. 12.3

* Generally, a spherical motion (see Sec. 2.1 of Chap. 9) of a rigid body consisting of a rotation about an axis rigidly connected with the body and a rotation of that axis about another axis, which intersects the former and is fixed relative to the frame of reference under consideration, is called *precession*. If both the rotations are uniform the precession is said to be *regular*.

EXAMPLE 12.2. The bevel gear (runner) shown in Fig. 12.3 runs round a fixed horizontal supporting gear $n = 150$ times per minute. The radius of the supporting gear is $R = 20$ cm and the vertex angle of the cone of the runner is equal to 60°. Determine the angular velocity ω_{rel} of the rolling motion of the runner on the supporting gear, the velocity of the point B and the acceleration of the point C of the runner.

Solution. Let us determine the positions of the axes of transportation, relative and absolute rotations of the runner (the axis of absolute rotation of the runner is its instantaneous axis). The velocity of the point C of the runner is equal to zero and therefore the instantaneous axis of rotation of the runner is the straight line OC. The axis of relative rotation lies in the straight line OA. Let us draw the vector $\boldsymbol{\omega}_{tr}$ of transportation angular velocity vertically upward from the point O; since the vectors $\boldsymbol{\omega}_{tr}$ and $\boldsymbol{\omega}_{rel}$ intersect at one point O we must construct the angular velocity parallelogram (see Fig. 12.3). For definiteness, let us suppose that the runner, when observed from above, runs round the supporting gear in counterclockwise direction. The modulus of the transportation angular velocity is

$$\omega_{tr} = \frac{\pi n}{30} = 5\pi \text{ rad/s}$$

From the parallelogram we find

$$\omega_{rel} = \frac{\omega_{tr}}{\sin 30^\circ} = 10\pi \text{ rad/s}, \qquad \omega_{abs} = \omega_{tr} \tan 60^\circ = 5\sqrt{3}\,\pi \text{ rad/s}$$

In order to find the absolute velocities and accelerations of the points of the runner let us consider its motion as a motion of a rigid body having one fixed point O (see § 2 of Chap. 9). The velocity of the point B is found with the aid of formula (9.12):

$$\boldsymbol{v}_B = [\boldsymbol{\omega}_{abs}, \boldsymbol{OB}]$$

The modulus of this velocity is

$$v_B = \omega_{abs} BD = 5\sqrt{3}\,\pi \cdot 0.1\sqrt{3} = 4.71 \text{ m/s}$$

where BD is the distance from the point B to the instantaneous axis. The velocity vector of the point B is perpendicular to the plane of the figure (its terminus lies behind that plane).

Before determining the accelerations of the points of the runner we must compute the angular acceleration of the runner in its absolute motion:

$$\boldsymbol{\varepsilon}_{abs} = \frac{d\boldsymbol{\omega}_{abs}}{dt}$$

The terminus of the vector $\boldsymbol{\omega}_{abs}$ (the point K) describes a circle of radius ω_{abs} in the horizontal plane. It is this circle that represents the hodograph of the angular velocity vector of the runner. The vector $\boldsymbol{\omega}_{abs}$ itself rotates about the vertical axis with angular velocity ω_{tr}. The angular acceleration of the runner is equal to the velocity of motion of the point K of the hodograph of the angular velocity:

$$\varepsilon_{abs} = v_K = \omega_{tr}\omega_{abs} = 5\pi \cdot 5\sqrt{3}\,\pi = 25\sqrt{3}\,\pi^2 \text{ rad/s}^2$$

The angular acceleration vector $\boldsymbol{\varepsilon}_{abs}$ is perpendicular to the plane of the figure (it goes from the point O towards the reader). The acceleration of the point C of the runner is determined by formula (9.17):

$$\boldsymbol{w}_C = [\boldsymbol{\varepsilon}_{abs}, \boldsymbol{OC}]$$

In the case under consideration the second component in formula (9.17) vanishes for the point C because this point lies on the instantaneous axis ($R = 0$).

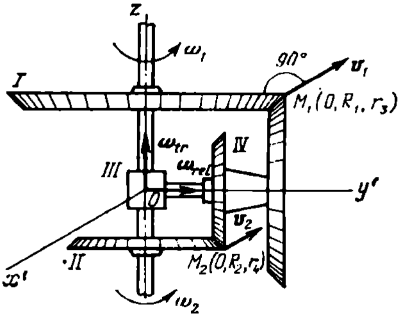

Fig. 12.4

Since the vectors ε_{abs} and OC are mutually perpendicular the modulus of the acceleration of the point C is

$$w_C = \varepsilon_{abs} OC = 25\sqrt{3}\,\pi^2 \cdot 0.2 = 85.5 \text{ m/s}^2$$

and its direction is specified by the right-hand screw rule for the vector product (namely, the acceleration vector goes vertically upward).

The example we have considered demonstrates a phenomenon of regular precession in engineering (see the footnote to the foregoing example).

EXAMPLE 12.3. Let us consider the differential gear shown in Fig. 12.4. Bevel gears I and II of radii R_1 and R_2 rotate with angular velocities ω_1 and ω_2 respectively about the vertical axes. Crank III also rotates about the vertical axis and carries a freely rotating differential pinion IV consisting of two rigidly connected bevel gears of radii r_3 and r_4. Find the angular velocity ω_{tr} of rotation of the crank and the angular velocity ω_{rel} of relative motion of the differential pinion with respect to the crank.

Solution. The moduli of the absolute velocities of the points M_1 and M_2 of gears I and II are

$$v_1 = R_1\,|\omega_1|, \qquad v_2 = R_2\,|\omega_2|$$

and the projections of these velocities on the axes of the fixed coordinate system $Ox'y'z$ are

$$v_{1x'} = -R_1\omega_1, \qquad v_{1y'} = v_{1z} = 0; \qquad v_{2x'} = -R_2\omega_2, \qquad v_{2y'} = v_{2z} = 0 \tag{1}$$

Now let us consider the motion of the differential pinion as a motion of a rigid body about the fixed point O. The resultant motion of the differential pinion is composed of two rotations:

(1) the transportation rotation—the rotation of the crank with angular velocity $\omega_{tr}\boldsymbol{k}$;

(2) the relative rotation—the rotation of the differential pinion with angular velocity $\omega_{rel}\boldsymbol{j}'$.

The instantaneous axes of both these rotations intersect at the point O, and therefore the absolute angular velocity $\boldsymbol{\omega}_{abs}$ of the differential pinion is

$$\boldsymbol{\omega}_{abs} = \omega_{rel}\boldsymbol{j}' + \omega_{tr}\boldsymbol{k}$$

The velocities of the points M_1 and M_2 of the differential pinion are found according to formula (9.12). The abscissa of each of the points M_1 and M_2 is equal

to zero, and therefore the velocity vector of any of these points is expressed by the formula

$$v = [\omega_{abs}, r] = \begin{vmatrix} i' & j' & k \\ 0 & \omega_{rel} & \omega_{tr} \\ 0 & y' & z \end{vmatrix} = (\omega_{rel} z - \omega_{tr} y')\, i'$$

that is

$$v_{x'} = \omega_{rel} z - \omega_{tr} y', \qquad v_{y'} = v_z = 0 \tag{2}$$

For the point M_1 we have $y_1' = R_1$ and $z_1 = r_3$; for the point M_2 we have $y_2' = R_2$ and $z_2 = -r_4$. Substituting these values into (2) and equating the resultant expressions to (1) we obtain the following system of two algebraic linear equations involving ω_{tr} and ω_{rel} as unknowns:

$$R_1\omega_{tr} - r_3\omega_{rel} = R_1\omega_1, \qquad R_2\omega_{tr} + r_4\omega_{rel} = R_2\omega_2$$

The solution of this system yields the required answer:

$$\omega_{tr} = \frac{R_1 r_4 \omega_1 + R_2 r_3 \omega_2}{R_1 r_4 + R_2 r_3}, \qquad \omega_{rel} = \frac{R_1 R_2 (\omega_2 - \omega_1)}{R_1 r_4 + R_2 r_3}$$

Here we everywhere used the algebraic values of the angular velocities. The solution is also valid for the case when bevel gears I and II rotate in opposite directions; in this case ω_1 and ω_2 have opposite signs. The signs of the sought-for quantities ω_{tr} and ω_{rel} specify the directions of rotation of the crank and of the differential pinion in its relative motion.

1.3. Example of Helical Motion of a Rigid Body. Let us consider the resultant motion of a rigid body composed of its rotation about a fixed axis with constant angular velocity ω and a uniform rectilinear translatory motion with velocity u parallel to ω. Each of the two motions can be regarded as the transportation (then the other must be regarded as the relative motion); this choice makes no difference because the instantaneous distribution of the velocities does not change if the roles of the motions are interchanged.

The absolute velocity of the point M (Fig. 12.5) is equal to the vector sum of the velocities of that point M in both the motions. Let us make use of formula (8.17) to write the absolute velocity vector of the point M in the form

$$v_M = [\omega, r] + u \tag{12.1}$$

The components $[\omega, r]$ and u of the vector v_M are shown in Fig. 12.5. Let us take the axis of rotation as the axis Oz and denote the coordinates of the point M relative to the fixed coordinate system by x, y and z. Then

$$\omega = \omega k, \quad u = uk, \quad r = xi + yj + zk$$

where i, j and k are unit vectors along the axes Ox, Oy and Oz. Formula (12.1) is

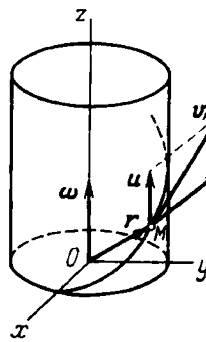

Fig. 12.5

written in full thus:

$$v_M = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} + uk = -\omega y i + \omega x j + uk$$

It follows that the projections of the absolute velocity of the point M on the coordinate axes are

$$v_x^M = -\omega y, \quad v_y^M = \omega x, \quad v_z^M = u \tag{12.2}$$

The modulus of the velocity of the point M is equal to

$$v_M = \sqrt{(v_x^M)^2 + (v_y^M)^2 + (v_z^M)^2} = \sqrt{\omega^2 R^2 + u^2} \tag{12.3}$$

where $R = \sqrt{x^2 + y^2}$ is the distance from the point M to the axis of rotation. By the way, formula (12.3) can also be derived directly if we take into account that the components of the vector v_M are mutually perpendicular and that their moduli are equal to ωR and u respectively.

Let us describe geometrically the motion under consideration. During the whole time of motion the point M remains on the surface of a right circular cylinder (see Fig. 12.5). If the point M is on an element of the cylinder at a given instant t then it again intersects that element at instant $t + T$, where $T = 2\pi/\omega$; the distance between the old position of the point on the element and the new position is equal to

$$h = uT = \frac{2\pi u}{\omega} \tag{12.4}$$

The distance h is called the *pitch of the screw*. The ratio

$$p = \frac{u}{\omega} \tag{12.5}$$

is called the *parameter of the screw*. The last formulas imply that

$$h = 2\pi p$$

Taking into account that $v_x = dx/dt$, etc. we can rewrite formulas (12.2) in the form

$$\frac{dx}{dt} = -\omega y, \quad \frac{dy}{dt} = \omega x, \quad \frac{dz}{dt} = u \tag{12.6}$$

Equalities (12.6) can be interpreted as a system of differential equations involving the coordinates of the moving point as the unknown functions. If the position of the moving point M at the initial instant $t = 0$ is $(R, 0, 0)$ then the corresponding solution of system (12.6) is

$$x = R\cos \omega t, \quad y = R\sin \omega t, \quad z = ut \tag{12.7}$$

This can be checked by differentiating expressions (12.7) and substituting them into equations (12.6).

This solution represents the equation of motion of the point M along the screw line; it was considered in Example 7.2. By virtue of formula (12.4), we have $u = \omega h/(2\pi)$. The velocity of the point and the arc length of the trajectory were found in Example 7.3.

§ 2. Composition of Rotations about Two Parallel Axes

2.1. Case of Rotations of Same Sense. Here we shall consider the case when, at a given instant, the transportation and the relative motions of a rigid body are rotations about parallel axes. We shall begin with the situation when the instantaneous angular velocity vectors ω_1 and ω_2 have the *same direction* (Fig. 12.6) (that is the corresponding rotations are of *same sense*).

The resultant motion is plane because the velocities of the points lying on a straight line parallel to the instantaneous axes are equal. Therefore it is sufficient to consider the instantaneous distribution of velocities in a plane Π perpendicular to ω_1 and ω_2. Let the plane Π intersect the lines of action of the vectors ω_1 and ω_2 at the points O_1 and O_2. For the sake of visualization let O_1 and O_2 be the points of application of the sliding vectors ω_1 and ω_2.

The velocities v_1 and v_2 of a point on the line segment O_1O_2 corresponding to the instantaneous rotations with angular velocities ω_1 and ω_2 respectively have opposite directions. Let us find a point C' on the line segment O_1O_2 whose velocity is equal to zero; for this point the equality $v_1 = v_2$ holds. We have $v_1 = O_1C'\omega_1$ and $v_2 = O_2C'\omega_2$ and therefore

$$\frac{O_1C'}{O_2C'} = \frac{\omega_2}{\omega_1} \tag{12.8}$$

The velocity of the resultant motion is equal to zero for each of the points lying on the straight line parallel to ω_1 and ω_2 and passing through the point C'. Consequently, the resultant motion is a rotation about that straight line which serves as the instantaneous axis. Let us find the modulus of the instantaneous angular velocity Ω. The mudulus of the velocity of the point O_2 is

$$v_{O_2} = O_1O_2\omega_1$$

On the other hand,

$$v_{O_2} = O_2C'\Omega$$

whence

$$\Omega = \frac{O_1O_2}{O_2C'}\omega_1 = \frac{O_1C' + C'O_2}{O_2C'}\omega_1 = \left(\frac{O_1C'}{O_2C'} + 1\right)\omega_1$$

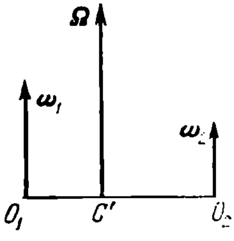

Fig. 12.6

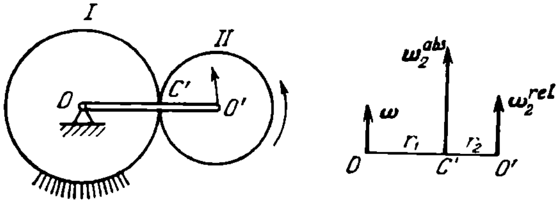

Fig. 12.7

Substituting the value of the ratio $(O_1C')/(O_2C')$ found from (12.8) into the last expression we finally obtain

$$\Omega = \left(\frac{\omega_2}{\omega_1} + 1\right)\omega_1 = \omega_1 + \omega_2 \tag{12.9}$$

Thus, in the case when the transportation and the relative motions are rotations of *same sense* with angular velocities ω_1 and ω_2 about parallel instantaneous axes the absolute motion of the rigid body is a rotation with instantaneous angular velocity $\Omega = \omega_1 + \omega_2$. The instantaneous axis of the resultant rotation lies in the plane of the instantaneous angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$; it is parallel to these velocities and divides the distance between them internally inversely proportionally to their moduli (see (12.8)).

EXAMPLE 12.4. The mechanism shown in Fig. 12.7 consists of two gears I and II of radii r_1 and r_2 and the carrier OO', the latter rotating with angular velocity ω. Gear I is fixed and gear II can freely rotate about the pin O' of the carrier. Compute the absolute angular velocity ω_2^{abs} of gear II and its angular velocity relative to the carrier.

Solution. At any given instant the point C' of gear II has zero velocity because it is the point of contact of gears I and II, the former being fixed. This point is the instantaneous centre of zero velocity of gear II; in other words, the instantaneous axis of rotation of gear II passes through that point C'.

We shall consider the rotation of the carrier as the transportation for gear II. Let us draw the vectors $\boldsymbol{\omega}$, $\boldsymbol{\omega}_2^{rel}$ and $\boldsymbol{\omega}_2^{abs}$ (they represent the transportation, the relative and the absolute angular velocities of gear II) perpendicularly to the line segment OO'. By formulas (12.8) and (12.9), we have

$$\frac{r_2}{r_1} = \frac{\omega}{\omega_2^{rel}}, \quad \omega_2^{abs} = \omega + \omega_2^{rel}$$

These relations make it possible to express ω_2^{rel} and ω_2^{abs} in terms of ω:

$$\omega_2^{rel} = \frac{r_1}{r_2}\omega, \quad \omega_2^{abs} = \omega + \frac{r_1}{r_2}\omega = \frac{r_1 + r_2}{r_2}\omega$$

2.2. Case of Rotations of Opposite Sense. Now let us consider the case when the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ representing the instantaneous angular velocities are *antiparallel*. By analogy with Fig. 12.6, let us take the plane of the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ as the plane of the figure (Fig. 12.8); here O_1O_2 is the line of intersection of the plane Π per-

pendicular to the vectors ω_1 and ω_2 with the plane of these vectors. Let $\omega_1 \neq \omega_2$ and let us suppose, for definiteness, that $\omega_1 < \omega_2$. The velocities v_1 and v_2 of a point M (these velocities correspond to the rotations with angular velocities ω_1 and ω_2 respectively) lying on the extension of the line segment O_1O_2 have opposite directions. The absolute velocity of the point C' is equal to zero when $C'O_1\omega_1 = C'O_2\omega_2$, that is when

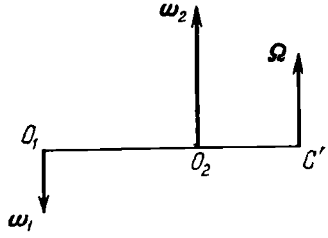

Fig. 12.8

$$\frac{C'O_1}{C'O_2} = \frac{\omega_2}{\omega_1} \tag{12.10}$$

Since this ratio exceeds unity, we have $C'O_1 > C'O_2$, which means that the point C' having zero velocity lies on the extension of the line segment O_1O_2 on the same side as the greater angular velocity. The straight line passing through the point C' and parallel to the vectors ω_1 and ω_2 is the instantaneous axis of rotation of the resultant motion. Let us compute the instantaneous angular velocity Ω of that motion. For the modulus of the velocity of the point O_2 we have

$$v_{O_2} = O_1O_2\omega_1 = C'O_2\Omega$$

whence

$$\Omega = \frac{O_1O_2}{C'O_2}\omega_1 = \frac{C'O_1 - C'O_2}{C'O_2}\omega_1 = \left(\frac{C'O_1}{C'O_2} - 1\right)\omega_1$$

Substituting the value of the ratio $(C'O_1)/(C'O_2)$ found from (12.10) into the last expression we finally obtain

$$\Omega = \left(\frac{\omega_2}{\omega_1} - 1\right)\omega_1 = \omega_2 - \omega_1 \tag{12.11}$$

Thus, in the case when the transportation and the relative motions are instantaneous rotations of *opposite sense* with angular velocities ω_1 and ω_2 $(\omega_2 > \omega_1)$ the absolute motion of the rigid body is a rotation with instantaneous angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. In the present case the last vector equality means that $\Omega = \omega_2 - \omega_1$ and that the instantaneous angular velocity $\boldsymbol{\Omega}$ goes in the direction of the greater angular velocity of the constituent rotations. The instantaneous axis of absolute rotation lies in the plane of the instantaneous angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$; it is parallel to them and divides the distance between $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ externally inversely proportionally to their moduli (see (12.10)).

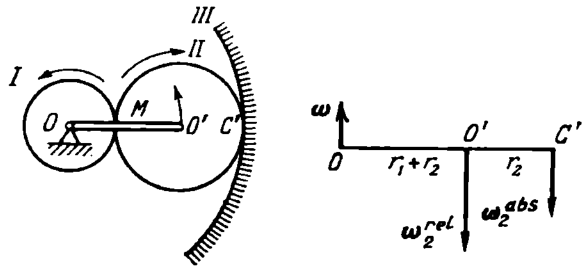

Fig. 12.9

EXAMPLE 12.5. The reduction gear shown in Fig. 12.9 consists of three gears I, II and III of radii r_1, r_2 and $r_3 = r_1 + 2r_2$ respectively and the carrier OO'. Gear I rotates about the fixed axis O; gear II can freely rotate about the pin O' of the carrier OO' and is in mesh with gear I and fixed gear III. What must be the angular velocity ω of rotation of the carrier for gear I to rotate with angular velocity ω_1? For this case what is the angular velocity ω_2^{rel} of gear II in its relative motion with respect to gear I?

Solution. The point C' is the instantaneous centre of zero velocity of gear II, in other words, the instantaneous axis of rotation of gear II passes through the point C'. Let us draw the vectors $\boldsymbol{\omega}$, $\boldsymbol{\omega}_2^{\text{rel}}$ and $\boldsymbol{\omega}_2^{\text{abs}}$ (they represent the transportation, the relative and the absolute angular velocities of gear II) perpendicularly to the line segment OC'. By formulas (12.10) and (12.11), we have

$$\frac{r_2}{r_1+2r_2}=\frac{\omega}{\omega_2^{\text{rel}}}, \qquad \omega_2^{\text{abs}}=\omega_2^{\text{rel}}-\omega$$

which yields the expressions of ω_2^{rel} and ω_2^{abs} in terms of ω:

$$\omega_2^{\text{rel}}=\frac{r_1+2r_2}{r_2}\,\omega, \qquad \omega_2^{\text{abs}}=\frac{r_1+2r_2}{r_2}\,\omega-\omega=\frac{r_1+r_2}{r_2}\,\omega$$

The modulus of the absolute velocity of the point of contact M of gears I and II is

$$v_M = 2r_2\omega_2^{\text{abs}} = 2\,(r_1+r_2)\,\omega$$

On the other hand, for the rotation of gear I we have

$$v_M = r_1\omega_1$$

Equating the expressions for v_M we obtain

$$\omega=\frac{r_1}{2\,(r_1+r_2)}\,\omega_1$$

whence, finally,

$$\omega_2^{\text{rel}}=\frac{r_1+2r_2}{r_2}\,\omega=\frac{(r_1+2r_2)\,r_1}{2r_2\,(r_1+r_2)}\,\omega_1$$

Here we have written the expression for the moduli of the angular velocities; the directions of the corresponding rotations are indicated in Fig. 12.9.

2.3. Couple of Rotations. It now remains to consider the case when the instantaneous angular velocities of the constituent motions *are antiparallel and have equal moduli*: $\boldsymbol{\omega}_1 = -\boldsymbol{\omega}_2$ $(\omega_1 = \omega_2 = \omega)$.

Such a combination of constituent motions is referred to as a *couple of rotations*. Our aim is to investigate the instantaneous motion corresponding to the couple of rotations.

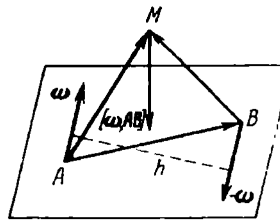

Fig. 12.10

Let us compute the velocity vector for an arbitrary point M of the body (Fig. 12.10):

$$v_M = [\boldsymbol{\omega}, \boldsymbol{AM}] + [-\boldsymbol{\omega}, \boldsymbol{BM}] = [\boldsymbol{\omega}, \boldsymbol{AM} - \boldsymbol{BM}] = [\boldsymbol{\omega}, \boldsymbol{AB}] \quad (12.12)$$

Since the vector v_M is independent of the coordinates of the point M the velocities of all the points of the body are equal at any given instant. Consequently, the *resultant instantaneous motion is translatory.*

From formula (12.12) it follows that the line of action of the velocity vector of the resultant translatory motion is perpendicular to the vectors $\boldsymbol{\omega}$ and $\boldsymbol{AB}$, that is perpendicular to the plane of the couple of rotations, and the direction of the velocity is specified by the right-hand screw rule. The modulus of this velocity is equal to the area of the parallelogram constructed on the vectors $\boldsymbol{\omega}$ and AB:

$$v = \omega h \quad (12.13)$$

where h is the distance between the vectors $\boldsymbol{\omega}$ and $-\boldsymbol{\omega}$; the quantity h is called the *arm of the couple of rotations*. The vector $\boldsymbol{v}$ is free because it represents the velocity of the instantaneous translatory motion: it is also referred to as the *(vector) moment of the couple of rotations*.

Conversely, a translatory motion with velocity $\boldsymbol{v}$ can be represented as the result of composition of two rotations forming a couple of rotations whose plane is perpendicular to $\boldsymbol{v}$, the arm h and the moduli of the angular velocities satisfying condition (12.13) and the directions of the velocity vectors corresponding to the right hand screw rule. The fact that a translatory motion can be represented as a couple of rotations means that when studying compound motion we can limit ourselves to the consideration of rotations only.

EXAMPLE 12.6. Let us consider a wheel rotating about a fixed horizontal axis (Fig. 12.11), the rod $O'A$ being hinged to the rim of the wheel. When the wheel rotates, the rod $O'A$ remains in vertical position. Investigate the character of the motion of the rod $O'A$ and the velocities of its points when the wheel is in a uniform rotation with constant angular velocity $\omega > 0$ (the positive value of the angular velocity indicates that the wheel rotates counterclockwise).

Solution. Shown in Fig. 12.11 are four positions of the rod $O'A$ corresponding to four values of the angle of rotation of the wheel differing by $\pi/2$. We shall choose the pin O' as the centre and consider the rotation of the rod about the pin O' as the relative motion. In order to investigate the relative motion let us imagine that positions *I*, *II*, *III* and *IV* of the rod are related to one and the

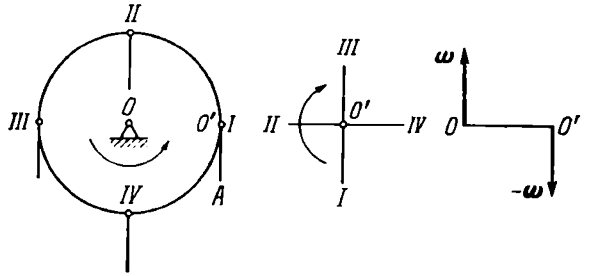

Fig. 12.11

same position of the pin O'. Then we see that during one revolution of the wheel the rod $O'A$ turns about the pin O' so that it also makes one revolution but in the clockwise direction. This means in fact that the relative motion of the rod is a rotation with angular velocity $-\omega$. Let us construct the vector $\boldsymbol{\omega}$ representing the transportation angular velocity and the vector $-\boldsymbol{\omega}$ representing the instantaneous angular velocity of the relative rotation.

Thus, the constituent motions of the rod $O'A$ form a couple of rotations and consequently the resultant (that is the absolute) motion of the rod is translatory. The vector $\boldsymbol{v}$ representing the instantaneous velocity of the rod is equal to the (vector) moment of the couple of rotations (see Fig. 12.11). The modulus v of the velocity is equal to ωR, where R is the radius of the wheel; this radius is the arm of the couple of rotations.

The fact that the motion of the rod $O'A$ is translatory is a direct consequence of the condition that the line segment $O'A$ remains vertical during the whole time of its plane motion. The velocity of the translatory motion of the rod is specified by the velocity of any of its points, for instance, by the velocity of the point O'. The point O' participates in the rotational motion of the wheel, and therefore the modulus of its velocity is equal to ωR, the velocity vector itself being directed along the tangent to the rim. In this simplest example our aim was to show that a couple of rotations is equivalent to a translatory motion.

DYNAMICS

Part 3

Introduction to Dynamics

1. Subject of Dynamics. As was mentioned in the Introduction to the present course, *dynamics studies the mechanical motion of material bodies in connection with the factors causing the motion*. These factors are the mechanical interaction between the bodies (whose measure is the force; see Sec. 2.2 of Chap. 1), the inertia of the bodies and the constraints imposed on the bodies. Dynamics deals with the general laws of mechanical motion of material bodies, and various particular kinds of their motion are considered from the point of view of the application of these general laws to the solution of special problems.

A particle, that is a material body whose dimensions are relatively small, gains an acceleration under the action of a force, and in dynamics we shall study the accelerating property of the force; in this connection we shall follow the ideas of Newton and speak of the accelerating forces. This does not mean that we consider forces different from those studied in statics. However, the notion of an accelerating force differs, for instance, from Leibniz's notion of a force measured by the kinetic energy (Leibniz suggested measuring the force by the quantity (1/2) mv^2, where m is the mass of the particle and v is the velocity). The so-called inertial forces (see Sec. 1.1 of Chap. 20) are a fictitious notion when we consider the forces acting on the body; hence the notion of an inertial force also differs from that of an accelerating force understood as the measure of the mechanical action of the other bodies on the particle (body) under consideration.

In mechanics when considering forces we abstract from their physical nature. The static measurement of a force is based on balancing that force by another force. For instance, when a dynamometer is used to measure a force this force is balanced by the elastic force of the spring. The magnitude of the force is indicated on the special graduated scale of the dynamometer.

We shall use only the International System of Units (SI) (see Sec. 2.3 of Chap. 1; on the relationship between the SI and the technical system of units see also Sec. 2.3 of Chap. 1). We remind the reader that in the SI the unit of force is the newton (N), which is

a force imparting an acceleration of 1 m/s^2 to a mass equal to that of the international standard of one kilogram-mass.

From the point of view of dynamics a force is measured by the acceleration which it produces. Such a measurement of the force is closely connected with the mass of the body understood as the measure of its inertia. The notion of a mass will be studied in greater detail in Sec. 1.1 of Chap. 13 where the laws of classical mechanics (Newton's laws) will be studied.

2. Historical Notes. The science of dynamics appeared in the 15th and the 16th centuries. That was the time when the solution of dynamic problems became particularly important for human practice and when it became clear that the investigation of the mechanical motion cannot be confined to abstract arguments only, which had been characteristic of the ancient scientists: Aristotle (384-322 B.C.) and his successor Hero of Alexandria (the 1st century A.D.). It turned out that it was necessary to resort to systematic observations and experiments. This was first realized by N. Copernicus (1473-1543) and J. Kepler (1571-1630) in connection with the study of planetary motion, that is in the field of celestial mechanics. J. Kepler thoroughly studied the extensive observation data found by the Danish astronomer T. Brahe (1546-1601) and deduced empirically the three fundamental laws of unperturbed planetary motion round the Sun. Later Kepler's laws were deduced theoretically by I. Newton (1643-1727) from the basic laws of classical mechanics and the law of gravitation.

One of the most distinguished predecessors of I. Newton was G. Galileo (1564-1642) who established the laws of free fall of bodies in vacuum. These laws included the first statement of the law of inertia and of Newton's second law concerning the accelerating forces.

The laws of oscillation of a simple pendulum (whose study had been initiated by G. Galileo) were established by Ch. Huygens (1629-1695) and R. Hooke (1635-1703).

The three basic laws of mechanics (they will be presented in Chap. 13) were established by the great English scientist I. Newton. Besides these laws he also stated a number of important consequences and solved many problems concerning the motion of particles under the action of given forces, namely the forces of attraction and the resistance forces of the medium. I. Newton also contributed many outstanding results to optics and calculus.

After the works of I. Newton the development of theoretical mechanics was closely connected with the application of mathematical methods and, first of all, of mathematical analysis. In this connection we mention the distinguished studies of L. Euler (1707-1783), particularly his famous work on analytic methods.

After I. Newton there remained some difficulties connected with setting the differential equations of motion. These difficulties were

to a certain extent overcome by J. D'Alembert (1717-1783) and J. L. Lagrange (1736-1813). The book by J. L. Lagrange devoted to analytic mechanics influenced very much the development of mechanics in the 19th century.

Among the Russian scientists of the 19th century who contributed much to the development of mechanics we must first of all mention M. V. Ostrogradsky (1801-1862) who initiated the development of analytic methods of mechanics in Russia and P. L. Chebyshev (1821-1894), a prominent mathematician who was also one of the creators of modern theory of mechanisms and machines and who himself invented more than 40 new mechanisms.

In 1888 the Russian mathematician S. V. Kovalevskaya (1850-1891) won the prize of the French Academy for her discovery of the last of the possible general cases of motion of a rigid body about a fixed point. This was one of the most difficult problems of dynamics which drew attention of such distinguished scientists as L. Euler, J. L. Lagrange, L. Poinsot (1777-1859), S. D. Poisson (1781-1840) and others. After S. V. Kovalevskaya this problem was studied by many Russian scientists: N. E. Joukowski, D. K. Bobylev, A. M. Lyapunov, P. V. Voronets, V. A. Steklov, D. N. Goryachev, S. A. Chaplygin and others.

In § 2 of Chap. 23 we shall dwell on the distinguished discoveries of K. E. Tsiolkovsky (1857-1935) and I. V. Meshchersky (1859-1935) in the theory of cosmic flights. In conclusion we mention the widely known results of A. M. Lyapunov (1857-1918), the creator of modern theory of stability of motion. The theory of stability inaugurated by A. M. Lyapunov has many highly important applications in the theory of automatic control, the theory of vibration, etc. In the development of the theory of stability in the USSR a very important role was played by N. G. Chetayev (1902-1959).

Chapter 13 Motion of a Free Particle

§ 1. Basic Laws of Classical Mechanics

1.1. Newton's Laws. A mechanical motion is a change with time of the position of one material body relative to other material bodies; thus, any motion must only be thought of as a relative one. Indeed, if there were only one particle in the infinite space then it would be impossible to specify its position, and hence the question of whether the particle moves or is at rest would be senseless. I. Newton, the creator of classical mechanics, postulated the existence of both absolute space (that is the absolutely fixed frame of reference) and absolute time, which make it possible to specify absolute motion.

However, I. Newton himself perhaps understood the limited character of these postulates (see Sec. 3 of Introduction to Kinematics).

To establish the relationship between the absolutely fixed frame of reference, absolute time and the natural phenomena it should be taken into account that the relative configuration of the stars that we observe varies so slightly that even the modern high-precision observation methods can hardly detect the variation during a year. That is why the stars, in contrast to the planets, are often said to be fixed. To describe the absolute motion of material bodies let us imagine a coordinate system whose axes occupy invariable position with respect to the fixed stars; it is these coordinate axes that will be referred to as *fixed*. The word "fixed" should be of course understood conditionally because we can give no proof of the invariability of the position of these coordinate axes. As absolute time we shall conditionally take the mean solar time whose course is uniform to within the precision of modern astronomical observations. As to the units of length and time, they were described in Sec. 2.3 of Chap. 1 and in Introduction to Kinematics. The displacement of a body relative to the system of fixed coordinate axes defined above will be called an *absolute motion*; as to the displacement of the body relative to any other coordinate system which is not invariably connected with the fixed stars, it will be spoken of as a *relative motion*. Below in Sec. 1.2 we shall see that these definitions are not completely sufficient.

The three basic general laws of mechanics were stated by I. Newton who formulated these laws for the motion of material bodies of infinitesimal dimensions and finite mass, that is, as we say, for particles.

NEWTON'S FIRST LAW (THE LAW OF INERTIA). *Any particle is in a state of rest or in a uniform rectilinear motion until some forces applied to it produce a change in the state of the particle.*

The first law indicates the property of substance known as inertia. That is why a uniform rectilinear motion of a particle is called an *inertial motion* (or *coasting*). According to the first law, a particle cannot by itself either start moving (if it is at rest) or stop moving or change the magnitude and the direction of its velocity (if it is in motion). Consequently, for the velocity of a particle to change it is necessary that an external factor should be applied to it. This external factor is the action of other bodies, that is the force.

NEWTON'S SECOND LAW (THE FUNDAMENTAL LAW OF DYNAMICS). *The change of the momentum of a particle is proportional to the force acting on the particle, the direction of the change coinciding with that of the force.*

To introduce the notion of the momentum of a particle we must define the notion of the mass of a body as a measure of the inertia of the rigid body in its translatory motion; I. Newton understood the mass as the amount of substance within the body. This is the

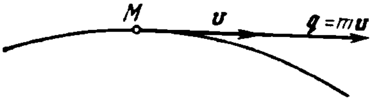

Fig. 13.1

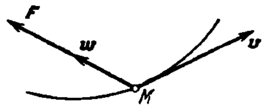

Fig. 13.2

notion of the so-called *inertial mass* which is inconvenient for quantitative measurement. Therefore the notion of the *gravitational mass* is introduced; it is defined as the ratio of the force of weight $\boldsymbol{p}$ to the acceleration of gravity $\boldsymbol{g}$ at the given place:

$$m = \frac{p}{g}$$

Physical experiments show that the inertial and the gravitational masses are equal, and in what follows we shall simply speak of the mass of a body or of a particle.

The unit of mass in the International System of Units (SI) (see Sec. 2.3 of Chap. 1) is one kilogram (kg).

By the momentum of a particle M is meant a vector $\boldsymbol{q}$ applied at the point of location of the particle and equal to the product of the mass of the particle by its velocity vector at the given instant (Fig. 13.1):

$$\boldsymbol{q} = m\boldsymbol{v}$$

Newton's second law is written in the form of the vector equality

$$\frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{F} \qquad (13.1)$$

If the mass of the particle does not change with time, that is if the mass is constant*, it follows from (13.1) that

$$m\boldsymbol{w} = \boldsymbol{F} \qquad (13.2)$$

Here $\boldsymbol{w} = d\boldsymbol{v}/dt$ is the acceleration vector of the particle (see Sec. 3.1 of Chap. 7) at the given instant. Newton's second law written in form (13.2) is referred to as the fundamental equation of particle dynamics; it reads thus:

The force is equal to the product of the mass of the particle by its acceleration.

In connection with this statement we should stipulate that the force acting on the particle and the acceleration of the particle lie in one straight line and have the same direction (Fig. 13.2).

NEWTON'S THIRD LAW (THE LAW OF ACTION AND REACTION). *To every action there corresponds an equal and opposite reaction*; in other words, two bodies exert on each other actions whose magnitudes are equal and whose directions are opposite.

* Throughout this course, except § 2 of Chap. 23, we shall suppose that the masses of the particles and of the bodies under consideration are constant.

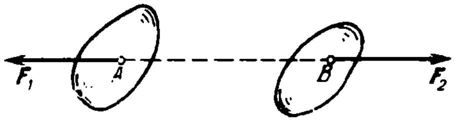

Fig. 13.3

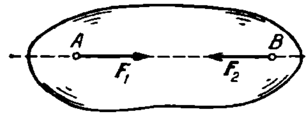

Fig. 13.4

Let us consider two bodies A and B of relatively small dimensions (Fig. 13.3). If the body B acts on the body A with a force $\boldsymbol{F}_1$ then the body A acts on the body B with a force $\boldsymbol{F}_2$, and there always holds the equality

$$\boldsymbol{F}_2 = -\boldsymbol{F}_1$$

One of these forces (any of them!) is referred to as "action" and the other is called "reaction". These forces are not mutually balanced because they are applied to different bodies (see Sec. 2.5 of Chap. 1, Axiom III).

REMARK. Let us consider a perfectly rigid body (Fig. 13.4). Suppose that a particle B of the body acts with an internal force $\boldsymbol{F}_1$ on a particle A of that body; then, according to Newton's third law, the particle B itself is acted upon by a force $\boldsymbol{F}_2$ exerted by the particle A, and there always holds the vector equality $\boldsymbol{F}_2 = -\boldsymbol{F}_1$. It follows that the *vector sum of the internal forces acting within one and the same rigid body is equal to zero.*

Below we state an important consequence established by I. Newton himself:

PRINCIPLE OF SUPERPOSITION. *If a particle is acted upon by several forces simultaneously then it gains an acceleration which is equal to the one that it would gain under the action of one force equal to the resultant of the given force system.*

If a particle M of mass m is acted upon by several forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_l$ then, according to Newton's second law and the principle of superposition, the acceleration $\boldsymbol{w}$ gained by the particle goes along the line of action of the resultant $\boldsymbol{R}$ of the given force system (Fig. 13.5). In this case the fundamental equation of particle dynamics takes the form

$$m\boldsymbol{w} = \boldsymbol{R}, \quad \text{where} \quad \boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \ldots + \boldsymbol{F}_l = \sum \boldsymbol{F} \qquad (13.3)$$

1.2. Inertial Frame of Reference. Relativity Principle of Classical Dynamics. According to Newton's law of inertia, every free particle retains its state of absolute rest or its absolute uniform rectilinear motion. Therefore there are no mechanical experiments by means of which one can find whether the given frame of reference is at absolute rest or is itself in an absolute translatory uniform rectilinear motion. We see that the idea of absolute motion loses its definiteness. Thus,

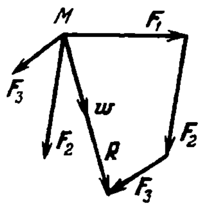

Fig. 13.5

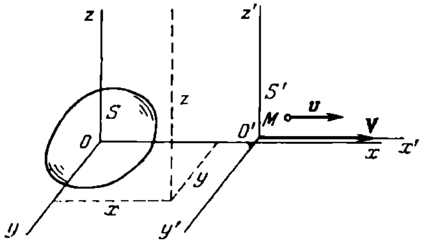

Fig. 13.6

all the frames of reference which are in rectilinear uniform translatory motion* are completely equivalent when the mechanical phenomena are studied; this is known as the relativity principle of classical dynamics established by G. Galileo. Let us dwell on this principle.

Let us consider two frames of reference S and S', for instance, the Earth** and a moving train (Fig. 13.6). Let the axis Ox go in the direction of rectilinear translatory motion of frame of reference S' relative to S. Let a particle M move along the axis Ox with velocity v and acceleration w relative to frame of reference S; then its velocity v' and acceleration w' relative to frame of reference S' are expressed by the formulas

$$v' = v - V, \quad w' = w - W \tag{13.4}$$

where V and W are the velocity and the acceleration of frame of reference S' (the train) relative to S (the Earth); for these formulas see Secs. 1.2 and 2.2 of Chap. 11.

If the particle is in an inertial motion relative to frame of reference S then for frame of reference S' we have $w' = -W \neq 0$. This means that the law of inertia loses definite meaning if we do not indicate to what frame of reference it is related. I. Newton stated the law of inertia for the absolute coordinate system. We shall assume that there exists the "fundamental" frame of reference for which the law of inertia holds.

A frame of reference S' which is in a uniform rectilinear translatory motion with velocity V relative to the fundamental frame of reference S (this means that the acceleration of S' is equal to zero: $W = 0$) is called *inertial.*

Any inertial frame of reference is equivalent to the fundamental

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* Previously it was thought that using physical experiments not involving the law of inertia, for instance, experiments with light, it was possible to detect a uniform rectilinear motion of a frame of reference; however, all the experiments of that kind gave negative results. The complete explanation of this fact was given by A. Einstein (1879-1955) who published in 1905 the special relativity principle.

** More precisely, a coordinate system with origin at the centre of the Earth and axes directed to the fixed stars.

frame of reference and can be chosen as that frame because, according to (13.4), a particle moving uniformly in S is also in a uniform motion in S'.

If the origins O and O' of two inertial frames of reference S and S' and their axes coincide at the initial instant $t = 0$ then the transformation from one inertial frame of reference to the other is expressed by the formulas

$$x = x' + Vt, \quad y = y', \quad z = z', \quad t = t' \quad (W = 0) \qquad (13.5)$$

Relations (13.5) represent the so-called *Galilean-Newtonian transformations*. Here t and t' denote time measured in the frames of reference S and S' respectively. The equality $t = t'$ means that in classical mechanics time is invariant with respect to the transformation from one inertial frame of reference to another (that is it remains invariable).

As was stated, in any inertial frame of reference S the motion of a particle is determined by Newton's second law: $m\boldsymbol{w} = \boldsymbol{F}$. When we pass from one inertial frame of reference S to another S' the acting force and the acceleration do not vary:

$$\boldsymbol{F}' = \boldsymbol{F}, \qquad \boldsymbol{w}' = \boldsymbol{w}$$

Therefore the law of motion does not change either:

$$m\boldsymbol{w}' = \boldsymbol{F}'$$

This conclusion leads to the **relativity principle of classical dynamics**: *the laws of dynamics are the same in all the inertial frames of reference (or, as we say, they are covariant with respect to transformations* (13.5)).

The meaning of this principle reduces to the assertion, which is confirmed experimentally, that a uniform rectilinear translatory motion of frame of reference S' does not produce the accelerations of bodies; besides, for the bodies moving with accelerations the uniform motion of S' does not violate the proportionality of these accelerations to the forces acting on the bodies. We shall come back to this question in Sec. 2.2 of Chap. 16.

§ 2. Differential Equations of Motion of a Free Particle

2.1. Equations of Motion in Cartesian Coordinates. Let a particle M of mass m be under the action of a given force system $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_l$ (see Fig. 13.5). We shall consider the motion of the particle M relative to an inertial (see Sec. 1.2) rectangular Cartesian coordinate system $Oxyz$. Let us write the fundamental equation of particle dynamics (see (13.3)):

$$m\boldsymbol{w} = \sum_{\lambda=1}^{l} \boldsymbol{F}_\lambda$$

Projecting this equation on the axes of the inertial coordinate system $Oxyz$ we obtain

$$m\frac{d^2x}{dt^2}=\sum_{\lambda=1}^{l}X_\lambda,\quad m\frac{d^2y}{dt^2}=\sum_{\lambda=1}^{l}Y_\lambda,\quad m\frac{d^2z}{dt^2}=\sum_{\lambda=1}^{l}Z_\lambda \qquad (13.6)$$

Here x, y and z are the coordinates of the moving particle M; d^2x/dt^2, d^2y/dt^2 and d^2z/dt^2 are the projections of the acceleration vector w of the particle M (see Sec. 3.2 of Chap. 7), and X_λ, Y_λ and Z_λ ($\lambda = 1, 2, \ldots, l$) are the projections of the λth force vector $\boldsymbol{F}_\lambda$ on the axes Ox, Oy and Oz respectively.

We shall analyse the structure of a system of differential equations (13.6); let us denote by R_x, R_y and R_z the projections of the resultant R of the forces acting on the particle M:

$$\boldsymbol{R}=\sum_{\lambda=1}^{l}\boldsymbol{F}_\lambda,\quad R_x=\sum_{\lambda=1}^{l}X_\lambda,\quad R_y=\sum_{\lambda=1}^{l}Y_\lambda,\quad R_z=\sum_{\lambda=1}^{l}Z_\lambda$$

Generally speaking, the forces acting on the particle M

(a) may depend on the position of the particle M, that is on x, y and z (for instance, such are the forces of Newtonian attraction);

(b) may depend on the velocity of the particle M, that is on $v_x = dx/dt$, $v_y = dy/dt$ and $v_z = dz/dt$ (for instance, such are resistance forces);

(c) may depend explicitly on time t (for instance, such is the perturbing (or disturbing) force producing the forced oscillation of a particle studied in Sec. 3.1 of Chap. 14). The Cases (a) and (b) involve an implicit dependence on time t, because x, y, z, dx/dt, dy/dt and dz/dt change with time.

Now, denoting the first derivatives with respect to time by the dot above and the second derivatives by two dots we conclude that the projections of the resultant are functions of x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$ and t. Thus, *a system of differential equations* (13.6) *describing the motion of the particle in terms of the projections on the inertial axes Oxyz can be written in the general form*

$$\begin{aligned}\ddot{x}&=\frac{1}{m}R_x(t;\,x,y,z;\,\dot{x},\dot{y},\dot{z})\\ \ddot{y}&=\frac{1}{m}R_y(t;\,x,y,z;\,\dot{x},\dot{y},\dot{z})\\ \ddot{z}&=\frac{1}{m}R_z(t;\,x,y,z;\,\dot{x},\dot{y},\dot{z})\end{aligned} \qquad (13.7)$$

Systems (13.6) and (13.7) are systems of ordinary differential equations of the sixth order. In the general case neither their general solution nor particular solutions can be expressed in terms of elementary functions, that is in the form of finite formulas involving power,

logarithmic, trigonometric functions, etc. of the independent variable t and integrals of these functions. That is why it is necessary to study various special classes of typical simplest problems; this question will be considered in § 3 of the present chapter and in Chaps. 14-16.

2.2. Natural Equations of Motion. Now let us introduce the natural axes (see Fig. 7.9) $M\tau nb$ instead of the Cartesian coordinate axes (here $M\tau$ is the tangent, Mn is the principal normal and Mb is the binormal to the trajectory at the point M; see Sec. 3.3 of Chap. 7). According to formulas (7.25a) and (7.26), the projections of the acceleration vector on these axes are

$$w_\tau = \frac{dv_\tau}{dt}, \quad w_n = \frac{v^2}{\rho}, \quad w_b \equiv 0$$

respectively. Projecting fundamental equation (13.3) on the natural axes we obtain the *natural equations of motion of a particle* (Euler's form of the equations of motion):

$$m\frac{dv_\tau}{dt} = \sum F_\tau, \quad m\frac{v^2}{\rho} = \sum F_n, \quad 0 = \sum F_b \qquad (13.8)$$

Here v_τ is the algebraic velocity of the particle (that is the projection of the velocity vector of the particle on the tangent; see (7.13)), ρ is the radius of curvature of the trajectory at the instant under consideration, and $\sum F_\tau$, $\sum F_n$ and $\sum F_b$ are the algebraic sums of the projections of all the forces applied to the particle on the tangent, the principal normal and the binormal respectively. From the last equation of system (13.8) it follows that the resultant force R as well as the acceleration w of the particle lie in the osculating plane (see Sec. 3.3 of Chap. 7).

2.3. First Basic Problem of Particle Dynamics. Each of the equations of system (13.6) connects two quantities: the projection of the acceleration of the particle and the projection of the resultant force on the corresponding axis of the inertial coordinate system. These equations make it possible to solve the so-called two basic problems of particle dynamics.

The *first basic problem of particle dynamics* is stated thus: knowing the mass and the character of the motion of the particle, that is knowing the equa ions of motion of the particle which have the form

$$x = f(t), \quad y = g(t), \quad z = h(t) \qquad (13.9)$$

in an inertial rectangular Cartesian coordinate system, it is required to find the force acting on the particle.

Solution. Differentiating twice equations (13.9) we find

$$\frac{d^2x}{dt^2} = \ddot{f}(t), \quad \frac{d^2y}{dt^2} = \ddot{g}(t), \quad \frac{d^2z}{dt^2} = \ddot{h}(t)$$

Substituting the projections of the acceleration vector of the particle thus found into equations (13.6) we obtain

$$m\ddot{f}(t)=\sum X, \quad m\ddot{g}(t)=\sum Y, \quad m\ddot{h}(t)=\sum Z \qquad (13.10)$$

Further, we have $\sum X = R_x$, $\sum Y = R_y$ and $\sum Z = R_z$, where R_x, R_y and R_z are the projections of the resultant force on the axes Ox, Oy and Oz respectively. The modulus of the resultant is found with the aid of the formula

$$R=\sqrt{R_x^2+R_y^2+R_z^2}=m\sqrt{\ddot{f}(t)^2+\ddot{g}(t)^2+\ddot{h}(t)^2} \qquad (13.11)$$

and its direction is specified by the direction cosines:

$$\cos(\widehat{R,x})=\frac{R_x}{R}=\frac{\ddot{f}(t)}{\sqrt{\ddot{f}(t)^2+\ddot{g}(t)^2+\ddot{h}(t)^2}}$$

$$\cos(\widehat{R,y})=\frac{R_y}{R}, \quad \cos(\widehat{R,z})=\frac{R_z}{R} \qquad (13.12)$$

Thus, the problem of finding the resultant force acting on the particle from the given equations of motion is solved by differentiating equations of motion (13.9). Hence, the solution of the first basic problem of particle dynamics is always possible and encounters no difficulties.

EXAMPLE 13.1. Let the equations of motion of a particle M under the action of one force F be

$$x=a\cos\omega t, \quad y=b\sin\omega t, \quad z=0 \qquad (13.13)$$

Find the acting force.

Solution. The particle M is in a plane motion along a trajectory whose equation is found by eliminating time t from (13.13):

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Thus, the particle M moves in an ellipse (Fig. 13.7). The projections of the force F are found using formulas (13.10):

$$X=m\ddot{x}=-ma\omega^2\cos\omega t$$

$$Y=m\ddot{y}=-mb\omega^2\sin\omega t, \quad Z=m\ddot{z}=0$$

By formulas (13.11) and (13.12), we have

$$F=m\omega^2\sqrt{a^2\cos^2\omega t+b^2\sin^2\omega t}$$

$$=m\omega^2\sqrt{x^2+y^2}=m\omega^2 r$$

$$\cos(\widehat{F,x})=-\frac{ma\omega^2\cos\omega t}{m\omega^2 r}=-\frac{x}{r}$$

$$\cos(\widehat{F,y})=-\frac{y}{z}$$

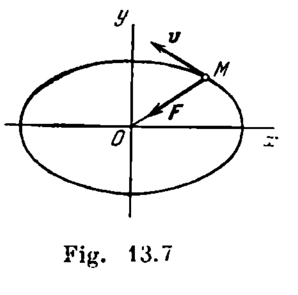

Fig. 13.7

Here r is the radius vector OM. We thus conclude that the modulus of the force is directly proportional to that of the radius vector, and the direction of the force is opposite to that of the radius vector (see Fig. 13.7). By the way, this result is an obvious consequence of the formula

$$F = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} = -m\omega^2(\boldsymbol{i}a\cos\omega t + \boldsymbol{j}b\sin\omega t) = -m\omega^2(x\boldsymbol{i} + y\boldsymbol{j}) = -m\omega^2\boldsymbol{r}$$

expressing the force.

It should be noted that the planets also move round the Sun in elliptic orbits; however, the Sun is not at the centre of the ellipse along which a planet moves but at one of its foci (this is known from Kepler's first law), and the attractive force is not proportional to the distance but is inversely proportional to its square (this is known from Newton's law of gravitation). Besides, the equations of motion of a planet are much more complicated than (13.13).

2.4. Second Basic Problem of Particle Dynamics. The second *basic problem of particle dynamics* is stated thus: the system of forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_l$ applied to a particle and the mass of the particle are known; it is required to determine the motion of the particle. To solve the problem it is sufficient to set the equations of motion of the particle of form (13.9) and then, using the methods of kinematics, to find from these equations the trajectory and also the velocity and the acceleration of the particle (the last two quantities are functions of time t). The second basic problem of particle dynamics reduces to the integration of a system of differential equations (13.7) whose right-hand sides are known since the force system $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_l$ is known. As was indicated in Sec. 2.1, in the general case it is impossible to solve this problem. The second basic problem of particle dynamics can be solved exactly only for some special cases. In more general cases the problem can be solved approximately using the methods of numerical integration involving electronic computers.

The integration of system (13.7) leads to the appearance of arbitrary constants, their number, in the general case, being equal to the order of the system, that is to six. These constants can be found from the initial conditions; for $t = t_0$ the initial conditions have the form

$$\begin{aligned} x(t_0) &= x_0, & y(t_0) &= y_0, & z(t_0) &= z_0 \\ \dot{x}(t_0) &= v_x^0, & \dot{y}(t_0) &= v_y^0, & \dot{z}(t_0) &= v_z^0 \end{aligned} \qquad (13.14)$$

The initial conditions uniquely determine the (particular) solution of system (13.7), that is the position of the particle in space and its velocity at any instant t:

$$\begin{aligned} x &= x(t), & y &= y(t), & z &= z(t) \\ v_x &= \dot{x}(t), & v_y &= \dot{y}(t), & v_z &= \dot{z}(t) \end{aligned} \qquad (13.15)$$

The two basic problems of particle dynamics mentioned above can also be solved using natural equations of motion (13.8).

EXAMPLE 13.2. A particle M of mass m kg is thrown at an angle α to the horizon with initial velocity v_0 m/s. During its motion the particle undergoes the action of the resistance force of the air $S = -\varkappa mgv$ N*, where $\varkappa$ is a constant proportionality factor and v is the velocity of the particle measured in m/s. Determine the maximum height H the particle attains, the time τ required to attain that maximum height and also the horizontal distance s at time τ.

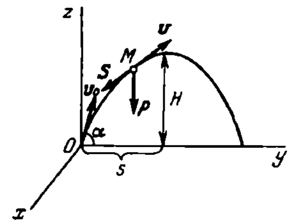

Fig. 13.8

Solution. We shall place the origin O at the initial position of the particle; let the axis Oz go vertically upward and the axis Oy be horizontal and lie in the plane passing through the axis Oz and the vector v_0 (Fig. 13.8).** The particle M is under the action of two forces: the force of gravity $p = -mgk$ (here k is unit vector along the axis Oz) and the resistance force S. Fundamental equation of particle dynamics (13.3) is written in the form

$$mw = -mgk - \varkappa mgv$$

Projecting this equation on the axes Ox, Oy and Oz and cancelling by m we arrive at a system of differential equations (13.7):

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -\varkappa g v_y, \quad \frac{d^2z}{dt^2} = -g - \varkappa g v_z$$

where $v_y = dy/dt$ and $v_z = dz/dt$ are the projections of the velocity vector v of the particle on the axes Oy and Oz. Let us take these projections and the quantity $v_x = dx/dt$ as intermediate variables and write

$$\frac{dv_x}{dt} = 0, \quad \frac{dv_y}{dt} = -\varkappa g v_y, \quad \frac{dv_z}{dt} = -g(1 + \varkappa v_z)$$

The separation of variables yields

$$dv_x = 0, \quad \frac{dv_y}{v_y} = -\varkappa g\, dt, \quad \frac{dv_z}{1 + \varkappa v_z} = -g\, dt \tag{13.16}$$

Integrating these equations we find

$$v_x = c_1, \ \ln v_y - \ln c_2 = -\varkappa gt, \qquad \frac{1}{\varkappa} \ln(1 + \varkappa v_z) - \frac{1}{\varkappa} \ln c_3 = -gt$$

where c_1, c_2 and c_3 are arbitrary constants. From the last formulas we find

$$v_x = c_1, \quad v_y = c_2 e^{-\varkappa gt}, \quad v_z = \frac{1}{\varkappa}(c_3 e^{-\varkappa gt} - 1)$$

Now let us make use of the last three initial conditions (13.14); namely, let

$$v_x(0) = 0, \quad v_y(0) = v_0 \cos\alpha, \quad v_z(0) = v_0 \sin\alpha$$

or $t = 0$. Then we find

$$c_1 = 0, \quad c_2 = v_0 \cos\alpha, \quad c_3 = \varkappa v_0 \sin\alpha + 1$$

* This force was investigated by I. Newton; see Sec. 2 of Introduction to Dynamics.

** The coordinate system we have chosen is not inertial; however, the error appearing due to this choice is negligibly small; in this connection see Sec. 2.2 of Chap. 16.

Thus, for any instant $t \geqslant 0$ we have

$$\frac{dx}{dt} = v_x = 0, \quad \frac{dy}{dt} = v_y = v_0 e^{-\varkappa g t} \cos\alpha$$
$$\frac{dz}{dt} = v_z = \frac{1}{\varkappa}\left[(1+\varkappa v_0 \sin\alpha)\, e^{-\varkappa g t} - 1\right] \tag{13.17}$$

It should be noted that it is simpler to introduce the initial conditions (when it is possible) into the limits of integration, that is to consider the variation of time from 0 to t and the variations of the projections of the velocity from $v_x^0 = 0$, $v_y^0 = v_0 \cos\alpha$ and $v_z^0 = v_0 \sin\alpha$ to v_x, v_y and v_z respectively; then we obtain from (13.16) the relations

$$v_x = 0, \quad \int\limits_{v_0 \cos\alpha}^{v_y} \frac{dv_y}{v_y} = -\varkappa g \int\limits_0^t dt, \quad \int\limits_{v_0 \sin\alpha}^{v_z} \frac{dv_z}{1+\varkappa v_z} = -g \int\limits_0^t dt$$

From these relations it is easier to derive formulas (13.17) expressing the last three equalities entering into the sought-for particular solution of form (13.15). The separation of variables in (13.17) yields

$$dx = 0, \; dy = v_0 e^{-\varkappa g t} \cos\alpha \cdot dt, \; dz = \frac{1}{\varkappa}\left[(1+\varkappa v_0 \sin\alpha)\, e^{-\varkappa g t} - 1\right] dt$$

Performing the integration we find

$$x = c_4, \; y = -\frac{v_0 \cos\alpha}{\varkappa g} e^{-\varkappa g t} + c_5, \; z = -\frac{1+\varkappa v_0 \sin\alpha}{\varkappa^2 g} e^{-\varkappa g t} - \frac{t}{\varkappa} + c_6$$

Let us use the first three initial conditions (13.14), namely the conditions $x(0) = y(0) = z(0) = 0$ for $t = 0$; this yields

$$c_4 = 0, \; c_5 = \frac{v_0 \cos\alpha}{\varkappa g}, \; c_6 = \frac{1+\varkappa v_0 \sin\alpha}{\varkappa^2 g}$$

For the first three equalities in the sought-for particular solution of form (13.15) we have

$$x = 0, \quad y = \frac{v_0 \cos\alpha}{\varkappa g}\left(1 - e^{-\varkappa g t}\right)$$
$$z = \frac{1+\varkappa v_0 \sin\alpha}{\varkappa^2 g}\left(1 - e^{-\varkappa g t}\right) - \frac{t}{\varkappa} \tag{13.18}$$

Equalities (13.18) represent parametric equations of the trajectory of the particle M. The particle moves upward during time τ; when it achieves the highest position the vertical component of its velocity vanishes; by virtue of (13.17) this yields the equation

$$(1 + \varkappa v_0 \sin\alpha)\, e^{-\varkappa g \tau} - 1 = 0$$

whence

$$\tau = \frac{1}{\varkappa g} \ln(1 + \varkappa v_0 \sin\alpha)$$

Substituting the value of τ thus found into (13.18) we determine s and H:

$$s = y(\tau) = \frac{v_0^2 \sin 2\alpha}{2g(1+\varkappa v_0 \sin\alpha)}$$
$$H = z(\tau) = \frac{v_0 \sin\alpha}{\varkappa g} - \frac{1}{\varkappa^2 g} \ln(1 + \varkappa v_0 \sin\alpha) \tag{13.19}$$

§ 3. Integration of Differential Equations of Motion of a Particle for Simplest Cases of Rectilinear Motion

In the case when the resultant force R (see (13.3)) has invariable direction and the initial velocity of a particle goes along the line of action of R (or is equal to zero) the motion of the particle is rectilinear. Let us choose a positive direction along the rectilinear trajectory and take this straight line as the axis Ox. When studying a rectilinear motion it is more convenient to consider not the force, the velocity and the acceleration vectors of the particle but the algebraic values of these quantities; the directions of the corresponding vectors are then indicated by the algebraic signs of the quantities. These algebraic values are the projections of the vectors under consideration on the axis Ox. Since the projections on any other axis perpendicular to Ox are identically equal to zero we can omit, for the sake of brevity, the subscript indicating the coordinate axis and write, for instance, v instead of v_x and denote the modulus of the velocity by $|v|$.

Let us write the differential equation of rectilinear motion of the particle in the form

$$m\frac{d^2x}{dt^2}=X(x,\ v,\ t) \tag{13.20}$$

(see (13.3) and (13.7)), where X is the algebraic value of the resultant force and $v = dx/dt$ is the algebraic value of the velocity. We have thus arrived at the second-order differential equation whose general solution

$$x = x(t,\ c_1,\ c_2)$$

involves two arbitrary constants. These constants can be found from the initial conditions; for instance, these conditions are of the form

$$x(0) = x_0, \qquad v(0) = v_0 \tag{13.21}$$

for $t = 0$. In the general case the solution of equation (13.20) is connected with considerable mathematical difficulties. However when the equation involves only one of the variables t, v or x it can be integrated by quadratures.

3.1. Case when Force Depends Solely on Time. In this case equation (13.20) has the form

$$\frac{dv}{dt}=\frac{1}{m}X(t)$$

The separation of variables yields

$$dv=\frac{1}{m}X(t)\,dt$$

Further, integrating the last equation for initial conditions (13.21) we obtain

$$\int_{v_0}^{v} dv = \frac{1}{m} \int_{0}^{t} X(t)\, dt$$

We have found a first integral of the equation of motion; denoting its right-hand side by $f(t)$ we write

$$v = v_0 + f(t) \tag{13.22}$$

Substituting $v = dx/dt$ and separating the variables once again we obtain

$$dx = [v_0 + f(t)]\, dt$$

Finally, performing the repeated integration for initial conditions (13.21) we find

$$x = x_0 + v_0 t + \int_{0}^{t} f(t)\, dt \tag{13.23}$$

EXAMPLE 13.3. Let the force X in the foregoing example be constant; we shall denote by a the constant algebraic value of the acceleration:

$$\frac{X}{m} = a$$

Then for the function $f(t)$ we have

$$f(t) = \frac{1}{m} \int_{0}^{t} X\, dt = at \quad \text{and} \quad \int_{0}^{t} f(t)\, dt = \int_{0}^{t} at\, dt = \frac{1}{2} at^2$$

First integral (13.22) and equation of motion (13.23) yield the formulas for the velocity and the abscissa of the particle in its uniformly variable rectilinear motion:

$$v = v_0 + at, \qquad x = x_0 + v_0 t + \frac{1}{2} at^2$$

EXAMPLE 13.4. A particle lies on a rough inclined plane with angle of inclination α; it is supported by a thread. At the initial instant $t = 0$ the thread is cut. Investigate the motion of the particle assuming that the friction force obeys the Amonton-Coulomb law (see Sec. 1.1 of Chap. 4).

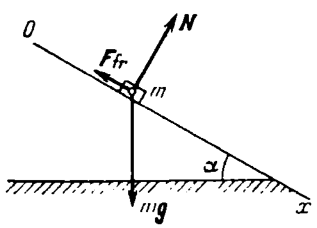

Fig. 13.9

Solution. Let us draw the axis Ox along the inclined plane (Fig. 13.9). The component of the force of weight perpendicular to the inclined plane is balanced by the normal component N of the reaction. Equation (13.20) is written in the form

$$m \frac{d^2x}{dt^2} = X = mg \sin \alpha - F_{fr}$$

By formulas (4.1) and (4.2) we have

$$F_{fr} \leqslant fN = fmg \cos \alpha$$

Therefore if

$$mg \sin \alpha \leqslant fmg \cos \alpha, \quad \text{that is} \quad \tan \alpha \leqslant f$$

then the particle remains at rest ($X = 0$). Let us suppose that $\tan \alpha > f$; then

$$X = mg (\sin \alpha - f \cos \alpha) > 0$$

In this case the particle starts moving with constant acceleration (see the foregoing example):

$$a = \frac{X}{m} = g (\sin \alpha - f \cos \alpha)$$

3.2. Case when Force Depends Solely on Velocity. Such cases are usually encountered when we consider the motion of a particle taking into account the resistance forces. Equation (13.20) takes the form

$$m \frac{dv}{dt} = X (v) \tag{13.24}$$

Equation (13.24) is integrated by quadratures after the separation of variables:

$$m \int_{v_0}^{v} \frac{dv}{X (v)} = \int_0^t dt$$

Let us denote by $\varphi (v)$ the left-hand member of the last equality and write the first integral thus obtained as

$$\varphi (v) = t \tag{13.25}$$

From (13.25) we find the inverse function $v = v (t)$; the substitution of the inverse function into the equality $v = dx/dt$ yields

$$\frac{dx}{dt} = v (t)$$

Performing the separation of variables once again and integrating (for the same initial conditions (13.21)) we obtain the equation of motion in the form

$$x = x_0 + \int_0^t v (t) \, dt \tag{13.26}$$

It should be noted that from equation (13.24) we can directly derive the relationship between x and v. Since

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx} v \tag{13.27}$$

equation (13.24) yields

$$m \frac{v \, dv}{X (v)} = dx$$

Consequently, the first integral has the form

$$m \int_{v_0}^{v} \frac{v \, dv}{X (v)} = x - x_0 \tag{13.28}$$

EXAMPLE 13.5. Let us come back to Example 13.4 and suppose that, besides the force of Coulomb friction, the particle is under the action of an additional friction force which depends linearly on the velocity, the proportionality factor being km.

Solution. Equation (13.20) takes the form

$$m\frac{dv}{dt}=mg(\sin\alpha-f\cos\alpha)-kmv$$

(here we suppose that $\tan\alpha>f$ because if $\tan\alpha\leqslant f$ the particle does not move). By analogy with Example 13.4, let us denote

$$g(\sin\alpha-f\cos\alpha)=a>0$$

Cancelling by m and separating the variables we write the differential equation of motion in the form

$$\frac{dv}{a-kv}=dt$$

The integration of this equation for the initial condition $v(0)=0$ yields

$$\int_0^v\frac{dv}{a-kv}=\int_0^t dt$$

We thus obtain first integral (13.25):

$$-\frac{1}{k}\ln(a-kv)\Big|_0^v=t$$

It follows that

$$\ln\frac{a-kv}{a}=-kt$$

whence we find

$$\frac{a-kv}{a}=e^{-kt},\quad\text{that is}\quad v=\frac{a}{k}(1-e^{-kt})$$

Since

$$\lim_{t\to\infty}e^{-kt}=0$$

we see that

$$\lim_{t\to\infty}v=\frac{a}{k},\quad v(t)<v(\infty)=\frac{a}{k}\quad(t>0)$$

By formula (13.26), the path travelled by the moving particle is equal to

$$x=\int_0^t\frac{a}{k}(1-e^{-kt})\,dt=\left[\frac{a}{k}t+\frac{a}{k^2}e^{-kt}\right]_0^t=\frac{a}{k}t-\frac{a}{k^2}(1-e^{-kt})$$

This is the law of rectilinear motion of a particle along an inclined plane for the case when the expression of the friction force involves additional linear dependence of the force on the velocity. If the force of Coulomb friction were not taken into account we would have $f=0$, that is $a=g\sin\alpha$.

EXAMPLE 13.6. A particle of mass m kg falls in the air without initial velocity. The resistance force of the air is $S=k^2mgv^2$ N, where v is the algebraic value of the velocity and k^2mg is the proportionality factor. Find the velocity of the particle at instant t s (the initial instant is $t=0$) and also the limiting velocity.

Solution. Let us place the origin at the initial position of the particle and draw the axis Ox in the positive direction of the force of gravity. The particle is in motion under the action of two forces: the force of gravity $\boldsymbol{p} = m\boldsymbol{g}$ directed vertically downward and the resistance force $\boldsymbol{S}$ directed vertically upward (Fig. 13.10) because the direction of a resistance force is always opposite to that of motion. Equation (13.24) takes the form

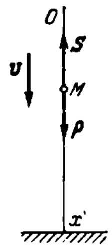

Fig. 13.10

$$m\frac{dv}{dt} = mg - S$$

Substituting the value $S = k^2mgv^2$ and cancelling by m we obtain

$$\frac{dv}{dt} = g(1 - k^2v^2)$$

Next we separate the variables and perform integration for the initial condition $v(0) = 0$:

$$\int_0^v \frac{dv}{1-k^2v^2} = g\int_0^t dt$$

The left-hand member of this equality involves a tabular integral; making use of the expression of this integral we find

$$\frac{1}{2k}\ln\frac{1+kv}{1-kv} = gt$$

because the substitution of the lower limit of integration results in zero. We have obtained a first integral of form (13.25); from this integral we find

$$\frac{1+kv}{1-kv} = e^{2kgt}$$

Let us resolve the last equation with respect to v:

$$v = \frac{1}{k}\frac{e^{kgt}-e^{-kgt}}{e^{kgt}+e^{-kgt}} = \frac{1}{k}\tanh(kgt)$$

(the fraction in this formula is a function known as the hyperbolic tangent). Let us pass to the limit for $t \to \infty$ and find the limiting value v_{lim} of the velocity:

$$v_{\text{lim}} = \lim_{t\to\infty}\frac{1}{k}\frac{e^{kgt}-e^{-kgt}}{e^{kgt}+e^{-kgt}} = \frac{1}{k}$$

It should be noted that in order to find the abscissa of the falling particle as function of time it is necessary to perform integration according to formula (13.26). In case we are interested in the dependence of the abscissa on the velocity (or in the inverse dependence) it is necessary to perform integration using formula (13.28).

3.3. Case when Force Depends Solely on the Position of the Moving Particle. By virtue of (13.27), equation (13.20) can be written in the form

$$mv\frac{dv}{dx} = X(x) \qquad (13.29)$$

We separate the variables and integrate the differential equation taking into account initial conditions (13.21):

$$\int_{v_0}^{v} v\,dv = \frac{1}{m}\int_{x_0}^{x} X(x)\,dx$$

Let us denote the integral on the right-hand side by $\psi(x)$; then the first integral we have obtained takes the form

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \psi(x), \quad \text{that is } v = \sqrt{v_0^2 + 2\psi(x)} \qquad (13.30)$$

Since v denotes the algebraic value of the velocity the sign in front of the square root must correspond to the physical meaning of the solution of the problem. Like in the cases studied in Secs. 3.1 and 3.2, in order to derive the equation of motion from the first integral we must substitute $v = dx/dt$ into it and once again separate the variables and integrate:

$$\int_{x_0}^{x} \frac{dx}{\sqrt{v_0^2 + 2\psi(x)}} = t$$

From the last relation we determine the inverse function and thus find the equation of motion

$$x = x(t)$$

EXAMPLE 13.7. A heavy particle M is thrown vertically upward from the surface of the Earth with a large initial velocity v_0. Determine the height the particle attains taking into account that the attractive force (of the Newtonian gravitation) is inversely proportional to the square of the distance from the particle to the centre of the Earth (the air resistance is neglected).

Solution. Let us draw the axis Ox as shown in Fig. 13.11. The point M on the vertical represents the position of the particle, for some instant t, at a distance x from the centre of the Earth for the upward motion of the particle. The vector $\boldsymbol{P}(x)$ represents the gravitational force applied to the particle. According to the law of gravitation we have

$$\frac{P(x)}{P(R)} = \frac{R^2}{x^2} \qquad (13.31)$$

where $P(R)$ is the value of the force of attraction on the surface of the Earth which we assume here to be equal to the force of gravity mg*. It follows that

$$P(x) = \frac{mgR^2}{x^2}$$

Now we write the differential equation of motion in form (13.29):

$$mv\frac{dv}{dx} = -\frac{mgR^2}{x^2}$$

whence

$$\int_{v_0}^{v} v\,dv = -gR^2\int_{R}^{x}\frac{dx}{x^2}$$

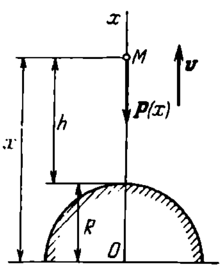

Fig. 13.11

* See Example 16.4 in Sec. 2.2 of Chap. 16.

Performing integration we obtain

$$\frac{1}{2}v^2-\frac{1}{2}v_0^2=gR^2\left.\frac{1}{x}\right|_R^x=gR^2\left(\frac{1}{x}-\frac{1}{R}\right)$$

This yields a first integral of form (13.30):

$$v=\sqrt{v_0^2-2gR\left(1-\frac{R}{x}\right)}$$

Equating v to zero we find

$$x=\frac{2gR^2}{2gR-v_0^2}$$

and hence the maximum height h the particle attains is

$$h=x-R=\frac{Rv_0^2}{2gR-v_0^2}$$

For $v_0^2 \to 2gR$ the maximum height h tends to infinity. The corresponding value

$$v_0=\sqrt{2gR}$$

of the initial velocity is termed the *escape velocity*; this is the velocity required for a heavy particle to overcome the action of the Earth's gravitational field. Let us compute its value:

$$v_{\mathrm{esc}}=\sqrt{2\cdot 9.81\cdot 6.37\cdot 10^6}=11\ 170\ \mathrm{m/s} \qquad (13.32)$$

In conclusion we mention that of great practical interest is the motion of a particle under the action of the elastic force of a spring proportional to the elongation of the spring; this question will be thoroughly discussed in Chap. 14.

Problems

PROBLEM 13.1. Let us consider a particle acted upon by its force of gravity only; let the initial velocity v_0 be directed along the vertical. We shall draw the axis Oz vertically upward. Determine the law of motion of the particle and find the instants $\tilde{t}$ at which the particle occupies the position $z = 0$.

Answer. $z=z_0+v_0t-\frac{1}{2}gt^2$, where $z_0=z(0)$;

if $z_0>0$ then $\tilde{t}=\frac{1}{g}\left(v_0+\sqrt{v_0^2+2z_0g}\right)$;

if $z_0\leqslant 0$ and $v_0^2+2z_0g\geqslant 0$ then $t_{1,2}=\frac{1}{g}\left(v_0\mp\sqrt{v_0^2+2z_0g}\right)$;

if $z_0<0$ and $v_0^2+2z_0g<0$ then the particle does not attain the position $z=0$ (here v_0 is the algebraic value of the initial velocity).

PROBLEM 13.2. A particle of mass 4 kg undergoes repulsion from a fixed centre, the repulsive force being $\boldsymbol{F}$. The particle moves away from the centre along a straight line and its velocity is proportional to the square of the distance r from the particle to the centre. Knowing that $r_0 = 2$ m and $v_0 = 3$ m/s at the initial instant find the modulus of the force at that instant.

Answer. $F_0 = 36$ N.

PROBLEM 13.3. In order to determine the law of resistance of the medium and to find the resistance force as function of the velocity a body of mass m is made to move under the action of a constant force $\boldsymbol{P}$; it turns out that the law of motion is $s = bt^3$. Find the expression for the resistance force.

Answer. $R=P-2m\sqrt{3bv}$.

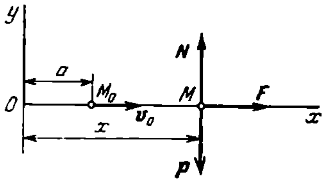

Fig. 13.12

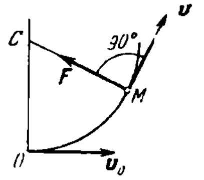

Fig. 13.13

PROBLEM 13.4. A particle M of mass m is in motion on a smooth horizontal plane away from a fixed centre O under the action of a repulsive force F. the magnitude F of the repulsive force is proportional to the distance from the particle to the centre: $F = k^2mx$. At the initial instant the particle is in the position M_0 (Fig. 13.12) at a distance a from the centre O and has a velocity v_0 directed to the right along the axis Ox. Find the law of motion of the particle M.

Answer. $x = \frac{1}{2} a\,(e^{kt} + e^{-kt}) + \frac{v_0}{2k}(e^{kt} - e^{-kt})$.

PROBLEM 13.5. A free particle of mass m receives an initial velocity v_0 and then moves under the action of a force which is directly proportional to the velocity and is perpendicular to the velocity, the proportionality factor being equal to km (Fig. 13.13). Find the trajectory of the particle and its law of motion along the trajectory.

Hint. Make use of Euler's equation (13.8).

Answer. The trajectory is a circle of radius v_0/k with centre C on the perpendicular to the velocity erected from the moving particle, the law of motion is $s = v_0t$.

Chapter 14 Rectilinear Oscillation of a Particle

§ 1. Harmonic Oscillation

There is a wide class of mechanical motions whose course is repeated again and again at certain regular intervals. A general motion of this kind is referred to as *oscillation* (*vibration*). When studying nonrepeating motions we are mainly interested in determining the position, the velocity and the acceleration of a moving particle at the various instants while a characteristic feature of the theory of vibration is that it predominantly studies the process as a whole. We are mainly interested not in the state of the bodies involved at the given instant but in the quantities specifying the repetition of the motion, namely

(a) the law according to which the motion is repeated;

(b) the time interval after which the body comes back to its previous position with the same direction of motion (this interval is the *period of oscillation*);

(c) the maximum deviation from the equilibrium position.

1.1. Differential Equation of Harmonic Oscillation. Let us consider a rectilinear horizontal motion of a particle M of mass m on a fixed smooth plane under the action of the elastic force of a spring. We shall place the origin O at the position of equilibrium of the particle, and draw the axis Ox to the right along the axis of the spring (Fig. 14.1). The point M in the figure shows one of the positions of the particle in its motion resulting from the violation of the state of equilibrium. The initial deviation and the initial velocity of the particle must be along the axis of the spring because, if otherwise, the motion would not be rectilinear. There are three forces acting on the particle M: the force of gravity mg, the reaction force N of the smooth plane and the elastic force X of the spring. Since there is no vertical displacement the first two vertical forces are mutually balanced: $N = mg$. Let the elastic force obey Hooke's law, that is let its algebraic value* be proportional to the magnitude of the deformation (elongation or contraction) of the spring:

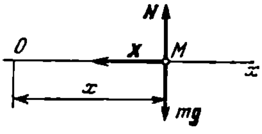

Fig. 14.1

$$X = -cx \tag{14.1}$$

The minus sign in formula (14.1) indicates that when the spring is elongated ($x > 0$) we have $X < 0$, that is the elastic force goes in the negative direction of the axis Ox; in the case of contraction the situation is opposite. The proportionality coefficient $c > 0$ whose dimension is N/m is called the *stiffness factor of the spring*. Let us put $x = 1$ m in (14.1); then c is equal to the modulus of X, that is the numerical value of the stiffness factor of the spring is equal to that of the force producing unit elongation (or contraction) of the spring. From formula (14.1) we obtain

$$c = \left| \frac{X}{x} \right| \tag{14.2}$$

Now let us proceed to the description of the motion. Suppose that the particle M receives an initial deviation to the right of the equilibrium position and that it gains no initial velocity. Under the action of the elastic force directed from right to left the particle starts moving with an acceleration to the left; when the particle returns to the equilibrium position it does not stop although $X = 0$ at that instant. Due to the velocity the particle M gains at that instant it continues to move to the left but with a deceleration (because the elastic force X has changed its direction); the motion to the left lasts until the velocity of the particle vanishes. After that the particle starts moving again but in the opposite direction.

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* We remind the reader that when studying a rectilinear motion we always use the algebraic values of the force, the velocity and the acceleration (see § 3 of Chap. 13).

A vibration (oscillation) of a particle which is subjected only to the action of an elastic restoring force is spoken of as *free* or *natural oscillation*. It should be noted that X may be any force attracting the particle to the fixed centre O so that the magnitude of the force is directly proportional to the distance from the particle to the centre.

By virtue of (13.1), in the case of rectilinear oscillation differential equation (13.20) is written thus:

$$m\frac{d^2x}{dt^2} = -cx$$

Let us rewrite this equation in the form

$$\ddot{x} + \omega^2 x = 0 \qquad \left(\omega^2 = \frac{c}{m}\right) \tag{14.3}$$

Equation (14.3) is a linear second-order differential equation with constant coefficients. The roots of its characteristic equation

$$\lambda^2 + \omega^2 = 0$$

are pure imaginary: $\lambda = \pm i\omega$ ($i^2 = -1$). The general solution of equation (14.3) can be rewritten as $x = c_1 \cos \omega t + c_2 \sin \omega t$ or in the equivalent form

$$x = a \cos (\omega t - \varphi_0) \tag{14.4}$$

where $a > 0$ and φ_0 are arbitrary constants. The free (natural) oscillation described by equation of motion (14.4) is called *harmonic*. Since the values of $\cos (\omega t - \varphi_0)$ are repeated when the argument gains an increment of 2π ($\omega T = 2\pi$) we see that, by virtue of (14.3), the *period T of oscillation* is expressed by the formula

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{c}} \tag{14.5}$$

The period of harmonic oscillation is independent of the initial conditions; this property is called *isochronism*. Irrespective of the deviation of the particle from the centre of oscillation O and of the initial velocity imparted to the particle it returns to the centre O after one and the same time. The quantity $\nu = 1/T$ is equal to the number of vibrations per second; it is called the *frequency of oscillation*; the unit for measuring the frequency is s^{-1} (one vibration per second). This unit is called the *hertz*. The quantity ω is called the *circular* (or *cyclic*) *frequency*, it is equal to the number of vibrations during 2π seconds.

Let us elucidate the mechanical meaning of the arbitrary constants. The positive quantity a (the maximum value of the deviation x of the particle from the equilibrium position) is called the *amplitude of oscillation*. The expression $\omega t - \varphi_0$ (it is a function of time) is

called the *phase of oscillation*, and the constant quantity φ_0 is the *initial phase*. In the general solution of equation (14.3) the quantities a and φ_0 are arbitrary constants. For every particular harmonic oscillation these constants assume certain definite values specified by the initial conditions. For instance, if the initial instant is $t = 0$ then the initial conditions are

$$x(0) = x_0, \qquad \dot{x}(0) = v_0 \tag{14.6}$$

1.2. Determination of Amplitude and Initial Phase from Initial Conditions. Now let us proceed to the determination of the quantities a and φ_0. Differentiating (14.4) we obtain the expression for the velocity of the oscillating particle for an arbitrary instant t:

$$\dot{x} = -a\omega \sin(\omega t - \varphi_0) \tag{14.7}$$

Putting $t = 0$ in (14.4) and (14.7) and using initial conditions (14.6) we obtain two equations determining the amplitude and the initial phase:

$$x_0 = a \cos \varphi_0, \quad \frac{v_0}{\omega} = a \sin \varphi_0$$

Let us square both the equations and add them together; this yields

$$x_0^2 + \frac{v_0^2}{\omega^2} = a^2$$

because $\sin^2 \varphi_0 + \cos^2 \varphi_0 = 1$. Further, we divide the first equation by the second and obtain

$$\cot \varphi_0 = \frac{\omega x_0}{v_0}$$

Thus, the amplitude and the initial phase of oscillation have been found from the initial conditions; they are expressed by the formulas

$$a = +\sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \quad \varphi_0 = \operatorname{arccot} \frac{\omega x_0}{v_0} \quad (0 \leqslant \varphi_0 \leqslant \pi) \tag{14.8}$$

In particular, when the initial velocity is equal to zero ($v_0 = 0$) formulas (14.8) imply that $a = |x_0|$, $\cot \varphi_0 = \pm\infty$ (here the upper sign corresponds to $x_0 > 0$ and the lower to $x_0 < 0$). Therefore for $v_0 = 0$ and $x_0 > 0$ we have

$$a = x_0, \qquad \varphi_0 = 0$$

The substitution of these values into (14.4) yields the law of motion in the form

$$x = x_0 \cos \omega t$$

In Fig. 14.2 the graph of oscillation is shown for $x_0 > 0$ and $v_0 = 0$.

For $v_0 = 0$ and $x_0 < 0$ we have

$$a = -x_0, \qquad \varphi_0 = \pi$$

In this case equation of motion (14.4) is again of form:

$$x = -x_0 \cos(\omega t - \pi) = x_0 \cos \omega t \tag{14.9}$$

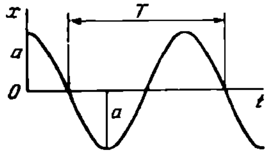


Fig. 14.2

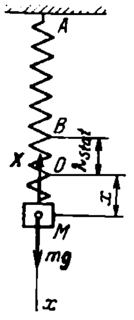


Fig. 14.3

1.3. Vertical Oscillation. Now we shall consider the oscillation of a weight M of mass m suspended from a vertical spring with stiffness factor c. Let the point B in Fig. 14.3 indicate the position of the lower end of the spring (whose upper end A is fixed) when the spring is not deformed (that is when it carries no weight). After the weight is suspended and then slowly ("statically") lowered, the end of the spring occupies an equilibrium position O at which we place the origin, the axis Ox going vertically downward. The length of the line segment BO is called the *static elongation of the spring* (we denote it by λ_{stat}). At the equilibrium position the magnitude of the elastic force is equal to that of the force of weight: $X = mg$; therefore formula (14.2) implies that

$$\lambda_{stat} = \frac{mg}{c} \tag{14.10}$$

Let us consider the position of the weight M in its oscillatory motion at an instant t. The particle M is acted upon by the force of gravity mg and the elastic force $X = -c\,(x + \lambda_{stat})$ of the spring because the elongation of the spring is $BM = BO + OM = \lambda_{stat} + x$. Differential equation (13.20) is written in the form

$$m\frac{d^2x}{dt^2} = mg - c\,(x + \lambda_{stat}) = -cx$$

because, by virtue of (14.10), we have $mg = c\lambda_{stat}$. Dividing by m we obtain

$$\ddot{x} + \omega^2 x = 0 \quad \left(\omega^2 = \frac{c}{m}\right)$$

We have thus derived the same differential equation (of form (14.3)) as in the case of horizontal harmonic oscillation. We intentionally placed the origin not at the end of the free spring but at the position of static equilibrium of the lower end of the spring when the weight is suspended from it. It is due to this choice of the origin that the

last equation coincides with the one describing horizontal oscillation. We see that the investigation of vertical harmonic oscillation reduces to the case of horizontal oscillation. By virtue of (14.10), we have

$$\omega^2 = \frac{c}{m} = \frac{g}{\lambda_{\text{stat}}} \tag{14.11}$$

and therefore formula (14.5) for the period T of oscillation is written in the form

$$T = 2\pi \sqrt{\frac{\lambda_{\text{stat}}}{g}} \tag{14.12}$$

Formulas (14.4), (14.7)-(14.9) remain unchanged.

EXAMPLE 14.1. Let the weight mentioned above be suspended from the point B (see Fig. 14.3) and be supported at that position. Now suppose that the constraint supporting the weight is instantaneously removed and the weight starts oscillating. Determine the maximum dynamic deviation λ_{dyn} of the weight assuming that the static elongation λ_{stat} of the spring is known.

Solution. Let us place the origin at the point O. Then the initial conditions for the oscillation are

$$x(0) = x_0 = -\lambda_{\text{stat}}, \quad \dot{x}(0) = v_0 = 0$$

for $t = 0$. According to formula (14.9) the law specifying the corresponding harmonic oscillation is

$$x = -\lambda_{\text{stat}} \cos \omega t$$

Substituting the value of ω from formula (14.11) we obtain

$$x = -\lambda_{\text{stat}} \cos \sqrt{\frac{g}{\lambda_{\text{stat}}}}\, t$$

The maximum values of x are attained when

$$\cos \sqrt{\frac{g}{\lambda_{\text{stat}}}}\, t = -1$$

This means that the particle attains its maximum deviation at the instants $t = T/2$, $3T/2$, $5T/2$, etc.; consequently we have

$$x_{\max} = \lambda_{\text{stat}}$$

The maximum dynamic deviation λ_{dyn} of the weight from its initial position is

$$\lambda_{\text{dyn}} = x_{\max} + OB = \lambda_{\text{stat}} + \lambda_{\text{stat}} = 2\lambda_{\text{stat}} \tag{14.13}$$

This simple formula is of use in engineering.

EXAMPLE 14.2. Two springs with stiffness factors c_1 and c_2 are in series connection, and a body M of weight P is attached to them as is shown in Fig. 14.4. Find the period T of free oscillation of the weight M.

Solution. Both the springs are in tension under the action of the force P. In the position of static equilibrium of the weight M the elongations of the springs are

$$\lambda_{\text{stat}}^{(1)} = \frac{P}{c_1} \quad \text{and} \quad \lambda_{\text{stat}}^{(2)} = \frac{P}{c_2}$$

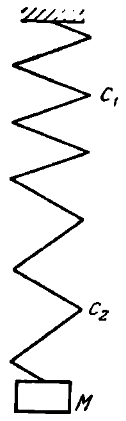


Fig. 14.4

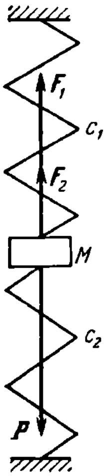


Fig. 14.5

respectively. The total elongation of both the springs is

$$\lambda_{stat} = \lambda_{stat}^{(1)} + \lambda_{stat}^{(2)} = \frac{P}{c_1} + \frac{P}{c_2} = P\,\frac{c_1+c_2}{c_1c_2} \qquad (1)$$

On the other hand, we can assume that the elongation λ_{stat} is produced by one spring with stiffness factor c_{red}:

$$\lambda_{stat} = \frac{P}{c_{red}} \qquad (2)$$

where the quantity c_{red} is called the reduced stiffness factor of the given springs. Equating the right-hand sides of equalities (1) and (2) we obtain

$$c_{red} = \frac{c_1c_2}{c_1+c_2}$$

The period of free oscillation of the weight M is found from formula (14.5), where $m = P/g$:

$$T = 2\pi\sqrt{\frac{P}{gc_{red}}} = 2\pi\sqrt{\frac{P(c_1+c_2)}{gc_1c_2}}$$

EXAMPLE 14.3. Solve the foregoing problem on condition that the weight M is attached between the same springs as is shown in Fig. 14.5.

Solution. Let the elongation of the upper spring corresponding to the position of static equilibrium of the weight M be equal to λ. Then the other spring undergoes contraction of the same magnitude. In the position of static equilibrium the force of weight P of the body M is balanced by the elastic forces F_1 and F_2 of both the springs (see Fig. 14.5), that is

$$P = c_1\lambda + c_2\lambda \qquad (3)$$

On the other hand, we can replace two springs by one for which we put

$$P = c_{red}\lambda \qquad (4)$$

where c_{red} is the reduced stiffness factor of both the given springs. Equating the right-hand sides of equalities (3) and (4) we find

$$c_{red} = c_1 + c_2$$

The period of free oscillation of the weight M is determined by formula (14.5) in which we put $m = P/g$:

$$T = 2\pi\sqrt{\frac{P}{gc_{red}}} = 2\pi\sqrt{\frac{P}{g(c_1+c_2)}}$$

§ 2. Damped Oscillation

2.1. Differential Equation of Damped Oscillation. When studying free harmonic oscillation we assume that the oscillating particle is acted upon only by the force $X = -cx$ restoring the equilibrium. However, in reality every oscillation of a particle which is not supported from outside undergoes damping. Damping is caused by the forces decelerating the motion such as the resistance force of the medium, the force of friction, etc.

Let us consider a particle M of mass m which is in a rectilinear motion under the action of a restoring force

$$X = -cx$$

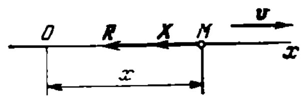


Fig. 14.6

and a resistance force R directed opposite to the motion of the particle (Fig. 14.6). The restoring force X attracts the particle and makes it move towards the equilibrium position O, the magnitude of the force being proportional to the distance $OM = x$. The restoring force can be realized with the aid of a spring with stiffness factor c obeying Hooke's law (see Sec. 1.1). The resistance force R can be of different magnitude depending on the physical nature of that force; however it is always a function of the velocity of the particle. Let us suppose that the resistance force depends linearly on the algebraic value $v = dx/dt$ of the velocity; in other words, it is directly proportional to v:

$$R = -bv$$

where b is the resistance coefficient.

Differential equation (13.20) is written in the form

$$m\frac{d^2x}{dt^2} = -cx - b\frac{dx}{dt}$$

We shall rewrite this equation thus:

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0 \quad \left(2\beta = \frac{b}{m}, \ \omega^2 = \frac{c}{m}\right) \tag{14.14}$$

The characteristic equation corresponding to (14.14) is

$$\lambda^2 + 2\beta\lambda + \omega^2 = 0$$

and its roots are

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega^2}$$

We shall limit ourselves to the case $\beta^2 < \omega^2$ (this condition excludes very large values of the resistance coefficient*). Let us denote

$$k^2 = \omega^2 - \beta^2 > 0 \tag{14.15}$$

Then the roots of the characteristic equation are

$$\lambda_{1,2} = -\beta \pm ki \qquad (i^2 = -1)$$

and the general solution of the differential equation has the form

$$x = a_0 e^{-\beta t}\cos(kt - \varphi_0) \tag{14.16}$$

where $a_0 > 0$ and φ_0 are arbitrary constants which can be determined from the initial conditions for every concrete oscillation.

* Inequality (14.15) is the condition guaranteeing the oscillatory character of the motion. When this condition is violated, that is when $\beta^2 \geqslant \omega^2$ the motion is aperiodic.

2.2. Properties of Damped Oscillation.

(1) The period of oscillation is expressed by the formula

$$T' = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{\omega^2 - \beta^2}} \tag{14.17}$$

where k is the circular frequency of damped oscillation. The constant period T' (it is independent of the initial conditions) is greater than the period $T = 2\pi/\omega$ of the harmonic oscillation of a particle with the same mass subjected to the action of the same restoring force $X = -cx$ and oscillating in a medium without resistance:

$$T' > T$$

Consequently, the *resistance of the medium increases the period of oscillation* (or, which is the same, *decreases the frequency*).

(2) The instantaneous value

$$A(t) = a_0 e^{-\beta t}$$

of the amplitude of oscillation is a variable quantity which asymptotically tends to zero for $t \to \infty$ (that is why the oscillation is said to be damped). At the initial instant $t = 0$ we have

$$A(0) = a_0$$

and at the end of the first half-period we have

$$A\left(\frac{1}{2} T'\right) = a_0 e^{-\frac{1}{2}\beta T'}$$

The ratio of the amplitudes of oscillation at the beginning and at the end of a half-period is equal to

$$\frac{A(0)}{A\left(\frac{1}{2} T'\right)} = e^{\frac{1}{2}\beta T'}$$

This ratio is called the *decrement of damped oscillation* or *damping factor*. The decrement is equal to the number of times the amplitude of oscillation decreases in every half-period. The natural logarithm of the decrement, that is the quantity

$$\Delta = \ln e^{\frac{1}{2}\beta T'} = \frac{1}{2}\beta T' \tag{14.18}$$

is called the *logarithmic decrement*. The logarithmic decrement can be determined directly from the observation by measuring the values of the amplitudes A_n and A_{n+1} of two successive maximum deviations from the equilibrium (Fig. 14.7). We have

$$\Delta = \ln \frac{A_n}{A_{n+1}} = \frac{1}{2}\beta T'$$

because the expression

$$\ln\left|\frac{x(t)}{x\left(t+\frac{1}{2}T'\right)}\right|$$

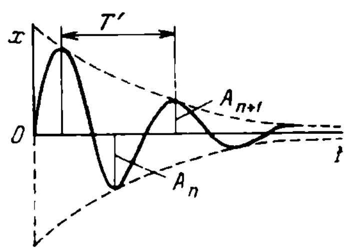


Fig. 14.7

assumes one and the same value equal to Δ irrespective of the instant t.

Knowing the value of Δ and the circular frequency ω of undamped oscillation we can determine the reduced resistance coefficient β. Substituting into the last equality the value of T' given by (14.17) we obtain

$$\Delta=\frac{\pi\beta}{\sqrt{\omega^2-\beta^2}}$$

whence

$$\Delta^2\omega^2 = (\pi^2 + \Delta^2)\,\beta^2$$

Finally,

$$\beta=\frac{\omega\Delta}{\sqrt{\pi^2+\Delta^2}} \tag{14.19}$$

Now let us substitute the values of β and ω given by (14.14); this results in the expression of the resistance coefficient b:

$$b=\frac{2\sqrt{cm}}{\sqrt{\pi^2+\Delta^2}}\,\Delta \tag{14.19'}$$

A sketch of the graph of damped oscillation is shown in Fig. 14.7.

(3) Let us find the initial amplitude a_0 and the initial phase φ_0 from the initial conditions. The differentiation of (14.16) yields the expression for the velocity of the particle in the damped oscillation:

$$\dot{x} = -a_0\beta e^{-\beta t}\cos(kt-\varphi_0) - a_0ke^{-\beta t}\sin(kt-\varphi_0) \tag{14.20}$$

Let us put $t = 0$ in (14.16) and (14.20); then for the given initial conditions (see (14.6)) we obtain

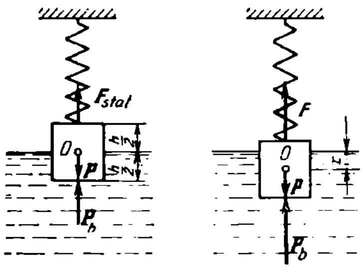
$$x_0 = a_0\cos\varphi_0, \quad v_0 = -a_0\beta\cos\varphi_0 + a_0k\sin\varphi_0$$

whence

$$a_0\cos\varphi_0 = x_0,\ \ a_0\sin\varphi_0=\frac{1}{k}(v_0+\beta x_0)$$

Solving these equations by analogy with Sec. 1.2 we find

$$\left.\begin{aligned} a_0 &= +\sqrt{x_0^2+\frac{1}{k^2}(v_0+\beta x_0)^2} \\ \varphi_0 &= \operatorname{arccot}\frac{kx_0}{v_0+\beta x_0} \quad (0\leqslant\varphi_0\leqslant\pi) \end{aligned}\right\} \tag{14.21}$$



(a) (b)

Fig. 14.8

For $\beta = 0$ formula (14.15) implies that $k = \omega$, and formulas (14.21) go into formulas (14.8). This is quite natural because the condition $\beta = 0$ indicates the absence of damping ($b = 0$), and without damping the oscillation becomes harmonic.

EXAMPLE 14.4. A cylinder of weight P N, radius r m and altitude h m is suspended from a spring whose upper end is fixed. The stiffness factor of the spring is c N/m. The cylinder is immersed in a fluid with specific weight γ. In the position of static equilibrium half of the altitude of the cylinder is immersed. At the initial instant the cylinder is immersed to 2/3 of its altitude and then let to move without initial velocity. Taking into account the resistance force $R = -bv$ of the fluid determine the motion of the cylinder. Find the condition under which the motion of the cylinder is oscillatory (Fig. 14.8).

Solution. In the position of static equilibrium the weight P of the cylinder is balanced by the elastic force $F_{\text{stat}} = c\lambda_{\text{stat}}$ of the spring and the buoyancy force $P_{\text{b}} = (1/2)\,\pi r^2 h\gamma$ directed vertically upward (Fig. 14.8*a*), that is

$$P = \frac{1}{2}\,\pi r^2 h\gamma + c\lambda_{\text{stat}} \tag{14.22}$$

Let us consider the position of the cylinder at an instant t when the centre O of the cylinder is displaced along the vertical by a distance x from the position of static equilibrium, the origin being taken at that position (Fig. 14.8*b*). At time t the cylinder is acted upon by the force of weight P, the elastic force $F = c\,(\lambda_{\text{stat}} + x)$ of the spring, the buoyancy force

$$P_{\text{b}} = \pi r^2 \left(\frac{h}{2} + x\right)\gamma$$

and the resistance force $R = -b\dot{x}$. The differential equation describing the motion of the cylinder has the form

$$\frac{P}{g}\,\ddot{x} = P - \pi r^2\left(\frac{h}{2} + x\right)\gamma - c\,(\lambda_{\text{stat}} + x) - b\dot{x}$$

Making use of equality (14.22) we obtain

$$\frac{P}{g}\,\ddot{x} = -(c + \pi r^2\gamma)\,x - b\dot{x}$$

Dividing by P/g and denoting

$$\omega_0^2 = \frac{c + \pi r^2\gamma}{P}\,g, \qquad 2\beta = \frac{gb}{P} \tag{14.23}$$

we arrive at the differential equation of damped oscillation of form (14.14):

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

For the motion of the cylinder to be oscillatory it is necessary that condition (14.15) should hold, that is

$$k^2 = \frac{c + \pi r^2\gamma}{P}\,g - \frac{1}{4}\left(\frac{gb}{P}\right)^2 > 0$$

We shall assume that this condition is fulfilled. The initial conditions are:

$$x(0)=\frac{1}{2}h-\frac{1}{3}h=\frac{1}{6}h, \quad \dot{x}(0)=v_0=0$$

From formulas (14.21) we find the initial amplitude and the initial phase

$$a_0=x_0\sqrt{1+\frac{\beta^2}{k^2}}=\frac{h\omega_0}{6\sqrt{\omega_0^2-\beta^2}}, \quad \varphi_0=\operatorname{arccot}\frac{\sqrt{\omega_0^2-\beta^2}}{\beta}$$

Hence, solution (14.16) takes the form

$$x=\frac{h\omega_0}{6\sqrt{\omega_0^2-\beta^2}}e^{-\beta t}\cos\left[\sqrt{\omega_0^2-\beta^2}\,t-\operatorname{arccot}\frac{\sqrt{\omega_0^2-\beta^2}}{\beta}\right]\text{m}$$

where ω_0^2 and β are determined by formulas (14.23).

EXAMPLE 14.5. A body suspended from a spring oscillates with period $T = 0.2\pi$ s when there is no resistance; when there is resistance proportional to the first degree of the velocity the period of oscillation is $T' = 0.25\pi$ s. Find the law describing the damped oscillation of the body on condition that at the initial instant the end of the spring is displaced by a distance of 0.06 m from the position of static equilibrium so that the spring gains an initial elongation, after which the body is let to move by itself.

Solution. By formula (14.5), the circular frequency ω of free oscillation of the body is

$$\omega=\frac{2\pi}{T}=\frac{2\pi}{0.2\pi}=10\text{ s}^{-1}$$

From formula (14.17) we find the coefficient β:

$$0.25\pi=\frac{2\pi}{\sqrt{100-\beta^2}}, \quad \beta=6\text{ s}^{-1}$$

Then, using formula (14.15), we determine

$$k=\sqrt{\omega^2-\beta^2}=8\text{ s}^{-1}$$

In the case under consideration general solution (14.16) describing the damped oscillation takes the form

$$x = a_0e^{-6t}\cos(8t-\varphi_0) \tag{1}$$

The initial amplitude a_0 and the initial phase φ_0 are determined from the initial conditions of the problem which are

$$x_0 = 0.06\text{ m}, \quad v_0 = 0$$

From these conditions, using formulas (14.21), we find:

$$a_0=\sqrt{0.06^2+\frac{36}{64}0.06^2}=0.075\text{ m}, \quad \varphi_0=\operatorname{arccot}\frac{8}{6}=0.643$$

Finally, substituting the values thus found into (1) we obtain the law of the oscillation:

$$x = 0.075e^{-6t}\cos(8t-0.643)\text{ m}$$

§ 3. Forced Oscillation

3.1. Forced Oscillation without Resistance. Let a particle M of mass m be in a rectilinear motion under the action of two forces applied to it: a restoring force $X = -cx$ and a *perturbing force* $\boldsymbol{F}$ (Fig. 14.9) whose algebraic value is a periodic function of time t,

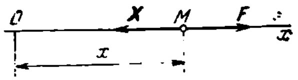


Fig. 14.9

for instance

$$F = H \cos (pt - \psi_0)$$

Here $H > 0$, p and ψ_0 are the amplitude, the circular frequency and the initial phase of the perturbing force respectively. The period of change of the perturbing force F is

$$\tau = \frac{2\pi}{p} \tag{14.24}$$

In the case under consideration differential equation (13.20) has the form

$$m \frac{d^2x}{dt^2} = -cx + H \cos (pt - \psi_0)$$

It can be rewritten thus:

$$\ddot{x} + \omega^2 x = h \cos (pt - \psi_0) \quad \left(\omega^2 = \frac{c}{m},\ h = \frac{H}{m}\right) \tag{14.25}$$

This is a nonhomogeneous linear differential equation of the second order with constant coefficients. Its general solution is the sum of the general solution x_1 of the corresponding homogeneous equation

$$\ddot{x}_1 + \omega^2 x_1 = 0$$

and a particular solution x_2 of nonhomogeneous equation (14.25). The general solution x_1 is of form (14.4):

$$x_1 = a_1 \cos (\omega t - \varphi_1)$$

The particular solution x_2 can be sought in the form

$$x_2 = B \cos (pt - \psi_0)$$

Differentiating twice x_2 and substituting x_2 and $\dot{x}_2$ into (14.25) we conclude that the identity

$$-p^2B \cos (pt - \psi_0) + \omega^2 \cos (pt - \psi_0) = h \cos (pt - \psi_0)$$

must hold. Whence, if $p \neq \omega$, we find

$$B = \frac{h}{\omega^2 - p^2}, \quad \text{that is } x_2 = \frac{h}{\omega^2 - p^2} \cos (pt - \psi_0) \tag{14.26}$$

Consequently, for $p \neq \omega$ the general solution of (14.25) has the form

$$x = x_1 + x_2 = a_1 \cos (\omega t - \varphi_1) + \frac{h}{\omega^2 - p^2} \cos (pt - \psi_0) \tag{14.27}$$

Let us elucidate the mechanical significance of the solution we have found. The motion of the particle M is composed of two oscillatory motions: the *oscillation x_1 with frequency equal to that of free harmonic oscillation* and the *forced oscillation x_2 with frequency equal to that of the perturbing force.* It should be stressed that the initial

data, that is the position and the velocity of the particle M at the initial instant, affect the amplitude a_1 and the initial phase φ_1 of the oscillation x_1 and do not affect the forced oscillation x_2. From formula (14.27) it follows that the amplitude and the initial phase of the oscillation x_1 whose frequency coincides with that of the free oscillation depend not only on the initial conditions but also on the parameters h, p and ψ_0 characterizing the perturbing force.

Let us investigate the properties of the forced oscillation x_2. The circular frequency of that oscillation is equal to p, that is it coincides with that of the perturbing force $\boldsymbol{F}$. Consequently, the period τ of the forced oscillation x_2 is also determined by formula (14.24). When $p < \omega$, that is when the frequency of the forced oscillation x_2 is *small*, the amplitude A_f of that oscillation is

$$A_f = \frac{h}{\omega^2 - p^2} \tag{14.28}$$

and the law describing the forced oscillation x_2 has the form

$$x_2 = \frac{h}{\omega^2 - p^2} \cos (pt - \psi_0) \tag{14.28a}$$

Consequently, if $p < \omega$ the forced oscillation x_2 and the perturbing force have the same phase. This means that the algebraic value of the force $\boldsymbol{F}$ and that of the forced oscillation x_2 are always of one sign.

If $p > \omega$, that is if the frequency of the forced oscillation x_2 is *large*, the amplitude of the oscillation is

$$A_f = \frac{h}{p^2 - \omega^2} \tag{14.29}$$

and the law describing the forced oscillation is

$$x_2 = \frac{h}{p^2 - \omega^2} \cos (pt - \psi_0 - \pi)$$

This of course coincides identically with (14.26). In the case $p > \omega$ the forced oscillation x_2 and the perturbing force have opposite phases, and consequently the projection of the force $\boldsymbol{F}$ on the axis Ox and the forced oscillation x_2 are always of different signs.

From formulas (14.28) and (14.29) it follows that the amplitude of the forced oscillation x_2 depends not only on the reduced amplitude $h = H/m$ of the perturbing force but also on the circular frequency p of the perturbing force (it is meant that the circular frequency ω of free oscillation is fixed). Let us consider the auxiliary coordinate system shown in Fig. 14.10; along the axis of abscissas the values of the ratio p/ω are set off and along the axis of ordinates the values of the ratio A_f/A_0 are set off, where $A_0 = h/\omega^2$ is the limiting value of the amplitude of the forced oscillation x_2 for $p \to 0$. In other words,

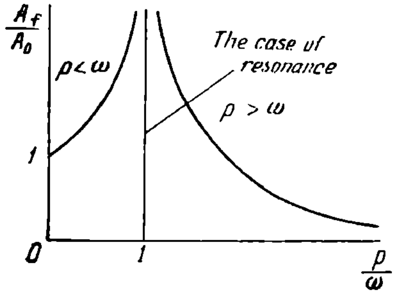


Fig. 14.10

we construct the graph of the function

$$\frac{A_f}{A_0}=\frac{1}{\left|1-\left(\frac{p}{\omega}\right)^2\right|}$$

This graph shows that for $0<p/\omega<$ <1 we have $1<A_f/A_0<\infty$ and for $1<p/\omega<\infty$ we have $\infty>$ $>A_f/A_0>0$. This means that when the circular frequency p of the perturbing force increases the amplitude of the forced oscillation x_2 increases monotonically starting with the value

$$A_0=\frac{h}{\omega^2}=\frac{H}{c}$$

It becomes infinitely large when p approaches ω, that is when the circular frequency of the perturbing force tends to that of free oscillation. When p becomes greater than ω and is increased further the amplitude of the forced oscillation x_2 decreases monotonically and tends to zero.

3.2. Resonance. When the frequency of the perturbing force coincides with that of free oscillation ($p=\omega$) the so-called *resonance phenomenon* sets in. In this case formula (14.26) is inapplicable, and the particular solution x_2 of equation (14.25) must be sought (for $p=\omega$) in the form

$$x_2=Dt\sin(\omega t-\psi_0)$$

Let us compute the derivatives $\dot{x}_2$ and $\ddot{x}_2$:

$$\dot{x}_2=D\sin(\omega t-\psi_0)+\omega Dt\cos(\omega t-\psi_0)$$

and

$$\ddot{x}_2=2\omega D\cos(\omega t-\psi_0)-\omega^2Dt\sin(\omega t-\psi_0)$$

Substituting these expressions into equation (14.25) (for $p=\omega$) and combining similar terms we obtain

$$2\omega D\cos(\omega t-\psi_0)=h\cos(\omega t-\psi_0)$$

whence

$$D=\frac{h}{2\omega}$$

Thus, in the case of resonance the law of the forced oscillation x_2 is

$$x_2 = \frac{h}{2\omega} t \sin(\omega t - \psi_0) \qquad (14.30)$$

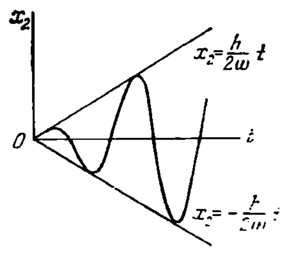


Fig. 14.11

The graph of the forced oscillation x_2 is shown in Fig. 14.11. From formula (14.30) and Fig. 14.11 it is seen that when resonance takes place the amplitude $ht/(2\omega)$ of the forced oscillation x_2 increases with time indefinitely according to a linear law*, that is directly proportionally to t. However, in reality the frequency of the perturbing force can never coincide exactly with that of free oscillation, and, besides, the resistance forces are always present. A resonance phenomenon should be understood as the one in which the amplitude of forced oscillation of an oscillatory system increases sharply when the frequency of the external action approaches one of the frequencies which the free (natural) oscillation of the system may have.

EXAMPLE 14.6. Let us consider the motion of a railway car. During the motion it undergoes shocks at the rail joints. Let the velocity of motion be 40 m/s and the length of the rail be $l = 12$ m. Find the magnitude of the static deflection of the carriage springs for which resonance does not set in.

Solution. The resonance sets in when the period of natural oscillation of the car coincides with that of the perturbing force; in the case under consideration this force is produced by the shocks at the rail joints.** The period of natural oscillation is determined by formula (14.12):

$$T = 2\pi \sqrt{\frac{\lambda_{stat}}{g}}$$

The period of the perturbing force is equal to the time during which the car travels a distance equal to the length of the rail:

$$\tau = \frac{l}{v}$$

From the inequality $T < \tau$ we obtain

$$\lambda_{stat} < \frac{gl^2}{4\pi^2 v^2}$$

* Here lies the distinction between an ordinary resonance phenomenon and a more complicated phenomenon known as the *parametric resonance* (*both the major* (or *main*) *resonance* and *the combination resonance*) for which the amplitude of oscillation obeys the exponential law.

** In the case when the external action on a linear system is described by a more complicated periodic function than a simple harmonic function the problem can be reduced to the one we have considered using the principle of superposition studied in the theory of vibration.

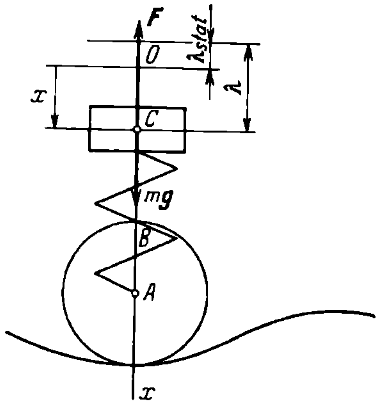


Fig. 14.12

For $v = 40$ m/s the value of the right-hand member of this inequality is

$$\frac{9.81 \cdot 12^2}{4 \cdot 9.87 \cdot 40^2} = 0.0223 \text{ m}$$

Thus, for the resonance not to set in it is necessary that $\lambda_{stat} < 2.23$ cm.

EXAMPLE 14.7. In Fig. 14.12 a vehicle (an automobile) is schematically shown as a mass m attached at the point C to the spring. The lower end A of the spring carries the axis of the wheel. The unevenness of the ground makes the wheel oscillate vertically according to the law

$$AB = 0.15 \sin 10t \text{ m}$$

Determine the forced oscillation of the vehicle; the mass of the vehicle is $m = 500$ kg and the stiffness factor of the spring is $c = 200\,000$ N/m.

Solution. We shall draw the axis Ox vertically downward placing the origin at the position of static equilibrium of the vehicle (on condition that the point A in Fig. 14.12 occupies the lowest position on the vertical). The vehicle is under the action of the force of weight $m\boldsymbol{g}$ and the elastic force $\boldsymbol{F}$ of the spring ($F = -c\lambda$). The differential equation of motion of the vehicle has the form

$$m\ddot{x} = mg - c\lambda \tag{1}$$

During time t the lowest point of the spring receives a displacement from the position A to a new position B. Consequently, the deformation of the spring is equal to

$$\lambda = \lambda_{stat} + x - AB = \lambda_{stat} + x - 0.15 \sin 10t \tag{2}$$

Substituting (2) into (1) and taking into account (14.11) we obtain

$$m\ddot{x} = -cx + 0.15c \sin 10t$$

Now we divide by m and bring the differential equation of motion of the vehicle to form (14.25):

$$\ddot{x} + \omega^2 x = h \sin 10t$$

where

$$\omega^2 = \frac{c}{m} = \frac{200\,000}{500} = 400 \text{ s}^{-2}, \quad h = \frac{0.15c}{m} = 0.15\omega^2 = 60 \text{ m/s}^2, \quad \psi_0 = \frac{\pi}{2}$$

Let us check whether resonance sets in. We have $p = 10$ s^{-1} and $\omega = 20$ s^{-1}, and hence $p \neq \omega$, that is there is no resonance. Note that here we deal with the case when $p < \omega$. Therefore the amplitude of forced oscillation (of type x_2) is determined by formula (14.28):

$$A_f = \frac{h}{\omega^2 - p^2} = \frac{60}{400 - 100} = 0.2 \text{ m}$$

Finally, the law describing forced oscillation is of form (14.28a):

$$x = 0.2 \sin 10t \text{ m}$$

3.3. Effect of Resistance on Forced Oscillation. As was already mentioned in Sec. 2.1, in every motion there appears a resistance

force; we assume that this force is proportional to the velocity: $R = -bv$. The differential equation of oscillation under the action of the restoring force $X = -cx$, the resistance force $R = -bv$ and the perturbing force $F = H \cos(pt - \psi_0)$ has the form

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = h \cos(pt - \psi_0)$$
$$\left(2\beta = \frac{b}{m}, \quad \omega^2 = \frac{c}{m}, \quad h = \frac{H}{m}\right) \tag{14.31}$$

(see (14.14) and (14.25)).

This is a nonhomogeneous linear differential equation of the second order with constant coefficients. Its general solution is the sum of the general solution x_1 of the corresponding homogeneous equation

$$\ddot{x} + 2\beta\dot{x} + \omega^2 x = 0$$

and a particular solution x_2 of nonhomogeneous equation (14.31). In the case of a small resistance force ($\beta < \omega$) the general solution x_1 is of form (14.16):

$$x_1 = a_1 e^{-\beta t} \cos(\sqrt{\omega^2 - \beta^2}\, t - \varphi_1)$$

The particular solution x_2 can be sought in the form

$$x_2 = \mathrm{E} \cos(pt - \psi_0 - \varepsilon)$$

where E and ε are constants which must be determined in such a way that this expression of x_2 satisfies equation (14.31).

To find E and ε let us compute the derivatives:

$$\dot{x}_2 = -p\mathrm{E} \sin(pt - \psi_0 - \varepsilon), \quad \ddot{x}_2 = -p^2\mathrm{E} \cos(pt - \psi_0 - \varepsilon)$$

The substitution of these expressions into differential equation (14.31) yields

$$\begin{aligned} -p^2\mathrm{E} \cos(pt - \psi_0 - \varepsilon) &- 2\beta p\mathrm{E} \sin(pt - \psi_0 - \varepsilon) \\ &+ \omega^2\mathrm{E} \cos(pt - \psi_0 - \varepsilon) = h \cos[(pt - \psi_0 - \varepsilon) + \varepsilon] \\ &= h \cos\varepsilon \cos(pt - \psi_0 - \varepsilon) - h \sin\varepsilon \sin(pt - \psi_0 - \varepsilon) \end{aligned}$$

Equating in this relation the coefficients in $\cos(pt - \psi_0 - \varepsilon)$ and $\sin(pt - \psi_0 - \varepsilon)$ we obtain

$$(\omega^2 - p^2)\,\mathrm{E} = h \cos\varepsilon, \quad 2\beta p\mathrm{E} = h \sin\varepsilon$$

Let us square each of the last equations and add them together:

$$\mathrm{E}^2 [(\omega^2 - p^2)^2 + 4\beta^2 p^2] = h^2$$

Further, the division of the first equation by the second results in

$$\frac{\omega^2 - p^2}{2\beta p} = \cot\varepsilon$$

Thus, in the case when the resistance force is small the general solution of differential equation (14.31) has the form

$$x = a_1 e^{-\beta t} \cos(\sqrt{\omega^2 - \beta^2}\, t - \varphi_1) + \mathrm{E} \cos(pt - \psi_0 - \varepsilon) \tag{14.32}$$

where

$$E = \frac{h}{\sqrt{(\omega^2 - p^2)^2 + 4\beta^2 p^2}}$$
$$\varepsilon = \operatorname{arccot} \frac{\omega^2 - p^2}{2\beta p} \qquad (0 < \varepsilon < \pi) \tag{14.33}$$

The first summand in (14.32) corresponds to the law of damped free oscillation (see (14.16)). Its amplitude a_1 and initial phase φ_1 are now found not from formulas (14.21) but by means of the substitution of the initial data into general solution (14.32) and into the expression for the derivative of the general solution. This oscillation is always damped and therefore when sufficiently large time elapses after the external perturbing force starts acting there only remains the forced oscillation of the form

$$x_2 = E \cos (pt - \psi_0 - \varepsilon)$$

The amplitude E and the initial phase shift ε are determined by formulas (14.33).

When investigating a steady-state regime we can confine ourselves to the consideration of the forced oscillation of type x_2; the period of this oscillation is

$$\tau = \frac{2\pi}{p}$$

(see Sec. 3.1).

Thus, the resistance force does not affect the period of the forced oscillation x_2. However, as is seen from the comparison of formulas (14.33) and (14.28) (or (14.29)), the resistance leads to the decrease of the amplitude of forced oscillation. When $p \to \omega$ the amplitude does not tend to infinity (which is the case when there are no resistance forces; see Fig. 14.10) but to the finite quantity $h/(2\beta p)$. For the given values of h/ω^2 and β/ω the amplitude of the forced oscillation x_2 is a function of the ratio p/ω:

$$E = E\left(\frac{p}{\omega}\right) = \frac{h/\omega^2}{\sqrt{\left[1 - \left(\frac{p}{\omega}\right)^2\right]^2 + 4\frac{\beta^2}{\omega^2}\left(\frac{p}{\omega}\right)^2}} \qquad \left(0 \leqslant \frac{p}{\omega} < \infty\right)$$

To determine the extrema of E we must find the extrema of the radicand, that is of the expression

$$f(\xi) = (1 - \xi^2)^2 + 4\frac{\beta^2}{\omega^2}\xi^2 \qquad \left(\xi = \frac{p}{\omega}\right)$$

Let us equate to zero the derivative of this expression:

$$f'(\xi) = -4\xi(1 - \xi^2) + 8\frac{\beta^2}{\omega^2}\xi = 0$$

From the last equation we find the stationary values

$$\xi_1 = 0, \qquad \xi_2 = \sqrt{1 - 2\frac{\beta^2}{\omega^2}}$$

(the third value ξ_3 is negative and is therefore discarded).

The first stationary value corresponds to the minimum of the amplitude of the forced oscillation:

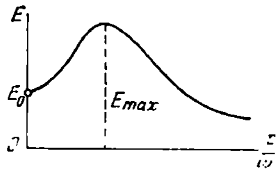


Fig. 14.13

$$E(0) = \frac{h}{\omega^2}$$

For $\xi = \xi_2$, that is for $p = \sqrt{\omega^2 - 2\beta^2}$, the amplitude attains its maximum value:

$$E_{max} = \frac{h/\omega^2}{2\frac{\beta}{\omega}\sqrt{1 - \left(\frac{\beta}{\omega}\right)^2}} = \frac{h}{2\beta\sqrt{\omega^2 - \beta^2}} \qquad (14.34)$$

The graph of the function $E(p/\omega)$ is shown in Fig. 14.13. When resonance takes place in the absence of the resistance forces the frequencies p and ω coincide: $p = \omega$; when the resistance forces are present resonance takes place for the circular frequency of the perturbing force equal to

$$p_{res} = \sqrt{\omega^2 - 2\beta^2} < \omega \qquad (14.35)$$

In mechanics resonance leads to a sharp increase in the amplitude of forced vibrations of various constructions (such as foundations, bridges, etc.). Resonance may even lead to the destruction of the construction. Here resonance is an undesirable phenomenon and must be avoided. In technical constructions special devices called dampers are often used in order to eliminate resonant vibrations.

EXAMPLE 14.8. A weight of mass 4 kg is suspended from a spring with stiffness factor c = 2000 N/m. The weight is acted upon by perturbing force F = 100 cos pt N. The oscillation of the weight encounters the action of the resistance force $R = -120v$ N (here v is measured in m/s). For what value of the circular frequency p of the forced oscillation does the amplitude achieve the maximum value? What is the maximum value of the amplitude of the forced oscillation?

Solution. The square of the circular frequency ω of the natural oscillation is found from formula (14.3):

$$\omega^2 = \frac{c}{m} = \frac{2000}{4} = 500 \text{ s}^{-2}$$

The damping factor β is determined by formula (14.31):

$$\beta = \frac{b}{2m} = \frac{120}{2 \cdot 4} = 15 \text{ s}^{-1}$$

In this case formula (14.35) determines the resonant value

$$p_{res}=\sqrt{\omega^2-2\beta^2}=\sqrt{500-2\cdot 225}=7.07\ \text{s}^{-1}$$

To find the maximum value of the amplitude of the forced oscillation using formula (14.34) we must first compute the ratio $h = H/m$:

$$h=\frac{H}{m}=\frac{100}{4}=25\ \text{m/s}^2$$

Finally, formula (14.34) yields

$$E_{max}=\frac{25}{2\cdot 15\sqrt{500-225}}=0.0503\ \text{m}$$

Problems

PROBLEM 14.1. Two bodies of the same weight are suspended from a spring whose upper end is rigidly fixed. This leads to an increase by 2 cm of the length of the spring. Then the lower body is instantaneously removed and the remaining upper body starts oscillating. Determine the amplitude a, the period T and the law of oscillation of the remaining body.

Hint. Place the origin O at the position of static equilibrium of the upper body when it is suspended from the spring alone and draw the axis Ox vertically downward.

Answer. $x=0.01\cos(10\sqrt{g}\,t)$ m, $a=0.01$ m, $T=\dfrac{0.2\pi}{\sqrt{g}}=0.200$ s.

PROBLEM 14.2. Two bodies A and B of weights P N and Q N respectively are joined together with a spring as is shown in Fig. 14.14. The weight A is in free vertical oscillation with amplitude a m and period T s. Determine the maximum pressure N the weight B exerts on the supporting plane.

Answer. $N=Q+P\left(1+\dfrac{4a\pi^2}{gT^2}\right)$ N.

PROBLEM 14.3. A body of mass $m = 10$ kg is suspended from a spring with stiffness factor $c = 4000$ N/m. The force of resistance of the medium is proportional to the first degree of the velocity. After three vibrations the amplitude of the oscillation decreases 10 times. Determine the logarithmic decrement Δ and the period T' of the damped oscillation.

Answer. $\Delta = 0.384$, $T' = 0.316$ s.

REMARK. The oscillations considered in Chap. 14 are called linear because they are described by linear differential equations. On nonlinear oscillations we refer the reader to [2], where some applied methods are developed; in this book the reader can also find extensive bibliography on nonlinear oscillations. Besides "ordinary" resonance studied in this chapter, in engineering an important role is played by the so-called parametric resonance, a phenomenon in which a periodic change with time of the parameters of an oscillatory system produces an increase in the amplitude of the oscillation. On mathematical theory of parametric resonance we refer the reader to [5].

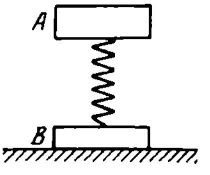


Fig. 14.14

Chapter 15 General Principles of Dynamics of Absolute Motion of a Particle

In this chapter we shall present three fundamental principles (theorems) for the case of absolute motion of a particle, that is for the motion which is considered relative to an inertial coordinate system (see Sec. 1.2 of Chap. 13).

§ 1. Principle of Momentum for a Particle

1.1. Impulse of a Force. Let a particle M of mass m move under the action of a force $\boldsymbol{F}$ (by $\boldsymbol{F}$ we shall mean the resultant of all the forces applied to the particle); let the velocity of the particle be equal to $\boldsymbol{v}$ at the given instant (Fig. 15.1). Now we construct the *vector* $\boldsymbol{q}$ representing the *momentum of the particle* (see Sec. 1.1 of Chap. 13):

$$\boldsymbol{q} = m\boldsymbol{v}$$

In the International System of Units (SI) the unit for measuring the modulus of momentum is kg·m/s. Momentum is one of the basic quantities characterizing particle motion.

By an elementary impulse $d\boldsymbol{S}$ of the force $\boldsymbol{F}$ is meant the vector equal to the product of the force vector $\boldsymbol{F}$ by infinitesimal time dt (see Fig. 15.1):

$$d\boldsymbol{S} = \boldsymbol{F}\,dt$$

Let the particle M pass under the action of the force $\boldsymbol{F}$ from the initial position M_0 to the position M during time interval $(0, t)$; generally speaking, this force may vary both in its modulus and direction (Fig. 15.2).

By the (*vector*) *impulse of the force* $\boldsymbol{F}$ during time interval $(0, t)$ is meant the integral of the vector function $\boldsymbol{F}(t)$ with respect to the scalar argument t:

$$\boldsymbol{S} = \int_0^t \boldsymbol{F}(t)\,dt \tag{15.1}$$

In the International System of Units (SI) the modulus of the impulse of a force is measured in N·s. The impulse of a force charac-

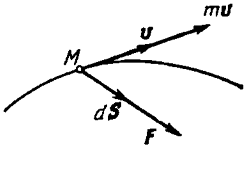


Fig. 15.1

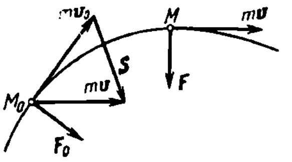


Fig. 15.2

terizes the action of the force depending on its modulus, direction and time of action.

We shall elucidate the meaning of vector integral (15.1). Let the interval $(0, t)$ be divided into n subintervals $(0, t_1)$, (t_1, t_2), ..., (t_{n-1}, t). Let us denote $t_\nu - t_{\nu-1} = \Delta t_\nu$ ($\nu = 1, 2, \ldots, n$; $t_0 = 0$, $t_n = t$); then, by definition,

$$S = \lim_{\substack{n\to\infty \\ \max \Delta t_\nu \to 0}} \sum_{\nu=1}^{n} F(t_\nu)\,\Delta t_\nu = \int_0^t F(t)\,dt$$

From the properties of the definite integral it follows that if

$$F(t) = X(t)\,i + Y(t)\,j + Z(t)\,k$$

then

$$S = i\int_0^t X(t)\,dt + j\int_0^t Y(t)\,dt + k\int_0^t Z(t)\,dt$$

Thus, the projection of the (vector) impulse of a force on an axis is equal to the impulse of the projection of the force on that axis. For the sake of simplicity, the functions $X(t)$, $Y(t)$ and $Z(t)$ are supposed to be continuous.

1.2. Vector and Coordinate Forms of the Principle of Momentum. Let us write down Newton's second law (13.1):

$$\frac{d}{dt}(mv) = F$$

Multiplying this equality by dt we see that the differential of the momentum of a particle is equal to the elementary impulse of the acting force:

$$d(mv) = F\,dt$$

The integration with respect to t from 0 to t and with respect to v from v_0 to v results in

$$mv - mv_0 = \int_0^t F\,dt \tag{15.2}$$

We have thus proved the *principle of momentum* for a particle (the theorem on change of momentum), this is the vector form of the principle, and it reads: *the increment of the (vector) momentum of a particle during some time is equal to the (vector) impulse of the force during that time.*

The meaning of the principle can be elucidated with the aid of Fig. 15.2. In order to find the increment of the (vector) momentum let us construct the vector mv at the point M_0 and draw a vector joining the terminus of the vector mv_0 and the terminus of the vec-

tor we have constructed. According to the principle, the increment of the vector mv is geometrically equal to the (vector) impulse S of the force.

Let us project equation (15.2) on the axes Ox, Oy and Oz of an inertial coordinate system; this results in

$$\begin{aligned} mv_x - mv_x^0 &= \int_0^t X\,dt \\ mv_y - mv_y^0 &= \int_0^t Y\,dt \\ mv_z - mv_z^0 &= \int_0^t Z\,dt \end{aligned} \tag{15.3}$$

We have thus derived the *scalar form* of the principle of momentum for a particle: *the increment of the projection of the momentum of a particle on a fixed axis of an inertial coordinate system during some time is equal to the impulse of the projection of the force on that axis during that time.*

The principle we have proved can be applied when the force F is dependent only on time (we remind the reader that by F is meant the resultant force); in particular, it can be applied when the force is constant both in its modulus and direction.

1.3. Corollaries.

(1) If $F \equiv 0$ then $mv - mv_0 \equiv 0$ and hence $v \equiv v_0$, that is the motion of the particle obeys Newton's first law (see Sec. 1.1 of Chap. 13) and is inertial.

(2) If $X \equiv 0$ then $mv_x - mv_x^0 \equiv 0$, that is $v_x \equiv v_x^0$.

Hence, if the projection of the acting force on an axis is equal to zero during the whole time of motion of the particle then the projection of the velocity on that axis remains constant.

(3) Let the acting force F retain invariable direction; for instance, let it be parallel to the axis Oz (that is $X = Y = 0$). Then from Corollary (2) we obtain $v_x = v_x^0$, and $v_y = v_y^0$, whence

$$\frac{dx}{dt} = v_x^0 \quad \text{and} \quad \frac{dy}{dt} = v_y^0$$

Multiplying the first identity by $v_y^0\,dt$ and the second identity by $-v_x^0\,dt$ and adding them together we obtain

$$v_y^0\,dx - v_x^0\,dy = 0$$

Finally, the integration of this equality yields

$$v_y^0 x - v_x^0 y = v_y^0 x_0 - v_x^0 y_0$$

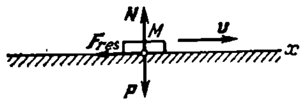


Fig. 15.3

Thus, in this case the motion of the particle is plane, the plane of motion is parallel to the force F and its position is specified by the initial conditions of motion. For instance, this is the case when a particle moves under the action of the force of gravity.

Corollaries (1)-(3) represent first integrals of differential equations (13.7) of motion of a particle.

EXAMPLE 15.1. An electric train approaches the station with velocity $v_0 = 20$ m/s. Determine the breaking period on condition that the resistance force acting during that period is equal to 0.12 of the weight of the train.

Solution. Let us consider the train as a particle (Fig. 15.3). The forces acting on the train are: the weight P of the train, the normal reaction N of the rails and the resistance force F_{res} directed opposite to the motion of the train (in Fig. 15.3 the resistance force goes along the negative x-axis). The first two forces are mutually balanced because there is no motion along the vertical. Let us apply the scalar form of the principle of momentum (as was said, the train is thought of as a particle). From the first formula (15.3) we obtain

$$\frac{P}{g} v_x - \frac{P}{g} v_x^0 = -\int_0^t 0.12P\,dt = -0.12Pt$$

The terminal velocity of the train is $v_x = 0$ and consequently

$$t = \frac{v_x^0}{0.12g} = \frac{20}{0.12 \cdot 9.81} = 17 \text{ s}$$

EXAMPLE 15.2. A ball of mass m moving with velocity v_0 falls on the horizontal plane AB at an angle α (Fig. 15.4). After the impact the ball is reflected from the plane at the same angle α and moves with velocity v ($v = v_0$). Determine the impulse of the force of impact transmitted from the ball to the plane.

Solution. By formula (15.2), we have

$$m v - m v_0 = S \quad \left(S = \int_0^\tau F\,dt\right)$$

where S is the impulse of the force of impact F imparted to the ball by the plane AB during the time of impact τ. From the vector triangle we find

$$S = 2mv_0 \sin\alpha \tag{1}$$

The impulse S is perpendicular to the plane AB. As to the impulse imparted by the ball to the plane, according to Newton's third law it is equal to $-S$.

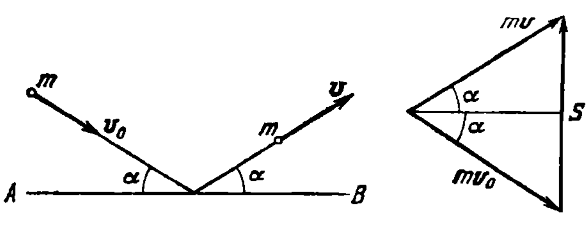


Fig. 15.4

From formula (1) it follows that among all the balls of mass m, the moduli of whose velocities are the same and are equal to v_0, falling on the plane at the various angles, the greatest impulse ($S = 2mv_0$) is imparted to the plane by the ball falling along the normal (at an angle $\alpha = 90°$).

§ 2. Principle of Angular Momentum for a Particle

2.1. Angular Momentum of a Particle about a Centre and an Axis. We shall consider a particle M moving along a trajectory; the momentum of the particle at the given instant is $m\boldsymbol{v} = \boldsymbol{MB}$ (Fig. 15.5). Let us choose arbitrarily a fixed centre O and drop from this centre a perpendicular to the line of action of the vector $m\boldsymbol{v}$; the length h of that perpendicular is called the arm of the vector $m\boldsymbol{v}$ with respect to the centre O.

The *angular momentum* (*the moment of momentum*) *of the particle* about the fixed centre O is the vector

$$\boldsymbol{K}_O = \mathbf{Mom}_O\,(m\boldsymbol{v})$$

applied at the point O whose modulus is equal to the product of the modulus of the vector $m\boldsymbol{v}$ by its arm and whose direction is perpendicular to the plane $\{O, m\boldsymbol{v}\}$ and is specified by the right-hand screw rule. This means that if we observe the plane OMB from the terminus of the vector $\boldsymbol{K}_O$ the rotation of the vector $m\boldsymbol{v}$ about the point O in the plane OMB is counterclockwise. By definition, the modulus of the vector $\boldsymbol{K}_O$ is

$$K_O = mvh$$

From this definition it follows that the *angular momentum vector* $\boldsymbol{K}_O$ *is equal to the vector product of the radius vector* $\boldsymbol{r}$ *of the particle* M *by the vector* $m\boldsymbol{v}$ *representing the momentum of the particle*:

$$\boldsymbol{K}_O = [\boldsymbol{r}, m\boldsymbol{v}] \tag{15.4}$$

Let us draw through the point O an axis Oz and a plane Π perpendicular to that axis. Let us project the vector $m\boldsymbol{v}$ on the plane Π and consider the component $m\boldsymbol{v}_p$ in that plane (in Fig. 15.5 this component is represented by the vector $\boldsymbol{ab}$). Finally, from the point O we drop the perpendicular h_1 to the line of action of the vector $m\boldsymbol{v}_p$.

The *angular momentum* K_z (*the moment of momentum*) *of the particle about the axis* Oz is the product of the modulus of the vector $m\boldsymbol{v}_p$ by the length of the perpendicular h_1 taken with plus or minus sign:

$$K_z = \mathrm{mom}_z\,(m\boldsymbol{v}) = \pm m v_p h_1 \tag{15.5}$$

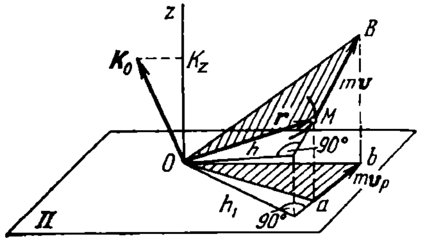


Fig. 15.5

If the direction of rotation of the vector $m v_p$ about the point O in the plane Π is counterclockwise, when observing from the positive direction of the axis Oz, the plus sign is taken in formula (15.5); if otherwise, the minus sign is taken.

We have

$$K_O = 2S_{OMB}, \qquad |K_z| = 2S_{Oab}$$

and, according to the well-known formula of solid geometry,

$$S_{Oab} = S_{OMB} \cos \gamma$$

where γ is the angle between the planes OMB and Π (that is the angle between the perpendiculars K_O and Oz to these planes; see Fig. 15.5). Thus, we obtain the formula

$$K_z = K_O \cos \gamma \tag{15.6}$$

Hence, *the angular momentum of a particle about an axis is equal to the projection on that axis of the angular momentum vector of the particle about any point lying on that axis.*

In the International System of Units (SI) the modulus of angular momentum is measured in $kg \cdot m^2/s$.

The material of the present section is completely analogous to what was presented in Secs. 1.1 and 1.2 of Chap. 5 in connection with the moment of a force. We see that the definitions we have stated for a localized vector (the vector $m v$) coincide with the corresponding definitions for a sliding vector (the force vector). In particular, formula (15.6) is analogous to formula (5.4). By analogy with Sec. 1.1 of Chap. 5, from formula (15.5) it follows that

$$\begin{gathered} \operatorname{mom}_z (m v) = 0 \quad \text{when} \quad h_1 = 0 \\ \text{or} \quad v_p = 0 \quad (\text{or both} \quad h_1 = 0 \quad \text{and} \quad v_p = 0) \end{gathered} \tag{15.7}$$

Hence, the angular momentum of a particle about an axis is equal to zero in the following two cases:

(a) when the line of action of the vector $m v$ passes through the axis;

(b) when the vector $m v$ is parallel to the axis.

In other words, the angular momentum about an axis vanishes when the vector representing the (linear) momentum and the axis lie in one plane.

2.2. Principle of Angular Momentum for a Particle. In this section we shall prove the principle of angular momentum for a particle (the theorem on change of angular momentum) which reads: *the time derivative of the angular momentum vector K_O of a particle about a centre O, which is fixed relative to an inertial coordinate system, is equal to the (vector) moment of the force F, acting on the particle, about that centre O,* that is

$$\frac{dK_O}{dt} = \mathbf{Mom}_O F \tag{15.8}$$

Proof. Let a particle M of mass m be in motion relative to an inertial coordinate system $Oxyz$ (Fig. 15.6) under the action of a force $\boldsymbol{F}$ (throughout the present chapter by $\boldsymbol{F}$ is meant the resultant force in case there are several forces acting on the particle). Let us write down Newton's second law:

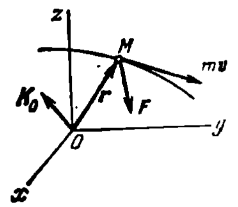


Fig. 15.6

$$\frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{F}$$

Now we perform the vector multiplication on the left of both the members of this identity by the radius vector $\boldsymbol{r}$ of the particle M:

$$\left[\boldsymbol{r}, \frac{d}{dt}(m\boldsymbol{v})\right] = [\boldsymbol{r}, \boldsymbol{F}] \tag{15.9}$$

We shall show that the left-hand side of (15.9) is identically equal to $d\boldsymbol{K}_O/dt$. Indeed, according to formula (15.4) and formula (3) of Introduction to Kinematics, we have

$$\frac{d\boldsymbol{K}_O}{dt} = \frac{d}{dt}[\boldsymbol{r}, m\boldsymbol{v}] = \left[\frac{d\boldsymbol{r}}{dt}, m\boldsymbol{v}\right] + \left[\boldsymbol{r}, \frac{d}{dt}(m\boldsymbol{v})\right]$$

The first summand on the right-hand side of the last equality is equal to zero because it is a vector product of two collinear vectors: $d\boldsymbol{r}/dt = \boldsymbol{v}$ and $m\boldsymbol{v}$. Therefore, formula (15.9) takes the form

$$\frac{d}{dt}[\boldsymbol{r}, m\boldsymbol{v}] = [\boldsymbol{r}, \boldsymbol{F}] \tag{15.10}$$

which is nothing other than formula (15.8) written in full. The principle is proved.

Projecting equality (15.8) on the axes of the inertial coordinate system $Oxyz$ we come to the *scalar form* of the principle of angular momentum for a particle: *the time derivative of the angular momentum of a particle about a fixed (or inertial) axis is equal to the moment of the force, acting on the particle, about that axis*, that is

$$\frac{dK_x}{dt} = \text{mom}_x \boldsymbol{F}, \qquad \frac{dK_y}{dt} = \text{mom}_y \boldsymbol{F}, \qquad \frac{dK_z}{dt} = \text{mom}_z \boldsymbol{F} \tag{15.11}$$

2.3. Corollaries.

(1) If the force $\boldsymbol{F}$ acting on the particle M passes through the fixed centre O during the whole time of motion (such a force is said to be *central*) then $\mathbf{Mom}_O \boldsymbol{F} \equiv 0$, and formula (15.8) implies that

$$\frac{d\boldsymbol{K}_O}{dt} \equiv 0 \text{ whence } \boldsymbol{K}_O(t) = \boldsymbol{K}_O(0)$$

This means that the angular momentum vector of the particle about the centre O is constant both in its modulus and direction.

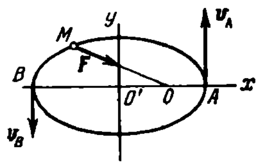


Fig. 15.7

(2) If the moment of the force $\boldsymbol{F}$ acting on the particle M about a fixed (or inertial) axis, say Oz, is identically equal to zero then the angular momentum K_z of the particle about that axis remains constant, that is

$$\text{if} \quad \text{mom}_z \boldsymbol{F} \equiv 0 \quad \text{then} \quad K_z(t) = K_z(0) \tag{15.12}$$

Indeed, the last formula (15.11) implies that $dK_z/dt \equiv 0$, which proves Corollary (2) (that is formula (15.12)).

Corollaries (1) and (2), when written in full, represent first integrals of differential equations of motion of a particle (13.7).

EXAMPLE 15.3. Determine the ratio of the velocity of the Earth at its perihelion A (that is at the point of the Earth's orbit which is nearest to the Sun; Fig. 15.7) to the velocity at the aphelion B (that is at the point of the orbit most distant from the Sun).

Solution. The Earth M moves round the Sun O under the action of the central force: the force of attraction of the Sun. Therefore, by Corollary (1), the angular momentum vector of the Earth about the Sun is constant both in its modulus and direction. The invariability of the direction of the angular momentum implies that the Earth's orbit is a plane curve because if the velocity vector deviated from the initial plane $\{MO, m\boldsymbol{v}_0\}$ this would mean that the direction of the vector $\boldsymbol{K}_O$ changed. Now let us make use of Kepler's first law according to which the Earth moves round the Sun in an elliptic orbit, the Sun being at one of the foci of the ellipse. The angular momentum of the Earth about the centre O (or, which is the same, about the axis Oz) when it is at the point A is

$$\text{mom}_O (m\boldsymbol{v}_A) = mv_A OA = mv_A (a - c)$$

where m is the mass of the Earth, $a = O'A$ is the semimajor axis of the ellipse and $2c = 2O'O$ is the focal distance. Similarly, at the point B the angular momentum is

$$\text{mom}_O (m\boldsymbol{v}_B) = mv_B OB = mv_B (a + c)$$

Since the angular momentum is constant we can equate these quantities, whence

$$mv_A (a-c) = mv_B (a+c), \quad \text{that is} \quad \frac{v_A}{v_B} = \frac{a+c}{a-c} = \frac{1+e}{1-e}$$

where $e = c/a$ is the eccentricity of the ellipse. For the Earth's orbit we have $e = 0.01674$ and thus

$$\frac{v_A}{v_B} = \frac{1+0.01674}{1-0.01674} = 1.034$$

§ 3. Principle of Energy for a Particle

3.1. Work and Power of a Force. The effect of the action of a force on a particle can also be characterized by using the notion of work. *Work* is a measure of the action of the force relative to the path travelled by the point of application of the force.

Let a particle M describe a curvilinear trajectory under the action of a given force $\boldsymbol{F}$ whose modulus and direction may be variable (Fig. 15.8). By $d\boldsymbol{r}$ we shall denote the *elementary displacement vector*

of the particle M during infinitesimal time. Since the velocity of the particle M is the vector $\boldsymbol{v} = d\boldsymbol{r}/dt$, we have

$$d\boldsymbol{r} = \boldsymbol{v}\, dt = \boldsymbol{i}\, dx + \boldsymbol{j}\, dy + \boldsymbol{k}\, dz \tag{15.13}$$

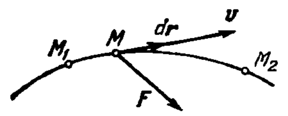


Fig. 15.8

which means that the direction of the elementary displacement coincides with that of the velocity. The projections dx, dy and dz of the elementary displacement vector on the axes Ox, Oy and Oz of an inertial coordinate system are equal to the increments of the corresponding coordinates of M during infinitesimal time dt. For the modulus of the elementary displacement we have

$$|d\boldsymbol{r}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = ds$$

(see Sec. 2.2 of Chap. 7), where ds is the differential of arc length of the trajectory at the point M.

By the *elementary work* δA* *of the force* $\boldsymbol{F}$ is meant the scalar product of the vector $\boldsymbol{F}$ by the vector $d\boldsymbol{r}$:

$$\delta A = (\boldsymbol{F}, d\boldsymbol{r}) \tag{15.14}$$

According to the definition of the scalar product (see (1.9) and (1.10)) the elementary work of the force can be written in the form

$$\delta A = F\, ds \cos(\widehat{\boldsymbol{F}, \boldsymbol{v}}) \qquad (ds = |d\boldsymbol{r}|) \tag{15.15}$$

and in the form

$$\delta A = X\, dx + Y\, dy + Z\, dz \tag{15.16}$$

Formula (15.15) represents the magnitude of the elementary work geometrically while formula (15.16) represents it analytically. In the latter formula X, Y and Z are the projections of the force $\boldsymbol{F}$ on the axes Ox, Oy and Oz of the inertial coordinate system. From formula (15.15) it follows that for $ds \neq 0$ we have $\delta A > 0$ when $0 \leqslant (\widehat{\boldsymbol{F}, \boldsymbol{v}}) < 90°$, $\delta A < 0$ when $90° < (\widehat{\boldsymbol{F}, \boldsymbol{v}}) \leqslant 180°$ and $\delta A = 0$ when $\boldsymbol{F} \perp \boldsymbol{v}$.

By the *work of the force* $\boldsymbol{F}$ *along a finite path* M_1M_2 (see Fig. 15.8) is meant the line integral of the elementary work δA along the arc M_1M_2 of the trajectory:

$$A = \int_{\smile M_1M_2} (\boldsymbol{F}, d\boldsymbol{r}) = \int_{\smile M_1M_2} F \cos(\widehat{\boldsymbol{F}, \boldsymbol{v}})\, ds \tag{15.17}$$

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* The symbol δA denotes an infinitesimal quantity which, in the general case, is not the differential of work. Only in some special cases is the elementary work of the force equal to the total differential of a function dependent on the coordinates.

This can also be written as

$$A=\int_{\smile M_1M_2}(X\,dx+Y\,dy+Z\,dz) \tag{15.18}$$

Formula (15.17) expresses the work of the force geometrically while formula (15.18) expresses it analytically.

In the International System of Units (SI) the unit of work (and of energy) is the *joule* (J) which is the work done by a force of one newton (N) along a rectilinear path of one metre on condition that the force acts in the direction of the displacement.

By the *power of the force* $\boldsymbol{F}$ is meant the ratio of the elementary work δA to the time dt during which this work is performed:

$$N=\frac{\delta A}{dt} \tag{15.19}$$

By formula (15.14), the power N at the given instant is equal to the scalar product of the force $\boldsymbol{F}$ by the velocity $\boldsymbol{v}$ of the particle M acted upon by that force:

$$N=(\boldsymbol{F},\ \boldsymbol{v})=X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt} \tag{15.20}$$

In the International System of Units (SI) the unit of power is the *watt* (W) which is the power of a force performing the work of one joule per second. Besides, up to the present time quite widespread use is made of a unit of power with the absurd name *horse power* (hp):

$$1\ \text{hp}\approx 745.7\ \text{W}$$

In the case when a particle is under the action of several forces $\boldsymbol{F}_1,\ \boldsymbol{F}_2,\ \ldots,\ \boldsymbol{F}_l$ (Fig. 15.9) there holds

LEMMA. *The work of the resultant of the forces applied to a moving particle along a path M_1M_2 is equal to the algebraic sum of the work performed by the constituent forces along that path.*

Proof. By formula (15.17), we have

$$A=\int_{\smile M_1M_2}(\boldsymbol{R},d\boldsymbol{r})=\int_{\smile M_1M_2}(\boldsymbol{F}_1+\boldsymbol{F}_2+\ldots+\boldsymbol{F}_l,\ d\boldsymbol{r})$$

$$=\int_{\smile M_1M_2}\{(\boldsymbol{F}_1,d\boldsymbol{r})+(\boldsymbol{F}_2,d\boldsymbol{r})+\ldots+(\boldsymbol{F}_l,\ d\boldsymbol{r})\}$$

A line integral of an algebraic sum of integrand functions is equal to the algebraic sum of the line integrals of each of the functions and therefore

$$A=\int_{\smile M_1M_2}(\boldsymbol{F}_1,\ d\boldsymbol{r})+\int_{\smile M_1M_2}(\boldsymbol{F}_2,\ d\boldsymbol{r})+\ldots+\int_{\smile M_1M_2}(\boldsymbol{F}_l,\ d\boldsymbol{r})$$

The lemma is proved.

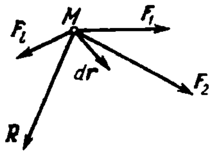

Fig. 15.9

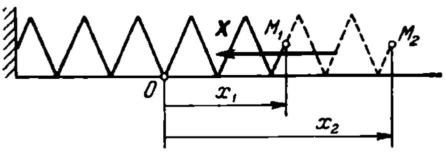

Fig. 15.10

The *kinetic energy T of a particle* is a scalar quantity equal to half the product of the mass of the particle by the square of its velocity:

$$T = \frac{1}{2} mv^2$$

The kinetic energy is another measure characterizing particle motion. It is a nonnegative quantity which vanishes only when the velocity of the particle is equal to zero, that is in the case of a state of rest.

EXAMPLE 15.4. *The Work of an Elastic Force.* For a rectilinear motion formula (15.18) takes the form

$$A = \int_{x_1}^{x_2} X\,dx$$

where x_1 and x_2 are the abscissas of the initial and the terminal positions of the particle. Let us compute the work of the elastic force $X = -cx$ of a spring (here c is the stiffness factor of the spring, see Sec. 1.1 of Chap. 14) when the particle moves from the position M_1 to the position M_2 (Fig. 15.10). By the above formula we have

$$A = -\int_{x_1}^{x_2} cx\,dx = \frac{1}{2} c\,(x_1^2 - x_2^2)$$

3.2. Principle of Energy. In this section we shall prove the *principle of energy* (the theorem on change of the kinetic energy) which reads: *the increment of the kinetic energy of a particle along a given path is equal to the work along that path of the forces acting on the particle.*

Proof. Let a particle M of mass m undergo a displacement from a position M_0 (at which its velocity is v_0) to a position M (at which its velocity is v) under the action of a force $\boldsymbol{F}$ (Fig. 15.11). We shall make use of the first of the natural equations of motion of a particle (see (13.8)):

$$m\frac{dv_\tau}{dt} = F_\tau = F\cos(\widehat{\boldsymbol{F}, \boldsymbol{v}})$$

Let us multiply this equation by $v_\tau dt = ds$:

$$mv_\tau\,dv_\tau = mv\,dv = F\cos(\widehat{\boldsymbol{F}, \boldsymbol{v}})\,ds$$

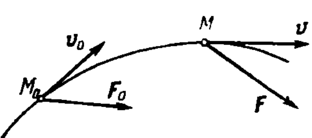

Fig. 15.11

According to (15.15) this can be written in the form

$$d\left(\frac{1}{2}mv^2\right)=\delta A \tag{15.21}$$

Thus, the differential of the kinetic energy T of the moving particle M is equal to the elementary work of the force F. Integrating the last equality within the limits corresponding to the positions M_0 and M of the moving particle we obtain

$$T-T_0=A=\int_{\smile M_0M}\delta A$$

The last equality can be written in full thus:

$$\frac{1}{2}mv^2-\frac{1}{2}mv_0^2=\int_{\smile M_0M}(X\,dx+Y\,dy+Z\,dz) \tag{15.22}$$

The principle is proved.

EXAMPLE 15.5. A sledge with steel runners on a horizontal ice surface is pushed, after which the path s travelled by the sledge until it stops and the time of motion t are measured. Determine the coefficient of sliding friction of steel on ice.

Solution. The force of weight mg of the sledge is balanced by the normal reaction N. During the sliding the modulus of the friction force assumes its greatest value $fN=fmg$ (see Sec. 1.1 of Chap. 4). According to the principle of energy, for the constant force of friction directed opposite to the motion we have

$$0-\frac{1}{2}mv_0^2=-fmgs \tag{1}$$

From this relation we cannot yet find f because the initial velocity v_0 is unknown. Let us make use of the principle of momentum (in the scalar form). The first equation (15.3) yields

$$0-mv_0=-fmgt \tag{2}$$

Eliminating t from equalities (1) and (2) we obtain

$$f=\frac{2s}{gt^2}$$

Thus the coefficient of friction is equal to the ratio of the path travelled by the sledge on the ice to the distance passed in the free fall during the same time

3.3. Potential Force Field. Force Function. Let us suppose that the force F acting on a particle depends solely on the position of the particle, that is

$$X=X(x,y,z),\ Y=Y(x,y,z),\ Z=Z(x,y,z)$$

The domain of the functions X, Y and Z is called a *force field*. If a particle moves in a force field and if the work of the forces of the field is independent of the shape of the path along which the particle travels and depends solely on the initial position M_1 and the ter-

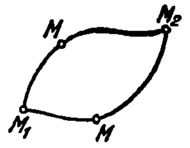

Fig. 15.12

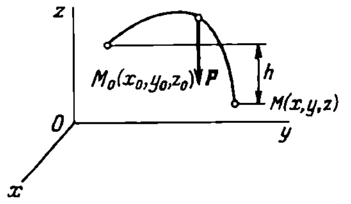

Fig. 15.13

minal position M_2 of the particle (Fig. 15.12) the force field is said to be *potential*. In a potential force field the work along any closed contour is equal to zero. As is proved in the theory of line integrals (see [1]), this condition is equivalent to the property that the elementary work of the force $\boldsymbol{F}$ is the total differential of a function $U(x, y, z)$, that is

$$\delta A = X\,dx + Y\,dy + Z\,dz = dU \tag{15.23}$$

The function $U(x, y, z)$ is called the *force function*. Since

$$dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz$$

the last two equalities and the independence of the differentials dx dy and dz imply that

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = \frac{\partial U}{\partial z} \tag{15.24}$$

Conditions (15.24) are necessary and sufficient for the force field to be potential.

The force function $U(x, y, z)$ can be defined as the work performed by the forces of the field when the particle M moves from a position $M_0(x_0, y_0, z_0)$ to an arbitrarily chosen position $M(x, y, z)$:

$$A = \int_{\smile M_0M} (X\,dx + Y\,dy + Z\,dz)$$

$$= \int_{\smile M_0M} dU = U(x, y, z) - U(x_0, y_0, z_0) \tag{15.25}$$

Below are given two examples of potential force fields.

EXAMPLE 15.6. *Homogeneous Gravitational Field*. If the axis Oz goes vertically upward (Fig. 15.13), the force of gravity is $\boldsymbol{P} = -mg\boldsymbol{k}$, that is

$$X = 0, \quad Y = 0, \quad Z = -mg$$

Let a particle M moving under the action of the force of gravity $\boldsymbol{P}$ travel a path along a curve from a point $M_0(x_0, y_0, z_0)$ to a point $M(x, y, z)$ (see Fig. 15.13). By formula (15.25), for the force function we have the expression

$$U(x, y, z) - U(x_0, y_0, z_0) = \int_{\smile M_0M} Z\,dz = -mg\int_{z_0}^{z} dz = -mgz + mgz_0$$

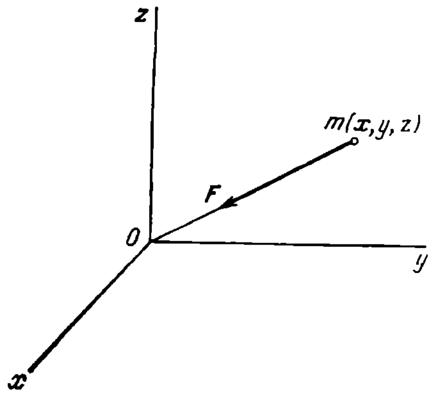

Fig. 15.14

It follows that $U = -mgz$, which corresponds to conditions (15.24). The last formula expressing the work performed by the gravity field can be written in the form

$$A = -mgh \qquad (15.26)$$

where $h = z - z_0$ is the difference between the heights of the terminal and the initial points. It is due to the fact that the gravity field is potential that the work of the force of gravity is independent of the shape of the trajectory and is specified by formula (15.26). The work is negative when the particle moves upward along the trajectory ($h > 0$) and is positive when it moves downward ($h < 0$).

EXAMPLE 15.7. *Central Gravitational Field.* Let us place the origin O at the centre of attraction (Fig. 15.14). Then the projections of the force of the Newtonian attraction are

$$X = -\frac{\mu m x}{r^3}, \quad Y = -\frac{\mu m y}{r^3}, \quad Z = -\frac{\mu m z}{r^3} \quad (r = \sqrt{x^2 + y^2 + z^2})$$

where m is the mass of the attracted particle; μ is the gravitational constant; x, y and z are the coordinates of the particle and r is its radius vector.

By formula (15.23),

$$dU = -\frac{\mu m}{r^3}(x\,dx + y\,dy + z\,dz) = -\frac{\mu m}{r^3} r\,dr = -\frac{\mu m}{r^2}\,dr$$

whence, performing integration, we find

$$U = \frac{\mu m}{r} + C_1$$

The function U is the sought-for force function of the Newtonian attraction.

3.4. Conservation of the Mechanical Energy of a Particle in Its Motion in a Potential Force Field. For a motion of a particle in a potential force field the principle of energy (15.22) can be written, by virtue of (15.25), in the form

$$\frac{1}{2} mv^2 - \frac{1}{2} mv_0^2 = U(x, y, z) - U(x_0, y_0, z_0)$$

Let us define the *potential energy* $V(x, y, z)$ *of the particle* as a function equal to the force function* with the opposite sign, that is

$$V(x, y, z) = -U(x, y, z)$$

Then the above formula can be written in the form

$$\frac{1}{2} mv^2 + V(x, y, z) = \frac{1}{2} mv_0^2 + V(x_0, y_0, z_0) = \text{const} \qquad (15.27)$$

* Both the functions U and V are determined to within an arbitrary additive constant.

Equality (15.27) represents an integral of differential equations of motion (13.7) for a motion of the particle in a potential force field. This is the so-called *kinetic energy integral*. The integral specifies the part of space where the motion is possible because

$$T + V = \text{const} \geqslant 0$$

What has been said will be elucidated at the end of Example 16.1.

The *total mechanical energy E of the particle* is equal to the sum of its kinetic energy and potential energy:

$$E = T + V$$

The kinetic energy integral can be written in the form

$$E = E_0 = \text{const} \tag{15.28}$$

Formula (15.28) expresses the *law of conservation of the total mechanical energy of a particle in its motion in a potential force field.* This law is a special case of the general law of conservation of energy studied in physics. In reality the particle is acted upon by resistance forces during its motion and therefore the concept of motion in a potential force field is an idealization of the real motion of the particle. This idealization provides an adequate description of the motion in case the resistance forces are small. However, in reality mechanical energy is always dissipated; more precisely, it is transformed into thermal, electric and other forms of energy in accordance with the general law of conservation of energy.

EXAMPLE 15.8. A body is thrown vertically upward from the Earth's surface. Determine the initial velocity v_0 required for the body to attain a height equal to the radius of the Earth R taking into account the known fact that the force of attraction is inversely proportional to the square of the distance from the body to the centre of the Earth (see Fig. 13.11).

Solution. The projections of the force of attraction on the coordinate axes shown in Fig. 13.11 are

$$X = -\frac{pR^2}{x^2}, \quad Y = Z = 0$$

where p is the weight of the body on the Earth's surface. Let us regard the body as a particle and apply the principle of energy (formula (15.22)):

$$-\frac{1}{2}\frac{p}{g}v_0^2 = -\int_R^{2R} \frac{pR^2}{x^2}\,dx$$

According to the condition of the problem, the terminal velocity is $v = 0$, and the abscissa x of the particle varies from R (on the Earth's surface) to $2R$ during the motion. Computing the definite integral

$$R^2\int_R^{2R} \frac{dx}{x^2} = R^2\left[-\frac{1}{x}\right]_R^{2R} = R^2\left[-\frac{1}{2R}+\frac{1}{R}\right] = \frac{1}{2}R$$

we find

$$v_0 = \sqrt{gR} = \sqrt{9.81\cdot 6.37\cdot 10^6} = 7910 \text{ m/s}$$

This value of the velocity is equal to circular (or orbital) velocity (see Example 20.2).

The answer to the problem can also be obtained in the solution of Example 13.7 where the integration of the differential equation of rectilinear motion of a particle was carried out. The aim of the present example is to show that the application of the general principles of dynamics in some cases makes it possible to avoid the integration of equations (13.7) of particle dynamics. These are the cases when the general principles of dynamics provide a member of first integrals of equations of particle dynamics sufficient for the solution of the problem. We recommend the reader to pay attention to this remark.

Problems

PROBLEM 15.1. A heavy body is let to move up an inclined plane with initial velocity $v_0 = 10$ m/s. The angle of inclination of the plane to the horizon is equal to 30° and the coefficient of friction is equal to 0.1. How long will the body move and what path will it travel until it stops?

Answer. $t = 1.74$ s, $s = 86.8$ m.

PROBLEM 15.2. A particle of mass m is in a harmonic oscillation along the axis Ox, the law of oscillation being $x = a \cos(\omega t - \varphi_0)$. Find the laws describing the variations of the kinetic energy T, the potential energy V and the total energy E of the moving particle as functions of the coordinate x; the value of the potential energy V for $x = 0$ is taken to be zero.

Answer. $T = \frac{1}{2} m\omega^2 (a^2 - x^2)$, $V = \frac{1}{2} m\omega^2 x^2$, $E = \frac{1}{2} m\omega^2 a^2$.

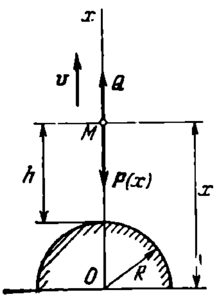

Fig. 15.15

PROBLEM 15.3. A body of mass m moves vertically upward with constant acceleration a starting from its state of rest on the surface of the Earth. The force of attraction $P(x)$ is inversely proportional to the square of the distance x from the body to the centre of the Earth (see Example 13.7); the air resistance is neglected. Determine the thrust force Q making the body move upward and the work of this force performed as the body achieves a height h (Fig. 15.15).

Answer. $Q = m\left(a + \frac{R^2}{x^2} g\right)$, $A = mh\left(a + \frac{R}{R+h} g\right)$.

Chapter 16 Motion of a Constrained Particle. Relative Motion of a Particle

Newton's laws only apply to a particle and, more precisely, only to a free particle. Similarly, all the general principles presented in Chap. 15 only apply to a free particle. Particle dynamics naturally splits into two divisions: dynamics of a free particle and dynamics of a constrained particle.

If the motion of a particle is limited by some additional conditions which impose some limits on the displacement of the particle, the particle is said to be *constrained.* The conditions imposed on the kinematic characteristics of the motion of a particle are called *constraints*, and the forces realizing these constraints are called the *constraint reactions*. Below we state a general principle which makes it possible to consider the motion of a constrained particle as that of a free particle.

AXIOM OF CONSTRAINTS. *A constrained particle can be considered as a free particle detached from its constraints if the constraint reactions (the passive forces) are added to the given (active) forces.*

According to this axiom the motion of a constrained particle can be treated as that of a free particle to which all the laws of motion, which were by now established for the motion of a free particle, can be applied if the constraint reactions (the passive forces) replacing the action of the constraints on the particle are added to the given forces applied to the particle (the active forces).

An essential distinction between the constraint reactions and the active forces is that the former are not given in the statement of the problem and must be determined, like the motion itself, in the process of the solution of the dynamic problem. Therefore in dynamics the constraints themselves are termed dynamic in order to stress their distinction from the constraints in statics (see Secs. 2.8 and 2.9 of Chap. 1).

§ 1. Motion of a Constrained Particle

1.1. Motion of a Particle on a Surface. Let a particle M be in motion on a fixed surface under the action of an active force $\boldsymbol{F}$ (Fig. 16.1). Here $Oxyz$ is an inertial coordinate system (see Sec. 1.2 of Chap. 13) relative to which the surface is fixed. The equation

$$f(x, y, z) = 0 \tag{16.1}$$

describing the surface is the constraint equation because, according to the condition, the coordinates x, y and z of the particle M must satisfy equation (16.1) at any instant t.

Let us consider the case of a perfect (perfectly smooth) surface, that is a surface whose reaction force $\boldsymbol{N}$ is normal to it at any instant during the motion of the particle on the surface:

$$\boldsymbol{N} \parallel \boldsymbol{\nu} \tag{16.2}$$

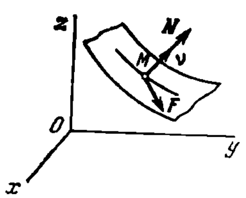

Fig. 16.1

The projections of unit normal vector $\boldsymbol{\nu}$ to the surface on the axes Ox, Oy and Oz are proportional to the corresponding partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$. Therefore condi-

tion (16.2) can be written in the form

$$\frac{N_x}{\frac{\partial f}{\partial x}} = \frac{N_y}{\frac{\partial f}{\partial y}} = \frac{N_z}{\frac{\partial f}{\partial z}} = \lambda \tag{16.3}$$

where λ is a proportionality factor. Differential equations of motion (13.6) take the form

$$m\frac{d^2x}{dt^2} = X + N_x, \quad m\frac{d^2y}{dt^2} = Y + N_y, \quad m\frac{d^2z}{dt^2} = Z + N_z \tag{16.4}$$

where X, Y, Z and N_x, N_y, N_z are the projections of the active force $\boldsymbol{F}$ and of the passive force $\boldsymbol{N}$ (the reaction force) on the axes Ox, Oy and Oz. Finding N_x, N_y and N_z from conditions (16.3) and adding constraint equation (16.1) we can write equations (16.4) in the form

$$\begin{gathered} m\frac{d^2x}{dt^2} = X + \lambda\frac{\partial f}{\partial x}, \quad m\frac{d^2y}{dt^2} = Y + \lambda\frac{\partial f}{\partial y} \\ m\frac{d^2z}{dt^2} = Z + \lambda\frac{\partial f}{\partial z}, \quad f(x, y, z) = 0 \end{gathered} \tag{16.5}$$

These are the *differential equations of motion of a particle on a fixed surface written in Lagrange's form involving the multiplier* λ. They form a system of four equations with the four unknowns x, y, z and λ.

By virtue of (16.1), the total differential of the function $f(x, y, z)$ is equal to zero:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$$

Making use of (16.3) we rewrite the last equality in the form

$$(\boldsymbol{N}, d\boldsymbol{r}) = N_x\,dx + N_y\,dy + N_z\,dz = 0 \tag{16.6}$$

Condition (16.6) expresses the fact that the reaction force $\boldsymbol{N}$ and the elementary displacement $d\boldsymbol{r}$ are mutually perpendicular. Hence, in the case of an ideal constraint the work of the reaction force is equal to zero for any real displacement. Consequently, the principle of energy and its corollaries (see § 3 of Chap. 15) must hold, their statement remaining the same as in the case of a free particle.

EXAMPLE 16.1. Let us consider the motion of a heavy particle on a fixed sphere of radius R (a spherical pendulum). We shall place the origin O at the centre of the sphere and draw the axis Oz vertically upward (Fig. 16.2). The constraint equation is

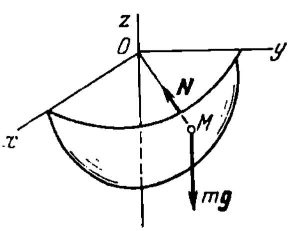

Fig. 16.2

$$f(x, y, z) \equiv x^2 + y^2 + z^2 - R^2 = 0$$

and therefore we have

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z$$

In the case under consideration the only active force is the force of gravity whose projections on the axes Ox, Oy and Oz are

$$X = Y = 0, \quad Z = -mg$$

Lagrange's equations (16.5) with multiplier λ are written in the form

$$m\ddot{x} = 2\lambda x, \quad m\ddot{y} = 2\lambda y, \quad m\ddot{z} = -mg + 2\lambda z, \quad x^2 + y^2 + z^2 - R^2 = 0$$

These are the differential equations of motion of a spherical pendulum. Since the particle moves in a potential force field (the gravity field) whose potential energy is

$$V = -U = mgz$$

(see Example 15.6), there exists energy integral (15.27) which has the form

$$\frac{1}{2} mv^2 + mgz = \frac{1}{2} mv_0^2 + mgz_0$$

It follows that the square of the velocity of the spherical pendulum is equal to

$$v^2 = v_0^2 - 2g(z - z_0)$$

at any instant t. Since $v^2 \geqslant 0$ the inequality

$$z \leqslant \frac{v_0^2}{2g} + z_0$$

holds during the whole time of motion of the spherical pendulum.

When investigating the motion of a particle on a fixed surface we can make use of natural equations of motion (13.8). Since the tangent is perpendicular to the normal to the surface at any point of the trajectory on the surface we have $N_\tau = 0$, and equations (13.8) take the form

$$m\frac{dv_\tau}{dt} = F_\tau, \quad m\frac{v^2}{\rho} = F_n + N_n, \quad 0 = F_b + N_b \qquad (16.7)$$

We remind the reader that here F_τ, F_n, F_b and 0, N_n, N_b are the projections of the active force and of the reaction force on the natural axes, that is on the tangent τ, the principal normal n and the binormal b of the trajectory.

Let us consider the case when $\boldsymbol{F} = \boldsymbol{0}$ (the case of an inertial motion of a particle on a fixed smooth surface). From the first equation (16.7) we obtain $v = v_0$, and the third equation implies $N_b = 0$, which means that the direction of the reaction force coincides with that of the principal normal to the trajectory. The curves on a surface at whose every point the principal normal coincides with the normal to the surface are called *geodesic lines*. Hence, the trajectory of an inertial motion of a particle on a fixed smooth surface is a geodesic line, the modulus of the velocity being constant.

EXAMPLE 16.2. A heavy particle M of mass m is in motion on a smooth surface of a hemisphere of radius R (Fig. 16.3), and there is no friction. At the initial instant the particle is at the point A and has an initial horizontal velocity $\boldsymbol{v}_0$. At what point will the particle M leave the surface of the hemisphere? Also determine the value of v_0 for which the particle M will leave the sphere at the initial instant.

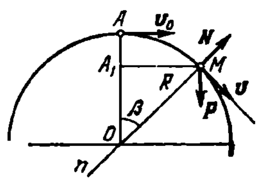

Fig. 16.3

Solution. Let us consider the motion of the particle M at an instant when the radius OM forms an angle β with the vertical. In this position the

particle M is acted upon by the force of gravity P directed vertically downward and the reaction force N of the hemisphere perpendicular to the surface of the hemisphere (because the surface of the hemisphere is perfectly smooth and there is no friction). Equations (16.7) are written in the form

$$m\frac{dv}{dt}=mg\sin\beta,\qquad m\frac{v^2}{R}=mg\cos\beta-N \tag{16.8}$$

When the particle M moves on the hemisphere the work of the reaction force N is equal to zero while, according to formula (15.26), the work of the force of gravity is equal to

$$mgAA_1 = mg\,(R - OA_1) = mgR\,(1-\cos\beta)$$

By the principle of energy (see (15.22)) we have

$$\frac{1}{2}mv^2-\frac{1}{2}mv_0^2=mgR\,(1-\cos\beta)$$

whence

$$v^2 = v_0^2 + 2gR\,(1-\cos\beta)$$

From the second equation (16.8) we express the modulus of the reaction force in terms of the angle β:

$$N=mg\cos\beta-\frac{mv^2}{R}=3mg\cos\beta-\frac{mv_0^2}{R}-2mg \tag{16.9}$$

The particle M remains on the surface of the hemisphere when $N > 0$. It leaves the surface of the hemisphere at the instant when the reaction N vanishes. Putting $N = 0$ in (16.9) we determine the angle β:

$$\cos\beta=\frac{2}{3}+\frac{v_0^2}{3gR}$$

In particular, if v_0 is negligibly small then

$$\beta=\arccos\frac{2}{3}=48°11'$$

The reaction N_0 of the hemisphere at the initial instant of motion of the particle (in the position A) can be found by putting $\beta = 0$ in formula (16.9), which yields

$$N_0=mg\left(1-\frac{v_0^2}{gR}\right)$$

For the particle M to separate from the surface of the hemisphere at the initial instant it is necessary that the condition $N_0 \leqslant 0$ should hold, whence follows the condition

$$v_0 \geqslant \sqrt{gR}$$

For a particle on the Earth's surface we have

$$\sqrt{gR}=\sqrt{9.81\cdot 6.37\cdot 10^6}=7910\ \text{m/s}$$

This limiting value is equal to the circular velocity.

1.2. Motion of a Particle along a Curve. We shall suppose that a particle M moves along a given fixed curve under the action of an active force $\boldsymbol{F}$ (Fig. 16.4). Let the curve be determined as the intersection of two surfaces; then the equations of the curve in the inertial coordinate system $Oxyz$ are

$$f_1(x, y, z) = 0,\quad f_2(x, y, z) = 0 \tag{16.10}$$

Equalities (16.10) are constraint equations because, according to the condition, the coordinates of the moving particle must satisfy the constraint equations at any instant during the motion of the particle. The curve is assumed to be smooth, that is, like in Sec. 1.1, we consider ideal constraints (the constraints without friction). This means that the reaction N of the constraint (the passive force) is expressed as

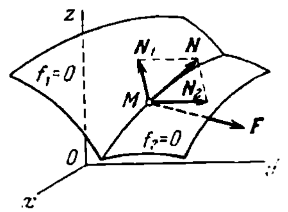

Fig. 16.4

$$\boldsymbol{N} = \boldsymbol{N}_1 + \boldsymbol{N}_2 = \lambda_1 \,\mathbf{grad}\, f_1 + \lambda_2 \,\mathbf{grad}\, f_2 \qquad (16.11)$$

where λ_1 and λ_2 are scalar multipliers, and $\mathbf{grad}\, f_1$ and $\mathbf{grad}\, f_2$ are the vectors

$$\mathbf{grad}\, f_1 = \frac{\partial f_1}{\partial x}\,\boldsymbol{i} + \frac{\partial f_1}{\partial y}\,\boldsymbol{j} + \frac{\partial f_1}{\partial z}\,\boldsymbol{k}$$

and

$$\mathbf{grad}\, f_2 = \frac{\partial f_2}{\partial x}\,\boldsymbol{i} + \frac{\partial f_2}{\partial y}\,\boldsymbol{j} + \frac{\partial f_2}{\partial z}\,\boldsymbol{k}$$

These vectors called ***gradients*** are directed along the normals to the surfaces $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$ respectively at the point M. From (16.11) it follows that the projections of the reaction force N on the axes Ox, Oy and Oz are

$$\begin{gathered} N_x = \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x}, \quad N_y = \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} \\ N_z = \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z} \end{gathered} \qquad (16.12)$$

Differential equations of motion (13.6) take the form

$$m\frac{d^2x}{dt^2} = X + N_x, \quad m\frac{d^2y}{dt^2} = Y + N_y, \quad m\frac{d^2z}{dt^2} = Z + N_z \qquad (16.13)$$

where X, Y and Z are the projections of the active force $\boldsymbol{F}$ on the axes Ox, Oy and Oz. Substituting into (16.13) the expressions of N_x, N_y and N_z found from (16.12) and adding constraint equations (16.10) we obtain

$$\begin{gathered} m\frac{d^2x}{dt^2} = X + \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x} \\ m\frac{d^2y}{dt^2} = Y + \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} \\ m\frac{d^2z}{dt^2} = Z + \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z} \\ f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0 \end{gathered} \qquad (16.14)$$

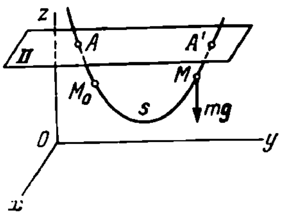

Fig. 16.5

These are the *differential equations of motion of a particle along a given fixed curve written in Lagrange's form involving the multipliers* λ_1 *and* λ_2. They form a system of five equations with the five unknowns x, y, z, λ_1 and λ_2.

In the case of ideal constraints the work of the reaction forces along any real path travelled by the particle is equal to zero. In the case under consideration this can be proved by analogy with Sec 1.1; in this connection we note that the indicated property can be taken as the definition of ideal constraints and is the so-called "axiom of ideal constraints" (see Sec. 1.2 of Chap. 17). Consequently, for the motion of a particle along a fixed smooth curve there holds the principle of energy and its corollaries (see § 3 of Chap. 15) with the same statement as in the case of a free particle.

The position of a particle on a fixed curve can be specified by one parameter, for instance, by the arc length. If the energy integral takes place in this case, it is itself sufficient for determining the position of the particle. Let us consider the motion of a heavy particle along a given curve (Fig. 16.5). The potential energy is $V = mgz$ (see Example 15.6) and energy integral (15.27) is written in the form

$$\frac{1}{2} mv^2 + mgz = \frac{1}{2} mv_0^2 + mgz_0 \equiv mga \qquad \left(a = z_0 + \frac{v_0^2}{2g} = \text{const}\right)$$

whence follows

$$v^2 = 2g\,(a - z) \tag{16.15}$$

To elucidate the mechanical meaning of the constant a let us consider the plane II: $z = a$. At the points A and A' of intersection of the plane $z = a$ with the given curve (see Fig. 16.5) there hold the equalities

$$v_A = v_{A'} = 0$$

which are implied by formula (16.15). Let us denote $a - z = \zeta$; then formula (16.15) takes the form

$$v^2 = 2g\zeta$$

We see that the velocity of the particle M coincides with that of free fall in the plane II without initial velocity. Let s denote the arc M_0M; then $v = ds/dt$, and (16.15) can be written as

$$\frac{ds}{dt} = \sqrt{2g}\,\sqrt{a - z}$$

For the points belonging to the given curve the function $z = z(s)$ is known. Therefore, separating variables in the last differential equation and performing integration, we obtain

$$\frac{1}{\sqrt{2g}} \int_0^s \frac{ds}{\sqrt{a - z(s)}} = t \qquad (16.16)$$

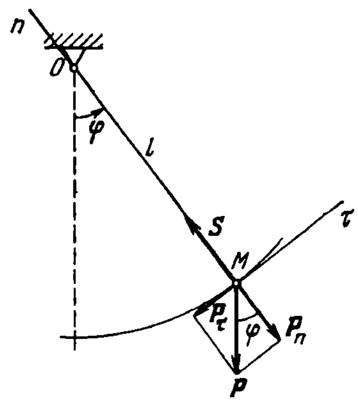

Fig. 16.6

We see that the problem reduces to quadrature (16.16) from which the function $t = t(s)$ can be found; its inverse function $s = s(t)$ specifies the law of motion. However, even in the case of a simple pendulum the exact solution we are speaking of reduces to elliptic integrals which cannot be expressed in terms of elementary functions. Therefore we shall limit ourselves to an approximate solution of the problem of motion of a simple pendulum.

1.3. Simple Pendulum. Let us consider an inextensible weightless thread of length l whose one end is attached to a fixed hinge O while from the other end a heavy particle M of mass m is suspended. We shall investigate the motion of the simple pendulum in the plane Oxy perpendicular to the axis of the hinge (Fig. 16.6).

The particle M moving along the arc of a circle of radius l is under the action of the force of gravity $\boldsymbol{P}$ (the active force) and the tension S of the thread (the passive force). Let φ be the angle of deviation of the thread from the vertical. We shall make use of natural equations of motion (13.8):

$$m \frac{dv_\tau}{dt} = -mg \sin \varphi, \quad m \frac{v^2}{l} = -mg \cos \varphi + S \qquad (16.17)$$

Since the algebraic value of the velocity of motion of the particle along the circle is $v_\tau = l\, d\varphi/dt$, the first of equations (16.17) is written in the form

$$l \frac{d^2\varphi}{dt^2} + g \sin \varphi = 0$$

For small oscillation of the pendulum we put $\sin \varphi \approx \varphi$; then, dividing by l, we arrive at the differential equation of small oscillation of a simple pendulum:

$$\ddot{\varphi} + k^2\varphi = 0 \qquad \left(k^2 = \frac{g}{l}\right) \qquad (16.18)$$

We have thus obtained the differential equation of harmonic oscillation (see Sec. 1.1 of Chap. 14) whose general solution is given by

formula (14.4):

$$\varphi = \alpha \cos (kt - \beta) \tag{16.19}$$

where the angular amplitude α and the initial phase β of oscillation are determined from the initial conditions using formulas (14.8). In particular, according to formula (14.9) expressing the law of oscillation, for the initial conditions

$$\varphi (0) = \varphi_0, \quad \dot{\varphi} (0) = 0$$

we have the expression

$$\varphi = \varphi_0 \cos \sqrt{\frac{g}{l}}\, t$$

The period T of oscillation is found from formula (14.5):

$$T = \frac{2\pi}{k} = 2\pi \sqrt{\frac{l}{g}} \tag{16.20}$$

The period T of small oscillation is independent of the initial conditions. This is the *property of isochronism of small oscillation of a simple pendulum.*

From the second of equations (16.17) we find the dynamic tension S of the thread. Since $v = l\dot{\varphi}$ we obtain

$$S = mg \cos \varphi + ml\dot{\varphi}^2$$

where $\dot{\varphi}$ is found by differentiating law of motion (16.19). For $\varphi = 0$ the derivative $\dot{\varphi}$ attains its maximum value

$$\dot{\varphi}_{\max} = \alpha k$$

and we have

$$S_{\max} = mg + ml\alpha^2k^2 = S_{\text{stat}} (1 + \alpha^2)$$

where α is the angular amplitude of oscillation measured in radians.

§ 2. Relative Motion of a Particle

In this section concluding particle dynamics we shall consider the motion of a particle M of mass m relative to a moving coordinate system $O'x'y'z'$ which itself is in motion relative to an inertial coordinate system $Oxyz$ (Fig. 16.7). The force $\boldsymbol{F}$ acting on the particle (here by $\boldsymbol{F}$ will be meant the resultant of all the forces both active and passive) is an accelerating force, it is determined relative to the inertial coordinate system $Oxyz$ which we conditionally consider as a fixed (absolute) frame of reference. However, we are going to determine the motion of the particle relative to the moving coordinate system $O'x'y'z'$, and here lies the essence of the statement of the problem.

2.1. Differential Equations of Relative Motion of a Particle. Let us write Newton's second law (13.1) for the particle M in the

form

$$m\boldsymbol{w}_{\mathrm{abs}} = \boldsymbol{F} \qquad (16.21)$$

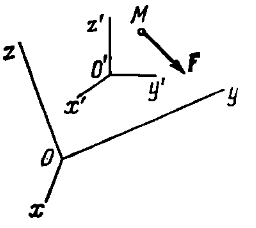

Fig. 16.7

The subscript abs indicates that on the left-hand side there is the absolute acceleration vector of the particle. By the Coriolis theorem (formula (11.11)), the acceleration of the absolute motion of the particle is equal to the vector sum of the transportation acceleration $\boldsymbol{w}_{\mathrm{tr}}$, the relative acceleration $\boldsymbol{w}_{\mathrm{rel}}$ and the Coriolis acceleration $\boldsymbol{w}_{\mathrm{C}}$:

$$\boldsymbol{w}_{\mathrm{abs}} = \boldsymbol{w}_{\mathrm{tr}} + \boldsymbol{w}_{\mathrm{rel}} + \boldsymbol{w}_{\mathrm{C}}$$

From (16.21) we obtain

$$m\boldsymbol{w}_{\mathrm{tr}} + m\boldsymbol{w}_{\mathrm{rel}} + m\boldsymbol{w}_{\mathrm{C}} = \boldsymbol{F}$$

Since we are interested in the study of the relative motion, let us separate out the product of the mass of the particle by the relative acceleration vector:

$$m\boldsymbol{w}_{\mathrm{rel}} = \boldsymbol{F} + (-m\boldsymbol{w}_{\mathrm{tr}}) + (-m\boldsymbol{w}_{\mathrm{C}}) \qquad (16.22)$$

On the right-hand side there is the accelerating force measured in the moving coordinate system $O'x'y'z'$. The expression $(-m\boldsymbol{w}_{\mathrm{tr}})$ is called the *transportation inertial force* and $(-m\boldsymbol{w}_{\mathrm{C}})$ is the *Coriolis inertial force.*

For the Coriolis acceleration vector we have (see formula (11.13))

$$\boldsymbol{w}_{\mathrm{C}} = 2\,[\boldsymbol{\omega}_{\mathrm{tr}},\ \boldsymbol{v}_{\mathrm{rel}}] = 2\begin{vmatrix} \boldsymbol{i}' & \boldsymbol{j}' & \boldsymbol{k}' \\ p & q & \tilde{r} \\ \frac{dx'}{dt} & \frac{dy'}{dt} & \frac{dz'}{dt} \end{vmatrix}$$

$$= 2\left(q\frac{dz'}{dt} - \tilde{r}\frac{dy'}{dt}\right)\boldsymbol{i}' + 2\left(\tilde{r}\frac{dx'}{dt} - p\frac{dz'}{dt}\right)\boldsymbol{j}'$$

$$+ 2\left(p\frac{dy'}{dt} - q\frac{dx'}{dt}\right)\boldsymbol{k}'$$

Here $\boldsymbol{i}'$, $\boldsymbol{j}'$ and $\boldsymbol{k}'$ are unit vectors along the axes of the moving coordinate system $O'x'y'z'$; p, q and $\tilde{r}$ are the projections on the axes $O'x'$, $O'y'$ and $O'z'$ of the vector $\boldsymbol{\omega}_{\mathrm{tr}}$ representing the instantaneous angular velocity of the moving coordinate system relative to the fixed system, and dx'/dt, dy'/dt and dz'/dt are the projections of the relative velocity vector $\boldsymbol{v}_{\mathrm{rel}}$ on the same axes.

Let us project vector equation (16.22) on the axes of the moving coordinate system $O'x'y'z'$:

$$
\begin{aligned}
m\frac{d^2x'}{dt^2} &= F_{x'} - mw_{x'}^{\mathrm{tr}} - 2m\left(q\frac{dz'}{dt} - \tilde{r}\frac{dy'}{dt}\right)\\
m\frac{d^2y'}{dt^2} &= F_{y'} - mw_{y'}^{\mathrm{tr}} - 2m\left(\tilde{r}\frac{dx'}{dt} - p\frac{dz'}{dt}\right)\\
m\frac{d^2z'}{dt^2} &= F_{z'} - mw_{z'}^{\mathrm{tr}} - 2m\left(p\frac{dy'}{dt} - q\frac{dx'}{dt}\right)
\end{aligned}
\qquad (16.23)
$$

These are the *differential equations of relative motion of the particle M*.

EXAMPLE 16.3. Find the Coriolis inertial force acting on a train going to the south along the meridian at the latitude of Leningrad ($\varphi = 60°$).

Solution. In Example 11.4 we computed the modulus of the Coriolis acceleration $w_C = 0.00252$ m/s² for $v = 20$ m/s. For the modulus of the Coriolis inertial force acting on a railway car whose mass is 60 000 kg (and whose length is approximately equal to that of the rail) moving with a velocity 20 m/s we have

$$mw_C = 60\,000 \cdot 0.00252 = 151 \text{ N}$$

In Example 11.4 we also showed that the Coriolis acceleration w_C is directed to the left from the direction of motion. The Coriolis force ($-mw_C$) is directed to the right and presses the wheel flanges to the inner side of the right rail.

Let us discuss the physical phenomena in this problem. For an observer in the fixed coordinate system the motion of the train is not rectilinear because, besides the motion along the meridian, it moves to the east due to the Earth's rotation. Consequently, from the point of view of this observer the train going to the south has an acceleration directed to the left (to the east). This means that there must be a force acting in this direction. This force is produced by the reaction of the right rail which undergoes deformation.

These conclusions are confirmed by practice. It is known that for the double-track railway in the northern hemisphere the inner side of the right rail is considerably worn. A similar effect is observed for the rivers flowing in the meridional direction. The right banks of the rivers flowing in the northern hemisphere are washed away and are steeper. Analogously, in the northern hemisphere the northern winds are deviated to the right (to the east), which accounts for the north-east trade winds in the northern hemisphere.

2.2. **Special Cases.** Let us consider some special cases of relative motion of a particle.

1. Suppose that the system $O'x'y'z'$ is in a *translatory motion* (that is $\omega_{\mathrm{tr}} \equiv 0$). In this case the Coriolis acceleration w_C is identically equal to zero, and equations (16.23) take the form

$$m\frac{d^2x'}{dt^2} = F_{x'} - mw_{x'}^{\mathrm{tr}}, \quad m\frac{d^2y'}{dt^2} = F_{y'} - mw_{y'}^{\mathrm{tr}}, \quad m\frac{d^2z'}{dt^2} = F_{z'} - mw_{z'}^{\mathrm{tr}}$$

In such a case the transportation acceleration w_{tr} of the particle M is equal to the acceleration $w_{O'}$ of the origin of the moving coordinate system, and therefore the differential equations of relative motion

are

$$m\frac{d^2x'}{dt^2}=F_{x'}-mw_{x'}^{O'},\quad m\frac{d^2y'}{dt^2}=F_{y'}-mw_{y'}^{O'}$$
$$m\frac{d^2z'}{dt^2}=F_{z'}-mw_{z'}^{O'} \tag{16.24}$$

Here $w_{x'}^{O'}$, $w_{y'}^{O'}$ and $w_{z'}^{O'}$ are the projections of the absolute acceleration vector of the point O' on the axes $O'x'$, $O'y'$ and $O'z'$ and they are known functions of time because the motion of the coordinate system $O'x'y'z'$ is assumed to be known.

2. Suppose that the coordinate system $O'x'y'z'$ is in a *translatory motion* with absolute *velocity* $\boldsymbol{v}_{O'}$ which has a *constant* modulus and an invariable direction. Then $\boldsymbol{w}_{O'}=\boldsymbol{0}$, and the differential equations of relative motion take the form

$$m\frac{d^2x'}{dt^2}=F_{x'},\quad m\frac{d^2y'}{dt^2}=F_{y'},\quad m\frac{d^2z'}{dt^2}=F_{z'}$$

which coincides with that of the equations of absolute motion. This implies the *equivalence of all the inertial frames of reference*; in other words, the laws of mechanics are stated quite similarly for all the frames of reference whose motion relative to one another is translatory, rectilinear and uniform. It is this property that constitutes the *relativity principle of classical dynamics* (the Galileo-Newton relativity principle; see Sec. 1.2 of Chap. 13).

Finally, let us consider the case when the particle is at *relative rest*, that is the particle is at rest relative to the moving coordinate system $O'x'y'z'$. In this case $\boldsymbol{v}_{\text{rel}}=\boldsymbol{w}_{\text{rel}}\equiv\boldsymbol{0}$ and consequently $\boldsymbol{w}_{\text{C}}\equiv\boldsymbol{0}$. From (16.22) we obtain

$$\boldsymbol{F}+(-m\boldsymbol{w}_{\text{tr}})=\boldsymbol{0} \tag{16.25}$$

Hence, in the case of relative rest the resultant of all the active and passive forces acting on the particle is balanced by the transportation inertial force.

EXAMPLE 16.4. Let a particle M be at relative rest on the Earth's surface (Fig. 16.8). We shall place the origin of the moving coordinate system at the centre O' of the Earth; let the axis $O'z'$ be directed to the North Pole and the axis $O'y'$ be directed to the point of intersection of the meridian on which the particle lies with the equator. The angle θ is called the *geocentric latitude*. We shall assume that the density of the Earth is constant for every spherical layer. Then the force of attraction $\boldsymbol{F}=m\boldsymbol{a}$ is in the direction to the centre of the Earth. In its transportation the particle M moves along a circle of radius $AM=R\cos\theta$, where R is the radius of the Earth, with constant angular velocity Ω. The transportation acceleration $\boldsymbol{w}_{\text{tr}}$ is directed to the point A, and its modulus is equal to $AM\Omega^2$. The modulus of the transportation inertial force $(-m\boldsymbol{w}_{\text{tr}})$ is equal to $mR\Omega^2\cos\theta$. Equation of relative rest (16.25) is written in the form

$$m\boldsymbol{a}+\boldsymbol{N}+(-m\boldsymbol{w}_{\text{tr}})=\boldsymbol{0}$$

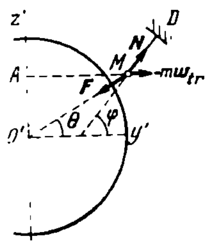

Fig. 16.8

where N is the reaction force. Projecting this vector equation on the axes $O'y'$ and $O'z'$ we obtain

$$\begin{aligned} -ma\cos\theta + N_{y'} + mR\Omega^2\cos\theta &= 0 \\ -ma\sin\theta + N_{z'} &= 0 \end{aligned} \tag{16.26}$$

The vector N specifies the direction of the local (gravity) vertical at the given place on the Earth's surface and specifies the angle φ called the ***astronomical latitude*** of that place. From equations (16.26) we find $N_{y'}$ and $N_{z'}$, and thus obtain

$$\tan\varphi = \frac{N_{z'}}{N_{y'}} = \frac{ma\sin\theta}{ma\cos\theta - mR\Omega^2\cos\theta} = \frac{\tan\theta}{1-\mu} \tag{16.27}$$

where

$$\mu = \frac{R\Omega^2}{a} = \frac{1}{289}$$

Formula (16.27) expresses the relationship between the astronomical latitude and the geocentric latitude.

The force of gravity is opposite to the reaction N and has the same modulus: $N = -m\boldsymbol{g}$. Therefore $N_{y'} = -mg\cos\varphi$ and $N_{z'} = -mg\sin\varphi$. Cancelling equations (16.26) by m we can write them in the form

$$g\cos\varphi = a(1-\mu)\cos\theta, \quad g\sin\varphi = a\sin\theta$$

Let us square these equations and add them together:

$$g^2(\cos^2\varphi + \sin^2\varphi) = a^2(\cos^2\theta - 2\mu\cos^2\theta + \mu^2\cos^2\theta + \sin^2\theta)$$

For the modulus of the acceleration of gravity we have

$$g = a\sqrt{1-2\mu\cos^2\theta+\mu^2\cos^2\theta} \approx a(1-\mu\cos^2\theta) \tag{16.28}$$

whence it is seen that the maximum value of the acceleration of gravity is attained at the poles ($\theta=\pi/2$) and the minimum value on the equator ($\theta=0$).

Now we can state strict definitions of the force of gravity and of weight. The *force of gravity acting on the particle* is equal to the product of the mass by the acceleration of gravity. The *weight of a body* is the numerical value (the modulus) of the resultant of the forces of gravity of the particles forming the body.

EXAMPLE 16.5. *Deviation to the East of Falling Bodies.*

Solution. Let us take a local coordinate system whose axis $O'z'$ is directed along the gravity vertical (see the foregoing example), the axis $O'x'$ lies in the meridional plane and is directed southward perpendicularly to the axis $O'z'$, and the axis $O'y'$ is directed eastward so that this coordinate system is rectangular and right-handed (Fig. 16.9). Now let us consider the free fall in vacuum of a particle of mass m (without taking into account the air resistance); we shall write a system of differential equations (16.23) for this particle. It is the resultant of the active force $\boldsymbol{F}$ and the transportation inertial force ($-m\boldsymbol{w}_{\text{tr}}$) (appearing due to the rotation of the Earth) that is the force of gravity at the given point on the Earth's surface:

$$\boldsymbol{F} - m\boldsymbol{w}_{\text{tr}} = m\boldsymbol{g} = -mg\boldsymbol{k}'$$

Consequently

$$F_{x'} - mw_{x'}^{\text{tr}} = F_{y'} - mw_{y'}^{\text{tr}} = 0, \quad F_{z'} - mw_{z'}^{\text{tr}} = -mg$$

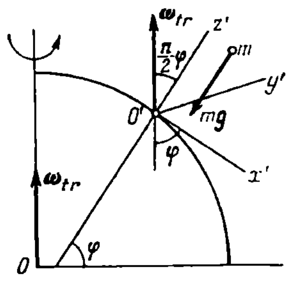

Fig. 16.9

The projections p, q and $\tilde{r}$ of the angular velocity vector $\boldsymbol{\omega}_{tr}$ of the Earth's rotation on the axes $O'x'$, $O'y'$ and $O'z'$ are

$$p = -\omega_{tr} \cos\varphi, \quad q = 0, \quad \tilde{r} = \omega_{tr} \sin\varphi$$

where the angle φ denotes the astronomical latitude of the given place. Hence, the differential equations of relative motion of a heavy particle in vacuum near the Earth's surface are written in the form

$$\frac{d^2x'}{dt^2} = 2\omega_{tr} \sin\varphi \frac{dy'}{dt}$$

$$\frac{d^2y'}{dt^2} = -2\omega_{tr}\left(\sin\varphi \frac{dx'}{dt} + \cos\varphi \frac{dz'}{dt}\right)$$

$$\frac{d^2z'}{dt^2} = -g + 2\omega_{tr} \cos\varphi \frac{dy'}{dt}$$

We shall apply the iterative method (the method of successive approximations) to find the particular solution of that system specified by the zero initial conditions

$$x'(0) = y'(0) = z'(0) = \dot{x}'(0) = \dot{y}'(0) = \dot{z}'(0) = 0$$

The substitution of the initial data into the right-hand sides results in the equations of first approximation for x_1', y_1' and z_1':

$$\frac{d^2x_1'}{dt^2} = 0, \quad \frac{d^2y_1'}{dt^2} = 0, \quad \frac{d^2z_1'}{dt^2} = -g$$

From these equations, for the given initial conditions, we find

$$x_1' = 0, \; y_1' = 0, \; z_1' = -\frac{1}{2} gt^2$$

Thus, the well-known law of free fall of bodies is accurate only to the first approximation. The substitution of the values of x_1', y_1' and z_1' thus found into the right-hand sides of the original system of differential equations yields the equations of the second approximation for x_2', y_2' and z_2':

$$\frac{d^2x_2'}{dt^2} = 0, \quad \frac{d^2y_1'}{dt^2} = 2\omega_{tr} \cos\varphi \cdot gt, \quad \frac{d^2z_2'}{dt^2} = -g$$

From this system we find, for the given initial conditions, the solution

$$x_2' = 0, \; y_2' = \frac{1}{3} \omega_{tr} g \cos\varphi \cdot t^3, \quad z_2' = -\frac{1}{2} gt^2$$

The axis $O'y'$ is directed eastward, and hence the second approximation describes the deviation of the falling bodies to the east produced by the Coriolis inertial force. For instance, after 10 seconds of free fall at the latitude of Leningrad ($\varphi = 60°$) the deviation to the east is equal to

$$y_2'(10) = \frac{2\pi}{3 \cdot 86\,400} \cdot 9.81 \cdot \frac{1}{2} \cdot 10^3 = 0.119 \text{ m}$$

Problems

PROBLEM 16.1. A small heavy ring of mass m is freely put on a smooth wire circle of radius R placed in the vertical plane. At the initial instant the ring is at the lowest point of the circle and gains an initial velocity v_0. Find the condition under which the ring makes one revolution along the circle and determine the pressure N which the ring exerts on the circle when it is at the upmost

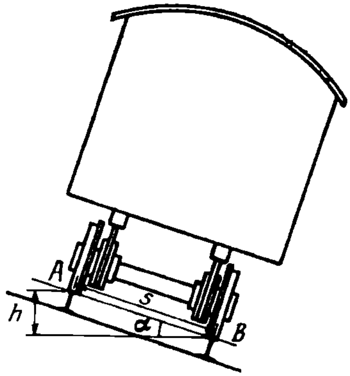

Fig. 16.10

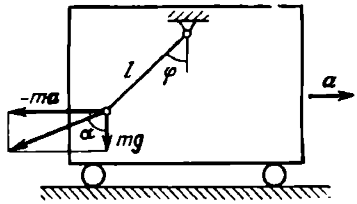

Fig. 16.11

point.

Answer. $v_0 \geqslant \sqrt{2gR}$, $N = 3mg - \frac{mv_0^2}{R}$.

PROBLEM 16.2. A heavy particle is suspended from two threads of equal length whose upper ends are attached to a horizontal supporting plane, the angle of inclination of each thread to the vertical being equal to α. At the initial instant one of the threads is cut. At that instant the tension of the remaining thread receives an instantaneous increase; find the ratio of the increased tension T of that thread to its initial tension T_0.

Answer. $T/T_0 = 2\cos^2\alpha$.

PROBLEM 16.3. Determine the superelevation h of the outer rail of the railway curve (Fig. 16.10). The radius of the curve is R, the railway gauge is s and the velocity of the train is v.

Hint. The resultant of the force of weight mg of the railway car and the transportation inertial force ($-mw_{tr}$) (the "centrifugal force"; it is also applied at the centre of gravity of the railway car) must be perpendicular to the straight line AB passing through the upper ends of the rails. From this condition find $\tan\alpha$ (see Fig. 16.10).

Answer. $h = s\sin\alpha = \frac{v^2}{\sqrt{v^4 + R^2g^2}}\, s$. ▮

PROBLEM 16.4. A simple pendulum is suspended inside a railway car which is in a rectilinear motion with constant acceleration a (Fig. 16.11). The length of the inextensible thread from which the pendulum is suspended is equal to l. Find the period T of oscillation of the pendulum.

Answer. $T = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a^2}}}$.

Chapter 17 Analytic Statics

§ 1. Principle of Virtual Work

1.1. Basic Definitions. In this chapter we shall consider the equilibrium of a system of particles. By a *system of particles* is understood a collection of particles subjected to the action of both the internal

forces of interaction between the particles of the system and the constraints imposed on the system.

By the *constraints* in mechanics are meant the restrictions (conditions) imposed on the position or on the motion of the particles forming the system. These restrictions are assumed to be given in advance and to be independent of either the applied active forces or the initial conditions of motion. Constraints are realized with the aid of surfaces, rods, threads, etc.

We shall consider only *geometric constraints*; moreover we shall suppose that they are *time independent* (*stationary*). In other words, we shall deal with those constraints whose equations do not involve the time derivatives of the coordinates and do not involve explicitly time t. Besides, we shall suppose that the particles cannot leave the constraints; such constraints are called *bilateral*. An example of a stationary bilateral geometric constraint imposed on the particles constituting an absolutely rigid body is the condition of the invariability of the distances between any two particles of the body.

When there are constraints imposed on the particles of a system, not every displacement is possible for that system.

By a *virtual displacement of a system* is meant every infinitesimal displacement of the particles of that system which is compatible with the constraints imposed on the system. In other words, by a virtual displacement of a system is understood every possible elementary displacement of the particles of the system which can be realized without violating the constraints imposed on that system. Let the coordinates of the νth particle M_ν of the system be x_ν, y_ν and z_ν. The virtual displacement of the νth particle is a vector

$$\delta \boldsymbol{r}_\nu = \delta x_\nu \boldsymbol{i} + \delta y_\nu \boldsymbol{j} + \delta z_\nu \boldsymbol{k}$$

Here $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors along the axes of an inertial rectangular Cartesian coordinate system $Oxyz$. The quantities δx_ν, δy_ν and δz_ν are the projections of the virtual displacement on these axes; they are called the *variations of the coordinates*.

The real displacement of the particles of the system during infinitesimal time in their real motion under the action of the forces acting on the system (in the case of stationary constraints) is one of the virtual displacements. However, not every virtual displacement of the particles of the system is a real displacement. The real displacement of the νth particle is a vector

$$d\boldsymbol{r}_\nu = dx_\nu \boldsymbol{i} + dy_\nu \boldsymbol{j} + dz_\nu \boldsymbol{k}$$

As usual, here $\boldsymbol{r}_\nu$ is the radius vector of the νth particle; dx_ν, dy_ν and dz_ν are the differentials of the coordinates.

1.2. Definition of Ideal Constraints. The action of constraints can be replaced by the reaction forces; this is the axiom which was

stated in elementary statics (see Sec. 2.8 of Chap. 1) and in Introduction to Chap. 16 where the motion of a constrained particle was studied.

AXIOM OF CONSTRAINTS FOR A SYSTEM OF PARTICLES. *If the constraint reactions (the passive forces) are added to the given (active) forces applied to a system of particles then the system can be thought of as being released from the constraints exerting these reactions.*

From this axiom it follows that we can replace by the reactions either all the constraints and then consider the system of particles as being free or a part of the constraints.

Let us suppose that there is a system of n particles $M_1, M_2, \ldots, M_n$. Let $R_x^{(\nu)}$, $R_y^{(\nu)}$ and $R_z^{(\nu)}$ be the projections of the resultant $\boldsymbol{R}_\nu$ of the reactions of the constraints applied to the particle M_ν on the axes Ox, Oy and Oz ($\nu = 1, 2, \ldots, n$). In what follows we shall always consider *ideal constraints.*

Definition of Ideal Constraints. The total elementary work (see Sec. 3.1 of Chap. 15) *done by the reactions of ideal constraints is equal to zero for every virtual displacement of the particles of the system,* that is

$$\sum_{\nu=1}^{n} (\boldsymbol{R}_\nu, \delta\boldsymbol{r}_\nu) = \sum_{\nu=1}^{n} (R_x^{(\nu)}\,\delta x_\nu + R_y^{(\nu)}\,\delta y_\nu + R_z^{(\nu)}\,\delta z_\nu) = 0 \qquad (17.1)$$

Some examples of ideal (frictionless) constraints are given below.

(a) A particle M is on a fixed smooth surface T (Fig. 17.1*a*) or on a fixed smooth curve L (Fig. 17.1*b*). In this case the reaction $\boldsymbol{N}$ of the constraint is along the normal $\boldsymbol{n}$ to the surface or along one of the normals $\boldsymbol{n}'$ to the curve. The virtual displacements $\delta\boldsymbol{r}$ of the particle M are in the tangent plane to the surface T or are directed along the tangent $\boldsymbol{\tau}$ to the curve at the point M. In both the cases $\boldsymbol{N} \perp \delta\boldsymbol{r}$ and therefore

$$(\boldsymbol{N}, \delta\boldsymbol{r}) = N_x\,\delta x + N_y\,\delta y + N_z\,\delta z = 0$$

(b) A rigid body has two fixed points (see Fig. 5.16). In this case the work of the reaction forces $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ of the constraints is equal to zero because the points of application of the reactions remain fixed for every virtual displacement of the rigid body (in this case a virtual displacement is a rotation through an infinitesimal angle $\delta\varphi$).

(c) The condition expressing the constraints imposed on the particles of an absolutely rigid body reduces to the requirement that the distances between any two particles of the body remain constant. The total work done by the internal forces of interaction is equal to zero for any virtual displacement of the rigid body.

Let the particles $M_1, M_2, \ldots, M_n$ of the system in question be acted upon by the corresponding active forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$

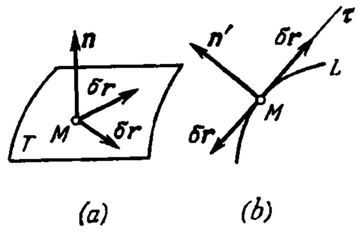

Fig. 17.1

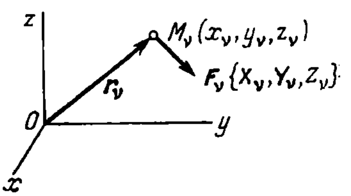

Fig. 17.2

(Fig. 17.2). If a particle M_ν of the system is acted upon by several forces then $\boldsymbol{F}_\nu$ is understood as the resultant force. The projections of the forces $\boldsymbol{F}_\nu$ ($\nu = 1, 2, \ldots, n$) on the axes Ox, Oy and Oz will be denoted by X_ν, Y_ν and Z_ν, that is

$$\boldsymbol{F}_\nu = X_\nu \boldsymbol{i} + Y_\nu \boldsymbol{j} + Z_\nu \boldsymbol{k} \quad (\nu = 1, 2, \ldots, n)$$

1.3. Principle of Virtual Work. This principle was introduced by Johann Bernoulli (1667-1748); it reads: *for a system of particles subjected to stationary bilateral ideal geometric constraints to be in equilibrium it is necessary and sufficient that the total elementary work done by the active forces should be equal to zero for any virtual displacement of the system from the equilibrium position under consideration*, that is

$$\delta A = \sum_{\nu=1}^{n} \delta A_\nu = \sum_{\nu=1}^{n} (\boldsymbol{F}_\nu, \ \delta \boldsymbol{r}_\nu)$$

$$= \sum_{\nu=1}^{n} (X_\nu \, \delta x_\nu + Y_\nu \, \delta y_\nu + Z_\nu \, \delta z_\nu) = 0 \quad (17.2)$$

on condition that at the initial instant the system is at rest.

Proof. Necessity. Let it be known that the system of particles is in equilibrium; it is required to prove that relation (17.2) holds. To this end we replace the constraints by the reactions, that is we add the constraint reactions $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\ldots$, $\boldsymbol{R}_n$ to the active forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\ldots$, $\boldsymbol{F}_n$. Since each of the particles of the system is in equilibrium we have

$$\boldsymbol{F}_\nu + \boldsymbol{R}_\nu = 0 \quad (\nu = 1, \ 2, \ \ldots, \ n) \quad (17.3)$$

In terms of the projections this is written in the form

$$X_\nu + R_x^{(\nu)} = 0, \quad Y_\nu + R_y^{(\nu)} = 0, \quad Z_\nu + R_z^{(\nu)} = 0$$
$$(\nu = 1, \ 2, \ \ldots, \ n) \quad (17.4)$$

Let us multiply scalarly each of equalities (17.3) by the virtual displacement vector $\delta \boldsymbol{r}_\nu$ of the particle M_ν ($\nu = 1, 2, \ldots, n$) and add together the resultant scalar products:

$$\sum_{\nu=1}^{n} (\boldsymbol{F}_\nu, \ \delta \boldsymbol{r}_\nu) + \sum_{\nu=1}^{n} (\boldsymbol{R}_\nu, \ \delta \boldsymbol{r}_\nu) = 0$$

In terms of the projections this is written as

$$\sum_{\nu=1}^{n} (X_\nu \, \delta x_\nu + Y_\nu \, \delta y_\nu + Z_\nu \, \delta z_\nu) + \sum_{\nu=1}^{n} (R_x^{(\nu)} \, \delta x_\nu + R_y^{(\nu)} \, \delta y_\nu + R_z^{(\nu)} \, \delta z_\nu) = 0$$

According to the definition of ideal constraints (17.1), the second sum is equal to zero and we thus obtain (17.2). The necessity is proved.

Sufficiency. Let condition (17.2) be fulfilled; it is required to prove that the system of particles is in equilibrium. We again add the reaction forces $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_n$ to the given active forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$. Then the system can be thought of as being free. We shall denote by

$$\tilde{\delta} \boldsymbol{r}_\nu = \tilde{\delta} x_\nu \boldsymbol{i} + \tilde{\delta} y_\nu \boldsymbol{j} + \tilde{\delta} z_\nu \boldsymbol{k} \quad (\nu = 1, 2, \ldots, n)$$

the virtual displacements of the *free* particles. For such a system the principle of virtual work is expressed in the form

$$\sum_{\nu=1}^{n} (\boldsymbol{F}_\nu + \boldsymbol{R}_\nu, \, \tilde{\delta} \boldsymbol{r}_\nu) = 0$$

Since for a free system the quantities $\tilde{\delta} x_\nu$, $\tilde{\delta} y_\nu$ and $\tilde{\delta} z_\nu$ ($\nu = 1, 2, \ldots, n$) can assume any small values we can put

$$\tilde{\delta} \boldsymbol{r}_\nu = \alpha \, (\boldsymbol{F}_\nu + \boldsymbol{R}_\nu) \quad (\nu = 1, 2, \ldots, n)$$

in the last equality, where $\alpha > 0$ is an infinitesimal. This yields

$$\alpha \sum_{\nu=1}^{n} (\boldsymbol{F}_\nu + \boldsymbol{R}_\nu)^2 = \alpha \sum_{\nu=1}^{n} [(X_\nu + R_x^{(\nu)})^2 + (Y_\nu + R_y^{(\nu)})^2 + (Z_\nu + R_z^{(\nu)})^2] = 0$$

Since the sum of the squares is equal to zero it follows that

$$X_\nu + R_x^{(\nu)} = 0, \quad Y_\nu + R_y^{(\nu)} = 0, \quad Z_\nu + R_z^{(\nu)} = 0 \quad (\nu = 1, 2, \ldots, n)$$

which is nothing other than equilibrium conditions (17.4) for all the particles of the system. Sufficiency is proved.

Now we shall consider the application of the principle of virtual work to the simplest mechanisms. Let at a point A of a mechanism a force $\boldsymbol{P}$ be applied, and let B be the point of application of a resistance force $\boldsymbol{Q}$ (Fig. 17.3). Both the forces and the virtual displacements $\delta \boldsymbol{r}_A$ and $\delta \boldsymbol{r}_B$ are directed along the tangents to the trajectories of the particles A and B. An example of this type is a lever of the first or of the second kind. By the principle of virtual work (17.2), for the equilibrium state of the mechanism we have

$$(\boldsymbol{P}, \, \delta \boldsymbol{r}_A) + (\boldsymbol{Q}, \, \delta \boldsymbol{r}_B) = P \, |\delta \boldsymbol{r}_A| - Q \, |\delta \boldsymbol{r}_B| = 0 \qquad (17.5)$$

Let us denote $\delta p = |\delta r_A|$ and $\delta q = |\delta r_B|$, and let the real displacements of the points (particles) A and B (which belong to the class of virtual displacements because the constraints are stationary) take place during time δt. Now we divide equality (17.5) by δt and write it in the form

$$Pu - Qv = 0, \text{ that is } \frac{P}{Q} = \frac{v}{u} \quad (17.6)$$

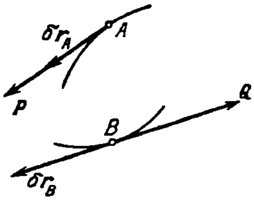

Fig. 17.3

where $u = \delta p/\delta t$ and $v = \delta q/\delta t$ are virtual velocities of the particles A and B (Johann Bernoulli himself stated the principle of virtual work in terms of virtual velocities). We have thus arrived at the *basic law for the simplest mechanisms*:

To any gain in force there corresponds an equal loss in displacement or in velocity.

Using this law we can readily derive equilibrium conditions and consequently the relationship between the moving force P and the resistance force Q for various mechanisms and appliances such as an inclined plane, a wedge, a screw, a gear drive, etc.

EXAMPLE 17.1. To the handle of a screw press (Fig. 17.4) a rotating moment $M_z = 2Pl$ (the moment of the driving force) is applied while the platform of the press undergoes the reaction Q of the compressed body which plays the role of the resistance force. Assuming that this is an ideal mechanism derive the equilibrium condition.

Solution. We suppose that the system is given a virtual displacement: let the screw turn through an angle $\delta\varphi > 0$; then the platform of the press receives a displacement $\delta q > 0$. The displacements of the points A_1 and A_2 are along the tangents to the circle, and their moduli are equal:

$$\delta p = l\,\delta\varphi$$

By formula (17.5), we have

$$2Pl\,\delta\varphi - Q\,\delta q = 0, \quad \text{that is } M_z\,\delta\varphi - Q\,\delta q = 0$$

We shall find the relationship between $\delta\varphi$ and δq. Let h be the pitch of the screw, that is the (upward or downward) path, travelled by the screw, corresponding to its one revolution. There holds the proportion

$$\frac{\delta\varphi}{2\pi} = \frac{\delta q}{h}, \text{ that is } \delta q = \frac{h}{2\pi}\,\delta\varphi$$

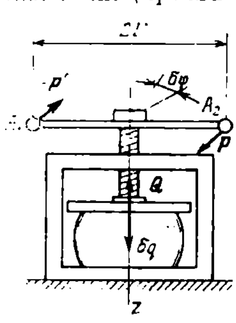

Fig. 17.4

Let us substitute δq into the equilibrium condition:

$$M_z\,\delta\varphi - Q\frac{h}{2\pi}\,\delta\varphi = 0$$

Cancelling by $\delta\varphi$ we obtain

$$M_z = \frac{Qh}{2\pi}$$

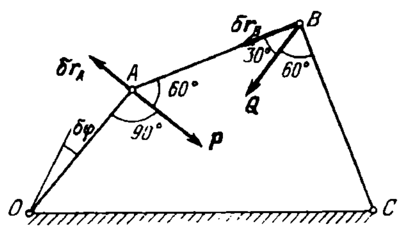

Fig. 17.5

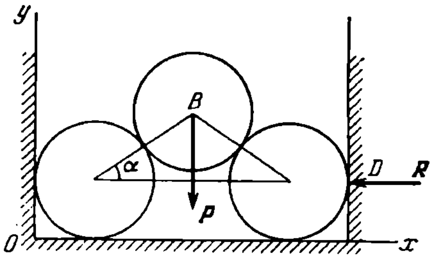

Fig. 17.6

It should be stressed that the reaction forces of constraints are not involved in the principle of virtual work, which makes it possible to solve statics problems without determining the reactions.

EXAMPLE 17.2. Let us consider the equilibrium of the hinged rod quadrilateral $OABC$ shown in Fig. 17.5. The side OC of the quadrilateral is fixed and the hinge A is acted upon by a force $\boldsymbol{P}$ applied at a right angle to the rod OA. To the hinge B a force $\boldsymbol{Q}$ is applied at an angle of 60° to the rod CB. The other data are: $\angle OAB = 150°$ and $\angle ABC = 90°$. Find the magnitude of the force Q.

Solution. Suppose that the system is given a virtual displacement: let the rod OA turn counterclockwise about the point O through an angle $\delta\varphi$. Then the point A receives a displacement $\delta \boldsymbol{r}_A$ perpendicular to OA, and the point B gains a displacement $\delta \boldsymbol{r}_B$ perpendicular to CB, the direction of $\delta \boldsymbol{r}_B$ coinciding with that of BA. Let us write equation of work (17.2):

$$(\boldsymbol{P}, \delta \boldsymbol{r}_A) + (\boldsymbol{Q}, \delta \boldsymbol{r}_B) = -P\,|\delta \boldsymbol{r}_A| + Q\,|\delta \boldsymbol{r}_B| \cos 30° = 0$$

On the basis of the theorem on the equality of the projections of the velocity vectors of the ends of a line segment on the direction of that line segment (formula (10.7)), we have

$$\mathrm{proj}_{BA}\, \delta \boldsymbol{r}_B = \mathrm{proj}_{BA}\, \delta \boldsymbol{r}_A$$

that is

$$|\delta \boldsymbol{r}_B| = |\delta \boldsymbol{r}_A| \cos 60° = \frac{1}{2}\, |\delta \boldsymbol{r}_A|$$

Substituting these values into the equation of work we obtain

$$-P\,|\delta \boldsymbol{r}_A| + \frac{1}{2}\, Q\,|\delta \boldsymbol{r}_A| \frac{\sqrt{3}}{2} = 0$$

whence

$$Q = \frac{4}{\sqrt{3}}\, P = 2.31P$$

The principle of virtual work also makes it possible to determine the constraint reactions. To this end all the constraints or their part are replaced by the corresponding reaction forces depending on whether all the reactions or their part must be found. We shall illustrate this by an example.

EXAMPLE 17.3. Three similar tubes of weight P each are placed as shown in Fig. 17.6. Determine the pressure which the tubes exert on the smooth walls.

Solution. Let us replace one of the constraints, say the right wall, by its reaction $\boldsymbol{R}$. We shall state the problem thus: what horizontal active force $\boldsymbol{R}$

must be applied to the right lower tube in order to preserve the balance of the tubes. If we impart a virtual displacement along the axis Ox to the right lower tube the work will be done by only two forces: P and R. According to the principle of virtual work (17.2), for the equilibrium position we have

$$\delta A = -P\,\delta y_B - R\,\delta x_D = 0 \tag{1}$$

Denoting the radius of the tube by r we see from the figure that

$$y_B = r + 2r \sin\alpha, \quad x_D = 2r + 4r\cos\alpha$$

From these equations we find the variations of the coordinates:

$$\delta y_B = \frac{dy_B}{d\alpha}\,\delta\alpha = 2r\cos\alpha\,\delta\alpha, \quad \delta x_D = \frac{dx_D}{d\alpha}\,\delta\alpha = -4r\sin\alpha\,\delta\alpha$$

Now we substitute these variations into (1) and obtain

$$-P\cdot 2r\cos\alpha\,\delta\alpha + R\cdot 4r\sin\alpha\,\delta\alpha = 0$$

whence, for $\delta\alpha \neq 0$, we find

$$R = \frac{1}{2} P\cot\alpha$$

In particular, if the lower tubes touch each other then $\alpha = 60°$ and

$$R = \frac{1}{2} P \cot 60° = \frac{\sqrt{3}}{6} P = 0.289P$$

§ 2. Equilibrium Conditions for a System of Particles in Generalized Coordinates

2.1. Number of Degrees of Freedom. Let us consider a system of n particles $M_1, M_2, \ldots, M_n$. We shall denote by $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ the resultants of the active forces applied to the corresponding particles of the system and by

$$X_\nu, Y_\nu, Z_\nu \quad (\nu = 1, 2, \ldots, n)$$

the projections of these resultants, for example $\boldsymbol{F}_\nu$ ($\nu = 1, 2, \ldots, n$), on the axes of an inertial coordinate system $Oxyz$. Let there be m stationary bilateral ideal geometric constraints (see Sec. 1.2) imposed on the system. Since in what follows we shall only consider such constraints we can say that there are $m < 3n$ constraints described by equations of the form

$$\begin{aligned} f_1(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) &= 0 \\ f_2(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n) &= 0 \end{aligned} \tag{17.7}$$

Here x_ν, y_ν and z_ν are the coordinates of the particle M_ν ($\nu = 1, 2, \ldots, n$). Equations (17.7) are supposed to be independent. Then the $3n$ Cartesian coordinates are connected by m equations (17.7), and the number k of the independent coordinates is

$$k = 3n - m \tag{17.8}$$

The number k of the independent coordinates specifying the position of a system of particles is called the number of degrees of freedom of that

system. For the constraints we are considering the number of independent coordinates coincides with that of independent virtual displacements of the system in question.

EXAMPLES.

(a) A system consisting of two particles connected with a rod has five degrees of freedom. Indeed, the configuration of the two particles is specified by the six coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) which are connected by one relation

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = l^2$$

expressing the constancy of the square of the distance between the particles. Consequently, in this case $n = 2$ and $m = 1$, and formula (17.8) yields $k = 3 \cdot 2 - 1 = 5$.

(b) For a particle moving on a surface we have $k = 2$. A particle moving along a curve possesses one degree of freedom. Indeed, a curve in space is described by two equations, that is $m = 2$ and $k = 3 \cdot 1 - 2 = 1$.

(c) A free rigid body has six degrees of freedom because it can be given three independent virtual translatory displacements along three mutually perpendicular coordinate axes and three independent virtual rotations about these axes.

(d) A rigid body with one fixed point possesses three degrees of freedom. A rigid body with two fixed points has only one independent virtual displacement: the rotation about the axis passing through these two points; hence it has one degree of freedom.

The Cartesian coordinates of the particles of a system on which m constraints are imposed can be expressed in terms of k independent parameters $q_1, q_2, \ldots, q_k$ which are called *generalized coordinates*

$$\begin{gathered} x_\nu = x_\nu(q_1, q_2, \ldots, q_k), \; y_\nu = y_\nu(q_1, q_2, \ldots, q_k) \\ z_\nu = z_\nu(q_1, q_2, \ldots, q_k) \quad (\nu = 1, 2, \ldots, n) \end{gathered} \tag{17.9}$$

The projections δx_ν, δy_ν and δz_ν of the virtual displacement δr_ν of the particle M_ν or, which is the same, the variations of the Cartesian coordinates of the νth particle of the system, are specified by formulas analogous to those expressing the total differentials of functions of several independent variables (see [1]):

$$\begin{aligned} \delta x_\nu &= \frac{\partial x_\nu}{\partial q_1}\delta q_1 + \frac{\partial x_\nu}{\partial q_2}\delta q_2 + \ldots + \frac{\partial x_\nu}{\partial q_k}\delta q_k \\ \delta y_\nu &= \frac{\partial y_\nu}{\partial q_1}\delta q_1 + \frac{\partial y_\nu}{\partial q_2}\delta q_2 + \ldots + \frac{\partial y_\nu}{\partial q_k}\delta q_k \\ \delta z_\nu &= \frac{\partial z_\nu}{\partial q_1}\delta q_1 + \frac{\partial z_\nu}{\partial q_2}\delta q_2 + \ldots + \frac{\partial z_\nu}{\partial q_k}\delta q_k \end{aligned}$$

that is

$$\delta x_\nu = \sum_{\varkappa=1}^{k} \frac{\partial x_\nu}{\partial q_\varkappa}\,\delta q_\varkappa, \quad \delta y_\nu = \sum_{\varkappa=1}^{k} \frac{\partial y_\nu}{\partial q_\varkappa}\,\delta q_\varkappa,$$
$$\delta z_\nu = \sum_{\varkappa=1}^{k} \frac{\partial z_\nu}{\partial q_\varkappa}\,\delta q_\varkappa \quad (\nu = 1, 2, \ldots, n) \tag{17.10}$$

where $\delta q_1, \delta q_2, \ldots, \delta q_k$ are the variations of the generalized coordinates.

2.2. **Generalized Forces.** Let us derive the expression of the virtual work δA in generalized coordinates. To this end we substitute the values of the variations of the coordinates determined by (17.10) into the left-hand side of formula (17.2); transforming the resultant expression we find:

$$\delta A = \sum_{\nu=1}^{n} (X_\nu\,\delta x_\nu + Y_\nu\,\delta y_\nu + Z_\nu\,\delta z_\nu)$$
$$= \sum_{\nu=1}^{n} \left[X_\nu \sum_{\varkappa=1}^{k} \frac{\partial x_\nu}{\partial q_\varkappa}\,\delta q_\varkappa + Y_\nu \sum_{\varkappa=1}^{k} \frac{\partial y_\nu}{\partial q_\varkappa}\,\delta q_\varkappa + Z_\nu \sum_{\varkappa=1}^{k} \frac{\partial z_\nu}{\partial q_\varkappa}\,\delta q_\varkappa \right]$$
$$= \sum_{\nu=1}^{n} \sum_{\varkappa=1}^{k} \left(X_\nu \frac{\partial x_\nu}{\partial q_\varkappa} + Y_\nu \frac{\partial y_\nu}{\partial q_\varkappa} + Z_\nu \frac{\partial z_\nu}{\partial q_\varkappa} \right) \delta q_\varkappa$$
$$= \sum_{\varkappa=1}^{k} \left[\sum_{\nu=1}^{n} \left(X_\nu \frac{\partial x_\nu}{\partial q_\varkappa} + Y_\nu \frac{\partial y_\nu}{\partial q_\varkappa} + Z_\nu \frac{\partial z_\nu}{\partial q_\varkappa} \right) \right] \delta q_\varkappa$$

The quantity

$$Q_\varkappa = \sum_{\nu=1}^{n} \left(X_\nu \frac{\partial x_\nu}{\partial q_\varkappa} + Y_\nu \frac{\partial y_\nu}{\partial q_\varkappa} + Z_\nu \frac{\partial z_\nu}{\partial q_\varkappa} \right) \tag{17.11}$$

is called the *generalized force* corresponding to the generalized coordinate $q_\varkappa$ ($\varkappa = 1, \ldots, k$).

Thus, by virtue of equations (17.9), the generalized forces $Q_1, Q_2, \ldots, Q_k$ corresponding to each of the generalized coordinates $q_1, q_2, \ldots, q_k$ are expressed in terms of the projections of the active forces with the aid of formulas (17.11). The expression of the virtual work now takes the form

$$\delta A = \sum_{\varkappa=1}^{k} Q_\varkappa\,\delta q_\varkappa = Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k \tag{17.12}$$

Formula (17.11) specifies a rule for determining a generalized force; expression (17.12) implies another rule for finding the generalized force.

Let us consider a virtual displacement for which

$$\delta q_1 = 0, \ldots, \delta q_{i-1} = 0, \ \delta q_i \neq 0, \ \delta q_{i+1} = 0, \ldots, \delta q_k = 0$$

Then

$$\delta_i A = Q_i \, \delta q_i$$

and consequently we find

$$Q_i = \frac{\delta_i A}{\delta q_i} \quad (i = 1, 2, \ldots, k) \tag{17.13}$$

RULE. To find the generalized force Q_i (corresponding to the generalized coordinate q_i) it is sufficient to impart to the system a virtual displacement for which only the coordinate q_i varies and then compute the corresponding total work $\delta_i A$ of all the active forces applied to the system. Then Q_i is determined by formula (17.13).

2.3. Equilibrium Conditions for a System of Particles in Generalized Coordinates. By virtue of (17.12), formula (17.2) expressing mathematically the principle of virtual work takes the form

$$\delta A = Q_1 \, \delta q_1 + Q_2 \, \delta q_2 + \ldots + Q_k \, \delta q_k = 0 \tag{17.14}$$

Condition (17.14) (which is equivalent to condition (17.2)) is necessary and sufficient for a system of particles subjected to the above-described constraints to be in equilibrium. Further, since the variations $\delta q_1, \delta q_2, \ldots, \delta q_k$ are independent (because such are the generalized coordinates $q_1, q_2, \ldots, q_k$) condition (17.14) implies that

$$Q_1 = 0, \quad Q_2 = 0, \ldots, Q_k = 0 \tag{17.15}$$

Indeed, let, for instance, the system be given a virtual displacement for which

$$\delta q_1 \neq 0, \quad \delta q_2 = \delta q_3 = \ldots = \delta q_k = 0$$

Then for the equilibrium position there must be

$$\delta_1 A = Q_1 \, \delta q_1 = 0$$

whence we conclude that equality $Q_1 = 0$ is necessary. The necessity of the remaining equalities (17.15) for the equilibrium of the system is established analogously.

The sufficiency of equalities (17.15) for the equilibrium of the system is readily proved by substituting (17.15) into formula (17.14). This substitution yields the equality $\delta A = 0$ which, according to the principle of virtual work, is sufficient for the system to be in equilibrium.

Thus, *for a system of particles subjected to constraints to be in equilibrium it is necessary and sufficient* that all generalized forces* (17.15)

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* The sufficiency is asserted under the assumption that at the initial instant the system is at rest.

should be equal to zero. Equalities (17.15) express the *equilibrium conditions for a system in (independent) generalized coordinates.*

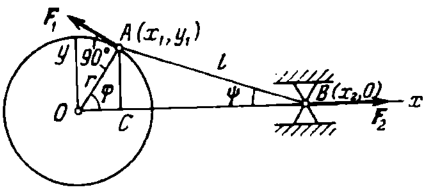

Fig. 17.7

EXAMPLE 17.4. To the hinge A of the slider-crank mechanism shown in Fig. 17.7 a force F_1 perpendicular to the crank OA is applied while the slide block B is acted upon by a horizontal force F_2. Neglecting the forces of weight and friction determine for what value of the angle φ the mechanism is in equilibrium.

Solution. The system of two ($n = 2$) particles A and B is subjected to five constraints ($m = 5$). Two of these constraints are trivial: since the motion is plane the z-coordinates of the points A and B are equal to zero:

$$z_1 = 0, \quad z_2 = 0$$

Besides, the slide block B moves between horizontal guides and therefore

$$y_2 = 0$$

Further, the lengths of the crank OA and the connecting rod AB remain invariable, that is

$$x_1^2 + y_1^2 = r^2, \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2$$

By formula (17.8) $k = 3\cdot 2 - 5 = 1$, that is the slider-crank mechanism has one degree of freedom or, which is the same, its state is described by one independent coordinate. As the generalized coordinate we can take the angle of rotation φ of the crank OA. To show this, let us express the Cartesian coordinates of the particles A and B in terms of φ:

$$x_2 = OB = OC + CB = r\cos\varphi + l\cos\psi$$

According to the law of sines, for the triangle OAB we have

$$\frac{\sin\psi}{r} = \frac{\sin\varphi}{l}, \text{ that is } \sin\psi = \frac{r}{l}\sin\varphi$$

Let us denote the ratio r/l by λ and compute

$$\cos\psi = \sqrt{1-\sin^2\psi} = \sqrt{1-\lambda^2\sin^2\varphi}$$

Thus, we have

$$\begin{gathered} x_1 = r\cos\varphi, \ y_1 = r\sin\varphi \\ x_2 = r\cos\varphi + l\sqrt{1-\lambda^2\sin^2\varphi}, \ y_2 = 0 \end{gathered} \tag{17.16}$$

Equations (17.16) are nothing other than equations (17.9) for the example under consideration. The only generalized force Q_φ present in this system is found from formula (17.11) in which the partial derivatives should be replaced by ordinary ones because the functions x_1, y_1, z_1, x_2, y_2 and z_2 depend on φ solely:

$$Q_\varphi = X_1\frac{dx_1}{d\varphi} + Y_1\frac{dy_1}{d\varphi} + Z_1\frac{dz_1}{d\varphi} + X_2\frac{dx_2}{d\varphi} + Y_2\frac{dy_2}{d\varphi} + Z_2\frac{dz_2}{d\varphi} \tag{17.17}$$

From Fig. 17.7 we conclude that

$$X_1 = -F_1\sin\varphi, \quad Y_1 = F_1\cos\varphi, \quad X_2 = F_2, \quad Y_2 = 0, \quad Z_1 = Z_2 = 0$$

Differentiating (17.16) we obtain

$$\frac{dx_1}{d\varphi} = -r\sin\varphi, \ \frac{dy_1}{d\varphi} = r\cos\varphi, \quad \frac{dx_2}{d\varphi} = -r\sin\varphi - l\frac{\lambda^2\sin\varphi\cos\varphi}{\sqrt{1-\lambda^2\sin^2\varphi}}$$

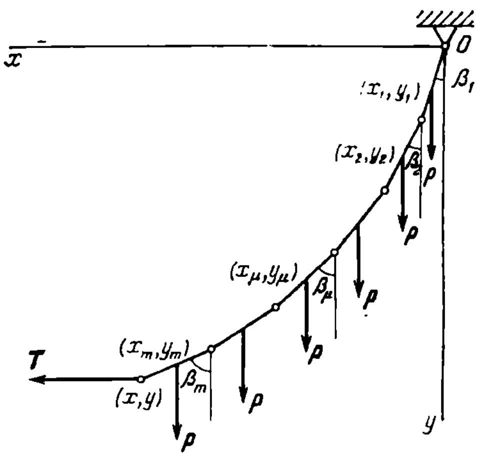

Fig. 17.8

and therefore formula (17.17) yields

$$Q_\varphi = -F_1 \sin\varphi(-r\sin\varphi) + F_1\cos\varphi\,(r\cos\varphi) + F_2\left[-r\sin\varphi - l\lambda\frac{\lambda\sin\varphi\cos\varphi}{\sqrt{1-\lambda^2\sin^2\varphi}}\right] = r\left[F_1 - F_2\sin\varphi\left(1+\frac{\lambda\cos\varphi}{\sqrt{1-\lambda^2\sin^2\varphi}}\right)\right]$$

As we know, in equilibrium all the generalized forces must be equal to zero. For our example conditions (17.15) yield the single equilibrium condition:

$$Q_\varphi = 0, \text{ that is } F_1 = \sin\varphi\left(1+\frac{\lambda\cos\varphi}{\sqrt{1-\lambda^2\sin^2\varphi}}\right)F_2$$

For the given forces F_1 and F_2 this condition can be regarded as an equation for determining the angle φ_0 in equilibrium position. Since the coefficient in F_2 does not exceed unity, for the equilibrium it is necessary that $F_1 \leqslant F_2$. In particular, if $F_1 = F_2$ then $\sin\varphi_0 = 1$, that is $\varphi_0 = \pi/2$.

It should be noted that if we solved this problem using the methods of elementary statics it would be necessary to introduce the reactions of the supporting hinge O and of the guides of the slide block B. Analytic statics makes it possible to solve equilibrium problems without determining the reactions of ideal constraints unless it is required in the condition of the problem.

EXAMPLE 17.5. Shown in Fig. 17.8 is a chain of m identical heavy rods hinged to one another; it is supposed that there is no friction in the joints. The rods are in tension under the action of the horizontal force T. The total length of the chain of the m rods is l, the weight of unit length of the rod is γ, the coordinates of the midpoint of the μth rod are denoted by (x_μ, y_μ) $(\mu = 1, 2, \ldots, m)$, the coordinates of the point of application of the force T are (x, y), and the angle between the μth rod and the axis Oy is β_μ. Determine the equilibrium position of this system.

Solution. We shall apply principle of virtual work (17.2):

$$\sum F_\nu\,\delta r_\nu = p\,\delta y_1 + p\,\delta y_2 + \ldots + p\,\delta y_m + T\,\delta x = 0 \tag{1}$$

Here $p = l\gamma/m$ is the weight of one rod; $y_1, y_2, \ldots, y_m$ are the ordinates of the midpoints of the rods; x is the abscissa of the point of application of the force T. Let us compute the coordinates $y_1, y_2, \ldots, y_m$ and x:

$$\begin{aligned}
&y_1 = \frac{1}{2}\frac{l}{m}\cos\beta_1, \quad y_2 = \frac{l}{m}\cos\beta_1 + \frac{1}{2}\frac{l}{m}\cos\beta_2, \ldots \\
&y_\mu = \frac{l}{m}\left(\cos\beta_1 + \ldots + \cos\beta_{\mu-1} + \frac{1}{2}\cos\beta_\mu\right), \ldots \\
&y_m = \frac{l}{m}\left(\cos\beta_1 + \ldots + \cos\beta_{m-1} + \frac{1}{2}\cos\beta_m\right) \\
&x = \frac{l}{m}\sin\beta_1 + \ldots + \frac{l}{m}\sin\beta_m
\end{aligned} \tag{2}$$

For the variations of these coordinates we obtain the expressions

$$\begin{aligned}
&\delta y_1 = \frac{\partial y_1}{\partial\beta_1}\delta\beta_1 = -\frac{1}{2}\frac{l}{m}\sin\beta_1\,\delta\beta_1 \\
&\delta y_2 = \frac{\partial y_1}{\partial\beta_1}\delta\beta_1 + \frac{\partial y_2}{\partial\beta_2}\delta\beta_2 = -\frac{l}{m}\sin\beta_1\,\delta\beta_1 - \frac{1}{2}\frac{l}{m}\sin\beta_2\,\delta\beta_2, \ldots \\
&\delta y_\mu = -\frac{l}{m}\left(\sin\beta_1\,\delta\beta_1 + \ldots + \sin\beta_{\mu-1}\,\delta\beta_{\mu-1} + \frac{1}{2}\sin\beta_\mu\,\delta\beta_\mu\right), \ldots \\
&\delta y_m = -\frac{l}{m}\left(\sin\beta_1\,\delta\beta_1 + \ldots + \sin\beta_{m-1}\,\delta\beta_{m-1} + \frac{1}{2}\sin\beta_m\,\delta\beta_m\right) \\
&\delta x = \frac{l}{m}(\cos\beta_1\,\delta\beta_1 + \ldots + \cos\beta_m\,\delta\beta_m)
\end{aligned}$$

Let us substitute these values into (1), cancel by $-l/2m$ and group the terms with $\delta\beta_1, \delta\beta_2, \ldots, \delta\beta_m$:

$$\begin{aligned}
&\{p[1 + 2(m-1)]\sin\beta_1 - 2T\cos\beta_1\}\,\delta\beta_1 + \{p[1 + 2(m-2)]\sin\beta_2 \\
&\quad - 2T\cos\beta_2\}\,\delta\beta_2 + \ldots + \{p[1 + 2(m-\mu)]\sin\beta_\mu - 2T\cos\beta_\mu\}\,\delta\beta_\mu \\
&\qquad + \ldots + \{p\sin\beta_m - 2T\cos\beta_m\}\,\delta\beta_m = 0
\end{aligned}$$

By virtue of (2), the angles $\beta_1, \beta_2, \ldots, \beta_m$ are generalized coordinates (see (17.9)). The comparison of the last equality with (17.14) implies that the expressions in braces are the generalized forces corresponding to the generalized coordinates. For the equilibrium it is necessary and sufficient that equalities (17.15) should hold, that is

$$Q_\mu \equiv p[1 + 2(m-\mu)]\sin\beta_\mu - 2T\cos\beta_\mu = 0 \qquad (\mu = 1, 2, \ldots, m)$$

whence we find

$$\tan\beta_\mu = \frac{2T}{[1 + 2(m-\mu)]\,p} = \frac{2mT}{[2m - (2\mu - 1)]\,l\gamma} \qquad (\mu = 1, 2, \ldots, m) \tag{3}$$

These formulas specify the values of $\beta_1, \beta_2, \ldots, \beta_m$ in the equilibrium state; this determines the equilibrium form of the rod polygon undergoing tension by the force T.

EXAMPLE 17.6. *Equilibrium Form of a Homogeneous Ideal Thread.*

By an ideal thread we shall understand an inextensible and absolutely flexible thread. As a model of such a thread let us consider a chain of $2m$ identical homogeneous rods of total length $2l$ connected by hinges without friction and suspended from the ends. It is supposed that the points of suspension A and B lie in one horizontal (Fig. 17.9); the notation coincides with that in Example 17.5. We shall consider the equilibrium problem for this system in the gravity field.

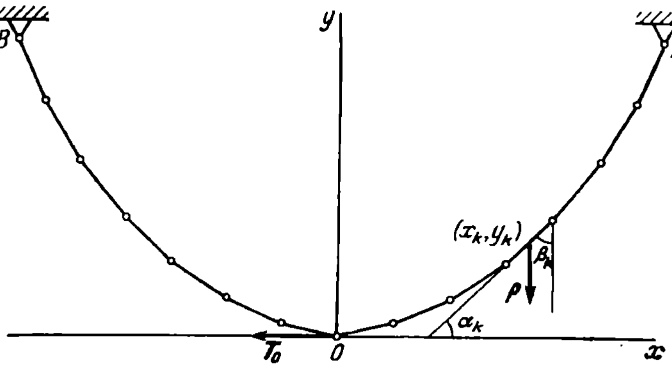

Fig. 17.9

Let s_k be the length of the rod polygon from the point A to the midpoint of the kth rod with coordinates (x_k, y_k); then

$$s_k = \frac{l}{m}(k-1) + \frac{1}{2}\frac{l}{m} = \frac{l}{2m}(2k-1)$$

The equilibrium form of a homogeneous ideal thread can be found as the limiting position of the equilibrium form of the rod polygon when the number $2m$ of the rods increases indefinitely while the length l/m of each rod tends to zero. Let us remove the left part of the rod polygon and replace its action on the remaining part by the tension T_0 (see Fig. 17.9). According to formula (3) of Example 17.5, for the tangent of the angle α_k of inclination of the kth rod we have

$$\tan\alpha_k = \cot\beta_k = \frac{l\gamma}{2mT_0}[2m-(2k-1)]$$

Let $y = y(x)$ be the equation of the curve specifying the equilibrium form of the homogeneous ideal thread; then

$$\frac{dy}{dx} = \lim_{k\to\infty}\tan\alpha_k = \frac{\gamma}{T_0}\lim_{k\to\infty}\left[l - \frac{l}{2m}(2k-1)\right] = \frac{\gamma}{T_0}[l - \lim_{k\to\infty} s_k] \tag{4}$$

The quantity in the brackets on the right-hand side is the length s of the thread from the origin to the point (x, y); it is determined by the formula

$$s = \int_0^x \sqrt{1+y'^2}\,dx$$

(see [1]). The substitution of the expression of s into (4) results in

$$y' = \frac{\gamma}{T_0}\int_0^x \sqrt{1+y'^2}\,dx$$

Differentiating this equality we obtain the differential equation of the curve specifying the equilibrium form of the thread:

$$\frac{dy'}{dx} = k\sqrt{1+y'^2} \quad \left(k = \frac{\gamma}{T_0}\right) \tag{5}$$

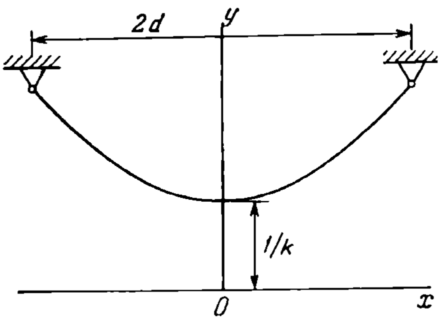

Fig. 17.10

Now we separate the variables and integrate:

$$\int_0^{y'} \frac{dy'}{\sqrt{1+y'^2}} = k\int_0^x dx$$

The integral on the left-hand side is tabular and we obtain

$$\ln\left(y'+\sqrt{1+y'^2}\right) = kx$$

Taking the inverse function we find

$$y'+\sqrt{1+y'^2} = e^{kx}$$

Let us transpose y' to the right-hand side and then square both the members of the equality:

$$\left(\sqrt{1+y'^2}\right)^2 = (e^{kx}-y')^2$$

After some simple transformations this yields

$$y' = \frac{1}{2}\,\frac{e^{2kx}-1}{e^{kx}}\,,\ \text{that is}\ \frac{dy}{dx} = \frac{1}{2}\,(e^{kx}-e^{-kx})$$

Now we separate the variables and integrate once again:

$$\int_0^y dy = \frac{1}{2}\int_0^x (e^{kx}-e^{-kx})\,dx$$

This yields

$$y = \frac{1}{2k}\,(e^{kx}+e^{-kx})\Big|_0^x = \frac{1}{k}\left(\frac{e^{kx}+e^{-kx}}{2}-1\right)$$

Let us transfer the origin along the axis Oy so that in the new coordinate system $y\,(0) = 1/k$ (Fig. 17.10). Finally we obtain

$$y = \frac{1}{k}\left(\frac{e^{kx}+e^{-kx}}{2}\right) = \frac{1}{k}\cosh kx \qquad (6)$$

The function in the parentheses is called the ***hyperbolic cosine*** (denoted by $\cosh kx$), and the curve specifying the equilibrium form of a homogeneous ideal thread is called the ***catenary***.

We denoted by T_0 the tension at the lowest point of the catenary:

$$T_0 = \frac{\gamma}{k} = \gamma y\,(0)$$

However, in every concrete problem we know neither the value of the tension T_0 nor the value of k before the catenary has been constructed. Besides the weight γ of unit length of the thread, we are given two more parameters: the length $2l$ of the thread and the distance $2d$ between the points of suspension (see Fig. 17.10). For the half-length of the thread we have

$$l = \int_0^d \sqrt{1+y'^2}\,dx = \int_0^d \sqrt{1+\sinh^2 kx}\,dx = \int_0^d \cosh(kx)\,dx = \frac{1}{k}\sinh kd$$

where the function

$$\sinh kx = \frac{1}{2}\left(e^{kx} - e^{-kx}\right)$$

is the so-called *hyperbolic sine*. Thus

$$l = \frac{1}{2k}\left(e^{kd} - e^{-kd}\right)$$

Solving this equation with respect to e^{kd} we find

$$e^{kd} = kl + \sqrt{1+k^2l^2}$$

For the given values of the parameters l and d the value of k is found from the last transcendental equation. After this the catenary can be constructed using formula (6).

Problems

PROBLEM 17.1. Shown in Fig. 17.11 is a tackle block, an appliance used for lifting weights. The block consists of two groups of tackle blocks one of which is fixed and the other is movable. Each group contains two tackle blocks fitted onto separate axes attached to a common pulley fork. One of the ends of the thread alternately pressing around the tackle blocks is attached to the fixed (upper) pulley fork while the other end is free and is acted upon by a tensile

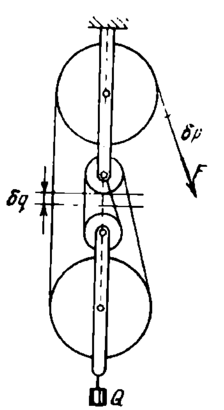

Fig. 17.11

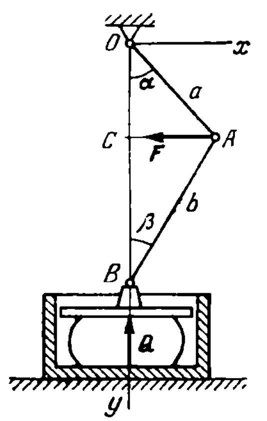

Fig. 17.12

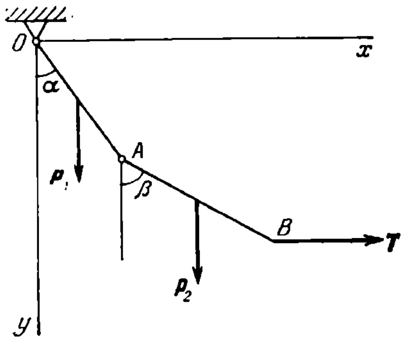

Fig. 17.13

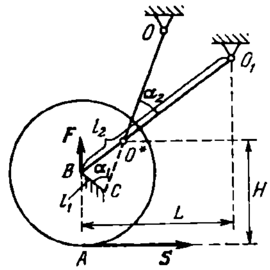

Fig. 17.14

force F. From the lower (movable) pulley fork a weight Q is suspended. Find the relationship between the magnitudes of the force F and the weight Q for the equilibrium of the system.

Hint. Establish the relationship between δq and δp.

Answer. $Q = 4F$.

PROBLEM 17.2. A toggle press OAB (Fig. 17.12) consists of two rods $OA = a$ and $AB = b$ placed in the vertical plane. To the hinge A a horizontal force F lying in the plane OAB is applied. Determine the resistance force Q of the compressed body balancing the force F.

Hint. Impart a virtual displacement to the system: let the rod OA turn through an angle $\delta\alpha$. Express the coordinates of the points A and B in terms of the lengths of the rods and the angles α and β, and then compute the variations of the coordinates. The relationship between the quantities $\delta\alpha$ and $\delta\beta$ is determined by the law of sines.

Answer. $Q = \dfrac{F}{\tan\alpha + \tan\beta}$.

PROBLEM 17.3. A homogeneous rod OA of weight p_1 has an immovable hinge at its end O and can rotate in the vertical plane about the point O (Fig. 17.13). The end A of this rod is hinged to another homogeneous rod AB of weight p_2. To the end B of the latter rod a horizontal force T is applied. Find the angles α and β for the state of equilibrium of this system of rods.

Hint. A more general situation than the one in Problem 17.3 was considered in Example 17.5.

Answer. $\tan\alpha = \dfrac{2T}{p_1 + 2p_2}$, $\tan\beta = \dfrac{2T}{p_2}$.

PROBLEM 17.4. To the joint B of the jointed four-link chain $OCBO_1$ shown in Fig. 17.14 a vertical force F is applied, and the link BC is rigidly connected with a disk whose centre is at the point B. At the point A of the disk a horizontal force S tangent to the disk is applied. The dimensions corresponding to the state of equilibrium of the system are indicated in the figure. Neglecting the weights of the rods and the disk and also the friction in the joints determine the relationship between the quantities F and S for the state of equilibrium.

Hint. Both the forces acting on the system are applied to one and the same rigid body: to the disk. The instantaneous motion of the disk reduces to rotation about the instantaneous centre O^*. When the virtual displacement of a system

reduces to rotation of the whole system (as one rigid body) about a fixed axis in space the generalized force corresponding to this virtual displacement is the sum of the moments of all the active forces about that axis.

Answer. $F = \frac{l_2 \sin \alpha_2}{l_1 \sin \alpha_1} \frac{H}{L} S.$

Chapter 18 General Equation of Dynamics. Lagrange's Equations

§ 1. General Equation of Dynamics

1.1. Statement of the Problem. We shall describe the motion of a system of particles relative to an inertial rectangular Cartesian coordinate system $Oxyz$ (see Sec. 1.2 of Chap. 13). Let $M_1, M_2, \ldots, M_n$ be the particles of the system with masses $m_1, m_2, \ldots, m_n$. The coordinates of the particles of the system at an instant under consideration will be denoted by $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$, and the projections on the axes Ox, Oy and Oz of the resultants $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ of the active forces applied to each of the particles of the system will be denoted by $\{X_1, Y_1, Z_1\}$, $\{X_2, Y_2, Z_2\}, \ldots, \{X_n, Y_n, Z_n\}$ respectively (see Fig. 18.1, where one of the particles M_ν ($\nu = 1, 2, \ldots, n$) of the system is shown).

We suppose that the given system of particles is not free, that is the motion of the system and the virtual displacements of its particles are subjected to constraints (see Sec. 1.1 of Chap. 17). As was stipulated, we assume that the constraints are

(a) *geometric*, that is independent of the velocities and the accelerations of the particles;

(b) *bilateral*, that is the particles cannot leave the constraints; Conditions (a) and (b) imply that the constraint conditions are expressed by equations (17.7);

(c) *ideal*, that is the total elementary work of their reactions is equal to zero for any virtual displacement (see (17.1)) irrespective of whether the system is in motion or at rest*.

Our aim is to write the equation of motion of the system in the form of an equation of work analogous to the one expressing the principle of virtual work in analytic statics (see (17.2)). It is natural that the work of the reactions of the ideal constraints must not enter into this equation.

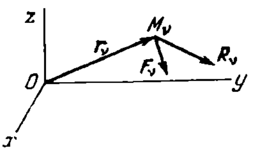

Fig. 18.1

1.2. Derivation of the General Equation of Dynamics (the D'Alembert-Lagrange Princi-

* The requirement that the constraints should be stationary (time independent) was important for statics; here we need not introduce this requirement.

ple). According to the axiom of constraints (see Sec. 1.2 of Chap. 17), a system of particles subjected to constraints can be regarded as free if unknown constraint reactions are added to the given active forces. The equations of motion of each of the particles of the system in terms of the projections on the axes Ox, Oy and Oz are written on the basis of Newton's second law:

$$\begin{gathered} m_\nu \frac{d^2x_\nu}{dt^2} = X_\nu + R_x^{(\nu)}, \quad m_\nu \frac{d^2y_\nu}{dt^2} = Y_\nu + R_y^{(\nu)} \\ m_\nu \frac{d^2z_\nu}{dt^2} = Z_\nu + R_z^{(\nu)} \quad (\nu = 1, 2, \ldots, n) \end{gathered} \tag{18.1}$$

where $R_x^{(\nu)}$, $R_y^{(\nu)}$ and $R_z^{(\nu)}$ are the projections of the resultant R_ν of the constraint reactions applied to the particle M_ν ($\nu = 1, 2, \ldots, n$). These equations can be rewritten in the form

$$-R_x^{(\nu)} = X_\nu - m_\nu \frac{d^2x_\nu}{dt^2}, \quad -R_y^{(\nu)} = Y_\nu - m_\nu \frac{d^2y_\nu}{dt^2}$$

$$-R_z^{(\nu)} = Z_\nu - m_\nu \frac{d^2z_\nu}{dt^2} \quad (\nu = 1, 2, \ldots, n)$$

The substitution of the expressions of $R_x^{(\nu)}$, $R_y^{(\nu)}$ and $R_z^{(\nu)}$ we have found into equation (17.1) results in

$$\sum_{\nu=1}^{n} \left[\left(X_\nu - m_\nu \frac{d^2x_\nu}{dt^2} \right) \delta x_\nu + \left(Y_\nu - m_\nu \frac{d^2y_\nu}{dt^2} \right) \delta y_\nu + \left(Z_\nu - m_\nu \frac{d^2z_\nu}{dt^2} \right) \delta z_\nu \right] = 0 \tag{18.2}$$

This is the *general equation of dynamics* expressing the *D'Alembert-Lagrange principle*.

The quantities

$$-m_\nu \frac{d^2x_\nu}{dt^2}, \quad -m_\nu \frac{d^2y_\nu}{dt^2}, \quad -m_\nu \frac{d^2z_\nu}{dt^2}$$

have the dimension of force and are called *inertial forces*. Equation (18.2) contains the vector quantities

$$\boldsymbol{F}_\nu - m_\nu \boldsymbol{w}_\nu \quad (\nu = 1, 2, \ldots, n)$$

and hence involves all the active forces and all the inertial forces applied to the particles of the system.

Equation (18.2) can be written in the form

$$\sum_{\nu=1}^{n} (\boldsymbol{F}_\nu - m_\nu \boldsymbol{w}_\nu, \delta \boldsymbol{r}_\nu) = 0$$

Here $\boldsymbol{w}_\nu$ is the acceleration vector of the particle M_ν

$$\boldsymbol{w}_\nu = \frac{d^2x_\nu}{dt^2} \boldsymbol{i} + \frac{d^2y_\nu}{dt^2} \boldsymbol{j} + \frac{d^2z_\nu}{dt^2} \boldsymbol{k}$$

and δr_ν is the virtual displacement vector of that particle:

$$\delta \boldsymbol{r}_\nu = \delta x_\nu \boldsymbol{i} + \delta y_\nu \boldsymbol{j} + \delta z_\nu \boldsymbol{k} \quad (\nu = 1, 2, \ldots, n)$$

THE D'ALEMBERT-LAGRANGE PRINCIPLE (THE GENERAL EQUATION OF DYNAMICS) reads: *for a system of particles subjected to bilateral ideal geometric constraints the total virtual work of all the active forces and all the inertial forces is equal to zero for any virtual displacement of the system.*

Above we proved the *necessity* of the general equation of dynamics.

Let us prove its *sufficiency*. Suppose that equation (18.2) holds; it is required to prove that the motion of each of the particles of the system obeys Newton's second law. Let us add the reaction forces $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_n$ to the given active forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$. Then, according to the axiom of constraints (see Sec. 1.2 of Chap. 17), the system of particles can be regarded as free under the action of the forces $\boldsymbol{F}_1 + \boldsymbol{R}_1, \boldsymbol{F}_2 + \boldsymbol{R}_2, \ldots, \boldsymbol{F}_n + \boldsymbol{R}_n$. The virtual displacements of the particles of the *free* system will be denoted by

$$\tilde{\delta} \boldsymbol{r}_\nu = \tilde{\delta} x_\nu \boldsymbol{i} + \tilde{\delta} y_\nu \boldsymbol{j} + \tilde{\delta} z_\nu \boldsymbol{k} \quad (\nu = 1, 2, \ldots, n)$$

By virtue of equation (18.2) (it applies to a free system of particles as well), we have

$$\sum_{\nu=1}^{n} \Big[\Big(X_\nu + R_x^{(\nu)} - m_\nu \frac{d^2 x_\nu}{dt^2} \Big) \tilde{\delta} x_\nu + \Big(Y_\nu + R_y^{(\nu)} - m_\nu \frac{d^2 y_\nu}{dt^2} \Big) \tilde{\delta} y_\nu + \Big(Z_\nu + R_z^{(\nu)} - m_\nu \frac{d^2 z_\nu}{dt^2} \Big) \tilde{\delta} z_\nu \Big] = 0 \tag{18.3}$$

For the free system the quantities $\tilde{\delta} x_\nu$, $\tilde{\delta} y_\nu$ and $\tilde{\delta} z_\nu$ can assume any small values and therefore we can put

$$\tilde{\delta} x_\nu = \alpha \Big(X_\nu + R_x^{(\nu)} - m_\nu \frac{d^2 x_\nu}{dt^2} \Big), \quad \tilde{\delta} y_\nu = \alpha \Big(Y_\nu + R_y^{(\nu)} - m_\nu \frac{d^2 y_\nu}{dt^2} \Big)$$

$$\tilde{\delta} z_\nu = \alpha \Big(Z_\nu + R_z^{(\nu)} - m_\nu \frac{d^2 z_\nu}{dt^2} \Big) \quad (\nu = 1, 2, \ldots, n)$$

in the last equality (here α is an infinitesimal). This yields

$$\alpha \sum_{\nu=1}^{n} \Big[\Big(X_\nu + R_x^{(\nu)} - m_\nu \frac{d^2 x_\nu}{dt^2} \Big)^2 + \Big(Y_\nu + R_y^{(\nu)} - m_\nu \frac{d^2 y_\nu}{dt^2} \Big)^2 + \Big(Z_\nu + R_z^{(\nu)} - m_\nu \frac{d^2 z_\nu}{dt^2} \Big)^2 \Big] = 0$$

Since in the last equality the sum of squares is equal to zero we obtain the relations

$$X_\nu + R_x^{(\nu)} - m_\nu \frac{d^2 x_\nu}{dt^2} = 0, \quad Y_\nu + R_y^{(\nu)} - m_\nu \frac{d^2 y_\nu}{dt^2} = 0$$

$$Z_\nu + R_z^{(\nu)} - m_\nu \frac{d^2 z_\nu}{dt^2} = 0 \quad (\nu = 1, 2, \ldots, n)$$

which is nothing other than equations of motion (18.1) for the particles of the system. The sufficiency of the D'Alembert-Lagrange principle (of the general equation of dynamics) is proved.

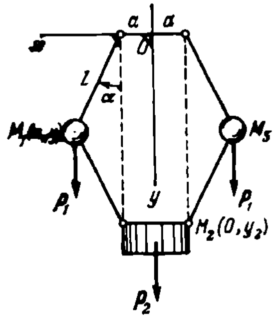

Fig. 18.2

The general equation of dynamics makes it possible to solve dynamic problems without determining the constraint reactions. However, when in the condition of the problem it is required to find some of the reactions, one should replace the action of the corresponding parts of the constraints by their reaction forces which must be added to the given forces. The general equation of dynamics can also be applied to a system for which the action of a part of its constraints is replaced by the corresponding reactions. It should be stressed that in all the three enumerated cases the virtual displacements are different; here we mean the following cases:

(1) the case of a system of particles subjected to all its constraints (it is described by equation (18.2));

(2) the case of a free system and also the case of a system subjected to constraints for which the constraints are (mentally) discarded and their action is replaced by the reactions (this case is described by equation (18.3));

(3) the case of a system the part of whose constraints is removed and their action is replaced by the corresponding reactions.

EXAMPLE 18.1. A centrifugal governor (Fig. 18.2) rotates about the vertical axis with constant angular velocity ω. The weight of each ball is p_1, the weight of the coupling is p_2, all the rods are of length l and are considered weightless; the suspension joints of the rods are at a distance a from the axis. For the steady-state regime of the governor determine the relationship between the angular velocity ω and the angle α of deviation of the rods from the vertical.

Solution. Let us place the origin at the fixed point O and draw the axis Oy downward and the axis Ox to the left in the plane of the governor at a fixed instant t. Since $\omega = \text{const}$ the ball M_1 has only normal acceleration (whose direction is opposite to that of the axis Ox):

$$\frac{d^2x_1}{dt^2} = -(a+l\sin\alpha)\,\omega^2, \qquad \frac{d^2y_1}{dt^2} = 0$$

In what follows we shall consider one of the balls, say M_1 (this is sufficient because of the symmetry of the system). The projections of the active forces, that is of the forces of weight acting on the ball M_1 and on the coupling M_2, are

$$X_1 = 0, \quad Y_1 = p_1; \quad X_2 = 0, \quad Y_2 = p_2$$

Equation (18.2) is written in the form

$$2\left(-\frac{p_1}{g}\frac{d^2x_1}{dt^2}\,\delta x_1 + p_1\,\delta y_1\right) + p_2\,\delta y_2 = 0 \tag{1}$$

where the coefficient 2 in the first summand is accounted for by the presence of two symmetric balls. Now let us compute the variations of the coordinates.

For the coordinates of the points M_1 and M_2 we have

$$x_1 = a + l \sin\alpha, \quad y_1 = l\cos\alpha; \quad x_2 = 0, \quad y_2 = 2l\cos\alpha$$

It follows that the variations of the coordinates (they are determined by the same formulas as the differentials) are

$$\delta x_1 = \frac{dx_1}{d\alpha}\delta\alpha = l\cos\alpha\,\delta\alpha, \quad \delta y_1 = \frac{dy_1}{d\alpha}\delta\alpha = -l\sin\alpha\,\delta\alpha$$

$$\delta y_2 = \frac{dy_2}{d\alpha}\delta\alpha = -2l\sin\alpha\,\delta\alpha$$

Substituting these values of the variations into (1) we obtain

$$2\left[\frac{p_1}{g}(a+l\sin\alpha)\,\omega^2 l\cos\alpha - p_1 l\sin\alpha\right]\delta\alpha - 2p_2 l\sin\alpha\,\delta\alpha = 0$$

whence

$$\omega^2 = \frac{(p_1+p_2)\,g\tan\alpha}{p_1(a+l\sin\alpha)}$$

It should be stressed that virtual displacements are thought of as taking place at the fixed instant t. In other words, these displacements occur as if the constraints were fixed and, consequently, for the points M_1 and M_2 they are in the plane of the figure. As to real displacements they take place during some time (however small). Therefore for real displacements the rotation of the plane of the centrifugal governor with angular velocity ω should also be taken into account. Hence, for the time-dependent constraints (in this case the system is said to be *rheonomic*) real displacements do not belong to the class of virtual displacements.

§ 2. Differential Equations of Motion of a System of Particles in Generalized Coordinates (Lagrange's Equations)

2.1. Statement of the Problem. The statement of the problem remains the same as in the foregoing section. We shall consider the motion of a system of n particles; the resultant of the active forces applied to the νth particle will be denoted by

$$\boldsymbol{F}_\nu = X_\nu \boldsymbol{i} + Y_\nu \boldsymbol{j} + Z_\nu \boldsymbol{k} \quad (\nu = 1, 2, \ldots, n)$$

where X_ν, Y_ν and Z_ν are the projections of the force $\boldsymbol{F}_\nu$ on the axes of an inertial coordinate system $Oxyz$. Let there be m bilateral ideal geometric constraints imposed on the system. Then the number k of degrees of freedom is determined by formula (17.8):

$$k = 3n - m$$

The Cartesian coordinates of the particles of the system can be expressed in terms of k generalized coordinates (see (17.9)):

$$\begin{gathered} x_\nu = x_\nu(t, q_1, q_2, \ldots, q_k), \quad y_\nu = y_\nu(t, q_1, q_2, \ldots, q_k) \\ z_\nu = z_\nu(t, q_1, q_2, \ldots, q_k) \quad (\nu = 1, 2, \ldots, n) \end{gathered} \tag{18.4}$$

Since we do not suppose that the constraints themselves are necessarily stationary the constraint equations (see (17.7)) may involve time explicitly. Therefore, in contrast to (17.9), in the general case the right-hand sides of (18.4) may involve time t.

The coordinates $q_1, q_2, \ldots, q_k$ possess the following properties:

(1) they are *real*, that is they cannot assume complex values;

(2) they are *independent*;

(3) each of them has an *independent geometrical meaning*.

The last property means that the numerical values of the generalized coordinates specify the configuration of the system, that is the values of the Cartesian coordinates of its particles before the equations of motion are written (and moreover before they are integrated). These coordinates are called *holonomic*. We shall refer to them as *Lagrange's generalized coordinates*.

2.2. Derivation of Lagrange's Equations. The variations of the Cartesian coordinates of the particles of the system are determined by formulas (17.10):

$$\delta x_\nu = \sum_{\varkappa=1}^{k} \frac{\partial x_\nu}{\partial q_\varkappa}\delta q_\varkappa, \quad \delta y_\nu = \sum_{\varkappa=1}^{k} \frac{\partial y_\nu}{\partial q_\varkappa}\delta q_\varkappa, \quad \delta z_\nu = \sum_{\varkappa=1}^{k} \frac{\partial z_\nu}{\partial q_\varkappa}\delta q_\varkappa$$
$$(\nu = 1, 2, \ldots, n)$$

where $\delta q_1, \delta q_2, \ldots, \delta q_k$ are the variations of the generalized coordinates.

The real motions of the system are described by general equation of dynamics (18.2) which we write in the form

$$\sum_{\nu=1}^{n}\left[\left(m_\nu \frac{d^2x_\nu}{dt^2} - X_\nu\right)\delta x_\nu + \left(m_\nu \frac{d^2y_\nu}{dt^2} - Y_\nu\right)\delta y_\nu + \left(m_\nu \frac{d^2z_\nu}{dt^2} - Z_\nu\right)\delta z_\nu\right] = 0$$

Let us substitute the expressions of δx_ν, δy_ν and δz_ν given by (17.10) into the last equality:

$$\sum_{\nu=1}^{n} m_\nu\left[\frac{d\dot{x}_\nu}{dt}\sum_{\varkappa=1}^{k}\frac{\partial x_\nu}{\partial q_\varkappa}\delta q_\varkappa + \frac{d\dot{y}_\nu}{dt}\sum_{\varkappa=1}^{k}\frac{\partial y_\nu}{\partial q_\varkappa}\delta q_\varkappa + \frac{d\dot{z}_\nu}{dt}\sum_{\varkappa=1}^{k}\frac{\partial z_\nu}{\partial q_\varkappa}\delta q_\varkappa\right]$$
$$- \sum_{\nu=1}^{n}\left(X_\nu\sum_{\varkappa=1}^{k}\frac{\partial x_\nu}{\partial q_\varkappa}\delta q_\varkappa + Y_\nu\sum_{\varkappa=1}^{k}\frac{\partial y_\nu}{\partial q_\varkappa}\delta q_\varkappa + Z_\nu\sum_{\varkappa=1}^{k}\frac{\partial z_\nu}{\partial q_\varkappa}\delta q_\varkappa\right) = 0$$

Changing the order of summation we obtain

$$\sum_{\varkappa=1}^{k} \delta q_\varkappa \sum_{\nu=1}^{n} m_\nu \left(\frac{d\dot{x}_\nu}{dt} \frac{\partial x_\nu}{\partial q_\varkappa} + \frac{d\dot{y}_\nu}{dt} \frac{\partial y_\nu}{\partial q_\varkappa} + \frac{d\dot{z}_\nu}{dt} \frac{\partial z_\nu}{\partial q_\varkappa} \right)$$

$$- \sum_{\varkappa=1}^{k} \delta q_\varkappa \sum_{\nu=1}^{n} \left(X_\nu \frac{\partial x_\nu}{\partial q_\varkappa} + Y_\nu \frac{\partial y_\nu}{\partial q_\varkappa} + Z_\nu \frac{\partial z_\nu}{\partial q_\varkappa} \right) = 0 \qquad (18.5)$$

Let us transform the first term. To this end we use the first formula (18.4) to compute $\dot{x}_\nu$:

$$\dot{x}_\nu = \frac{dx_\nu}{dt} = \frac{\partial x_\nu}{\partial t} + \sum_{i=1}^{k} \frac{\partial x_\nu}{\partial q_i} \dot{q}_i \quad (\nu = 1, 2, \ldots, n) \qquad (18.6)$$

Now we take the partial derivatives with respect to $\dot{q}_\varkappa$:

$$\frac{\partial \dot{x}_\nu}{\partial \dot{q}_\varkappa} = \frac{\partial x_\nu}{\partial q_\varkappa} \quad (\nu = 1, 2, \ldots, n; \varkappa = 1, 2, \ldots, k) \qquad (18.7)$$

On the other hand, taking the partial derivatives of both the sides of equality (18.6) with respect to $q_\varkappa$ we obtain

$$\frac{\partial \dot{x}_\nu}{\partial q_\varkappa} = \frac{\partial^2 x_\nu}{\partial t\, \partial q_\varkappa} + \sum_{i=1}^{k} \frac{\partial^2 x_\nu}{\partial q_i\, \partial q_\varkappa} \dot{q}_i = \frac{\partial}{\partial t} \left(\frac{\partial x_\nu}{\partial q_\varkappa} \right) + \sum_{i=1}^{k} \frac{\partial \left(\frac{\partial x_\nu}{\partial q_\varkappa} \right)}{\partial q_i} \dot{q}_i$$

Consequently,

$$\frac{\partial \dot{x}_\nu}{\partial q_\varkappa} = \frac{d \left(\frac{\partial x_\nu}{\partial q_\varkappa} \right)}{dt} \quad (\nu = 1, 2, \ldots, n; \varkappa = 1, 2, \ldots, k) \qquad (18.8)$$

Thus, the first term in formula (18.5) is expressed thus:

$$\frac{d\dot{x}_\nu}{dt} \frac{\partial x_\nu}{\partial q_\varkappa} = \frac{d}{dt} \left(\dot{x}_\nu \frac{\partial x_\nu}{\partial q_\varkappa} \right) - \dot{x}_\nu \frac{d \left(\frac{\partial x_\nu}{\partial q_\varkappa} \right)}{dt} = \frac{d}{dt} \left(\dot{x}_\nu \frac{\partial \dot{x}_\nu}{\partial \dot{q}_\varkappa} \right) - \dot{x}_\nu \frac{\partial \dot{x}_\nu}{\partial q_\varkappa}$$

Here, when passing to the last equality, we have used formulas (18.7) and (18.8). Finally, we write

$$\frac{d\dot{x}_\nu}{dt} \frac{\partial x_\nu}{\partial q_\varkappa} = \frac{d}{dt} \left[\frac{\partial \left(\frac{1}{2} \dot{x}_\nu^2 \right)}{\partial \dot{q}_\varkappa} \right] - \frac{\partial}{\partial q_\varkappa} \left(\frac{1}{2} \dot{x}_\nu^2 \right)$$

$$(\nu = 1, 2, \ldots, n; \varkappa = 1, 2, \ldots, k)$$

Now we substitute these expressions into equation (18.5) together with the analogous expressions for the second and the third summands

in the first parentheses of (18.5):

$$\sum_{\varkappa=1}^{k} \delta q_\varkappa \left\{ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_\varkappa} \sum_{\nu=1}^{n} \frac{1}{2} m_\nu (\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2) \right] \right.$$

$$\left. - \frac{\partial}{\partial q_\varkappa} \sum_{\nu=1}^{n} \frac{1}{2} m_\nu (\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2) - \sum_{\nu=1}^{n} \left(X_\nu \frac{\partial x_\nu}{\partial q_\varkappa} + Y_\nu \frac{\partial y_\nu}{\partial q_\varkappa} + Z_\nu \frac{\partial z_\nu}{\partial q_\varkappa} \right) \right\} = 0$$

The last sum is called the *generalized force* corresponding to the generalized coordinate $q_\varkappa$ (see (17.11)):

$$Q_\varkappa = \sum_{\nu=1}^{n} \left(X_\nu \frac{\partial x_\nu}{\partial q_\varkappa} + Y_\nu \frac{\partial y_\nu}{\partial q_\varkappa} + Z_\nu \frac{\partial z_\nu}{\partial q_\varkappa} \right) \quad (\varkappa = 1, 2, \ldots, k)$$

The quantity

$$T = \frac{1}{2} \sum_{\nu=1}^{n} m_\nu (\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2) = \sum_{\nu=1}^{n} \frac{1}{2} m_\nu v_\nu^2 \tag{18.9}$$

is the sum of the values of the kinetic energy for all the particles of the system; it is called the *kinetic energy of the system of particles* in its absolute motion.

Thus, for the system in question the transformations we have performed make it possible to bring the general equation of dynamics to the form

$$\sum_{\varkappa=1}^{k} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\varkappa} \right) - \frac{\partial T}{\partial q_\varkappa} - Q_\varkappa \right] \delta q_\varkappa = 0 \tag{18.10}$$

Since condition (18.10) has been derived from general equation of dynamics (18.2) by means of identical transformations it is necessary and sufficient for the real motion of the system at any instant t. Further, since the coordinates $q_1, q_2, \ldots, q_k$ are independent so are the variations $\delta q_1, \delta q_2, \ldots, \delta q_k$, and therefore condition (18.10) implies

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\varkappa} \right) - \frac{\partial T}{\partial q_\varkappa} = Q_\varkappa \quad (\varkappa = 1, 2, \ldots, k) \tag{18.11}$$

Indeed, let the system be given a virtual displacement

$$\delta q_1 \neq 0, \quad \delta q_2 = \delta q_3 = \ldots = \delta q_k = 0$$

Then from (18.10) it follows that at the given instant the real motion satisfies the equality

$$\left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} - Q_1 \right] \delta q_1 = 0$$

whence it follows that equality

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right)-\frac{\partial T}{\partial q_1}=Q_1$$

is necessary. The necessity of the remaining equalities (18.11) for the real motion of the system at any instant t is established in a similar way. The sufficiency of equalities (18.11) at any instant for the real motion of the system is established in an obvious way: equalities (18.11) should be substituted into condition (18.10).

We have thus derived differential equations of motion in generalized coordinates (18.11) for the system of particles; these equations are called *Lagrange's equations.*

2.3. Lagrange's Equations for the Case of a Potential Force Field. In practical problems we often encounter forces for which the force function exists (see Sec. 3.3 of Chap. 15). Let us consider the form of Lagrange's equations for this case.

A differentiable function

$$U = U\,(x_1, y_1, z_1;\ x_2, y_2, z_2;\ \ldots;\ x_n, y_n, z_n)$$

is called the *force function* if the projections of the active forces applied to the particles of the system can be represented in the form

$$X_\nu=\frac{\partial U}{\partial x_\nu},\quad Y_\nu=\frac{\partial U}{\partial y_\nu},\quad Z_\nu=\frac{\partial U}{\partial z_\nu}\quad (\nu=1,\ 2,\ \ldots,\ n)$$

In this case the *force field* (the domain of the function U) is said to be *potential.*

Let there exist the force function; then, taking into account (18.4), we obtain the expression

$$Q_\varkappa=\sum_{\nu=1}^{n}\left(X_\nu\frac{\partial x_\nu}{\partial q_\varkappa}+Y_\nu\frac{\partial y_\nu}{\partial q_\varkappa}+Z_\nu\frac{\partial z_\nu}{\partial q_\varkappa}\right)$$

$$=\sum_{\nu=1}^{n}\left(\frac{\partial U}{\partial x_\nu}\frac{\partial x_\nu}{\partial q_\varkappa}+\frac{\partial U}{\partial y_\nu}\frac{\partial y_\nu}{\partial q_\varkappa}+\frac{\partial U}{\partial z_\nu}\frac{\partial z_\nu}{\partial q_\varkappa}\right)=\frac{\partial U}{\partial q_\varkappa}\qquad(18.12)$$

$$(\varkappa=1,\ 2,\ \ldots,\ k)$$

for generalized force (17.11). In this case Lagrange's equations (18.11) take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\varkappa}\right)-\frac{\partial T}{\partial q_\varkappa}=\frac{\partial U}{\partial q_\varkappa}\quad(\varkappa=1,\ 2,\ \ldots,\ k)\qquad(18.11\text{a})$$

The sum of the kinetic energy and the force function of the system of particles is called the *Lagrangian function*; this function (denoted by L) is expressed in terms of the generalized coordinates and the

velocities:

$$L = T + U$$

Here, by virtue of (18.4), $T = T(q_1, q_2, \ldots, q_k; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k)$ and $U = U(q_1, q_2, \ldots, q_k)$.

In the case of a potential force field the introduction of the Lagrangian function makes it possible to write Lagrange's equations in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\varkappa}\right) - \frac{\partial L}{\partial q_\varkappa} = 0 \quad (\varkappa = 1, 2, \ldots, k) \qquad (18.11\text{b})$$

because we obviously have $\partial U / \partial \dot{q}_\varkappa \equiv 0 \quad (\varkappa = 1, 2, \ldots, k)$.

2.4. **Conclusion.** A system of differential equations (18.1) describing the motion of a system of particles in terms of the motions of each of its particles is of order $6n$ and involves the unknown constraint reactions. In contrast to (18.1), Lagrange's equations form a system of ordinary differential equations (and not a system of partial differential equations!) of order $2k = 6n - 2m$ and do not involve the constraint reactions. Lagrange's equations have similar form for all the systems of particles under consideration and can be written for any generalized coordinates in terms of which the Cartesian coordinates of the particles of the system are expressed.

In order to write Lagrange's equations (18.11) we must know expression (18.9) of the kinetic energy T of the system in terms of the generalized coordinates and the velocities as well as expressions (17.11) for the generalized forces $Q_1, Q_2, \ldots, Q_k$. It should also be noted that the generalized forces can be computed not only with the aid of formulas (17.11) as was done in Example 17.4 but also with the aid of formulas (17.13) which were stated and explained in that example.

Lagrange's equations are also applicable when there are constraints not satisfying Conditions (b) and (c) stated in Sec. 2.1. In such a case one should discard these constraints and add the corresponding reactions to the external active forces.

Lagrange's equations can also be applied to a free system of n particles. In this case the coordinates $x_1, y_1, z_1; x_2, y_2, z_2; \ldots; x_n, y_n, z_n$ of the particles are themselves the generalized coordinates, and the projections

$$X_1, Y_1, Z_1; X_2, Y_2, Z_2; \ldots; X_n, Y_n, Z_n$$

of the active forces applied to each of the particles of the system are the corresponding generalized forces. The kinetic energy of such

a system is

$$T=\frac{1}{2}\sum_{\nu=1}^{n} m_\nu(\dot{x}_\nu^2+\dot{y}_\nu^2+\dot{z}_\nu^2)$$

Let us compute the derivatives of the kinetic energy:

$$\frac{\partial T}{\partial \dot{x}_\nu}=m_\nu\dot{x}_\nu,\quad \frac{\partial T}{\partial \dot{y}_\nu}=m_\nu\dot{y}_\nu,\quad \frac{\partial T}{\partial \dot{z}_\nu}=m_\nu\dot{z}_\nu$$

$$\frac{\partial T}{\partial x_\nu}=\frac{\partial T}{\partial y_\nu}=\frac{\partial T}{\partial z_\nu}=0 \quad (\nu=1,\ 2,\ \ldots,\ n)$$

Now we substitute these values into equations (18.11):

$$\frac{d}{dt}(m_\nu\dot{x}_\nu)=X_\nu,\quad \frac{d}{dt}(m_\nu\dot{y}_\nu)=Y_\nu,\quad \frac{d}{dt}(m_\nu\dot{z}_\nu)=Z_\nu$$

$$(\nu=1,\ 2,\ \ldots,\ n)$$

Performing the differentiation we obtain

$$m_\nu\frac{d^2x_\nu}{dt^2}=X_\nu,\quad m_\nu\frac{d^2y_\nu}{dt^2}=Y_\nu,\quad m_\nu\frac{d^2z_\nu}{dt^2}=Z_\nu \quad (\nu=1,\ 2,\ \ldots,\ n)$$

We see that for a free system of particles Lagrange's equations reduce to the system of $3n$ differential equations expressing Newton's second law for each of the particles of the system.

EXAMPLE 18.2. Using Lagrange's equations write in polar coordinates (Fig. 18.3) the equations of plane motion of a free particle of mass m under the action of a force F.

Solution. Let us choose the polar coordinates of the particle M, that is the polar radius $r = OM$ and the polar angle φ as the independent generalized coordinates of the particle. Thus,

$$q_1=r,\quad q_2=\varphi$$

The expression of the square of the velocity of the particle in terms of its polar coordinates was given in Sec. 1.3 of Chap. 11:

$$v^2=\dot{r}^2+r^2\dot{\varphi}^2$$

The kinetic energy T of the particle M is

$$T=\frac{1}{2}mv^2=\frac{1}{2}m(\dot{r}^2+r^2\dot{\varphi}^2)$$

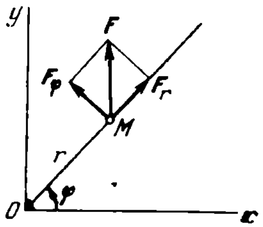

Fig. 18.3

To compute the generalized forces $Q_1 = Q_r$ and $Q_2 = Q_\varphi$ we shall make use of formulas (18.4) which in the case under consideration are the well-known formulas expressing the transformation from the polar coordinates to the Cartesian coordinates:

$$x=r\cos\varphi,\quad y=r\sin\varphi$$

By formulas (17.11), we have

$$Q_r = X\frac{\partial x}{\partial r} + Y\frac{\partial y}{\partial r} = X\cos\varphi + Y\sin\varphi = F_r$$

$$Q_\varphi = X\frac{\partial x}{\partial \varphi} + Y\frac{\partial y}{\partial \varphi} = -Xr\sin\varphi + Yr\cos\varphi = rF_\varphi$$

Here F_r and F_φ are the projections of the force $\boldsymbol{F}$ on the radius vector $\boldsymbol{r}$ and on the direction perpendicular to it (in the latter case the positive direction is the one in which the polar angle increases). Lagrange's equations corresponding to the independent coordinates r and φ take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} = Q_\varphi$$

We have

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r}, \quad \frac{\partial T}{\partial r} = mr\dot{\varphi}^2; \quad \frac{\partial T}{\partial \dot{\varphi}} = mr^2\dot{\varphi}, \quad \frac{\partial T}{\partial \varphi} = 0$$

and consequently Lagrange's equations are written thus:

$$m\frac{d^2r}{dt^2} - mr\left(\frac{d\varphi}{dt}\right)^2 = F_r, \quad m\frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right) = rF_\varphi \tag{18.13}$$

In particular, if $\boldsymbol{F}$ is the force of the Newtonian attraction

$$\boldsymbol{F} = -\frac{\mu m}{r^3}\boldsymbol{r}$$

whence

$$F_r = -\frac{\mu m}{r^2}, \quad F_\varphi = 0$$

In this case the second equation (18.13) is written as

$$\frac{d}{dt}\left(r^2\frac{d\varphi}{dt}\right) = 0$$

From this equation we obtain the equality

$$r^2\frac{d\varphi}{dt} = \text{const} \tag{18.14}$$

expressing the so-called *cyclic integral*. The left-hand member of this equality is equal to $2d\sigma/dt$, where $d\sigma$ is the area of the sector swept by the polar radius $r = OM$ during time dt (Fig. 18.4). Therefore first integral (18.14) can be written in the form

$$2\frac{d\sigma}{dt} = \text{const}$$

The derivative $d\sigma/dt$ is called the *sector velocity*, and first integral (18.14) itself is called the *area integral*. Formula (18.14) applies to the motion of planets round the Sun (see Example 15.3): it expresses Kepler's second law stating that the radius vector of a planet sweeps over equal areas during equal time.

EXAMPLE 18.3. Write the equation of motion of a pendulum of mass m suspended from a weightless spring with stiffness factor c, the length of the unloaded spring being l (Fig. 18.5).

Solution. The pendulum moves in the plane perpendicular to the axis of the joint O. The particle of mass m moving in that plane possesses two degrees of

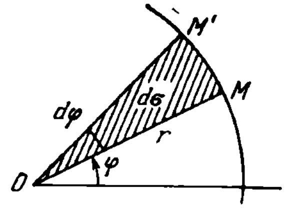

Fig. 18.4

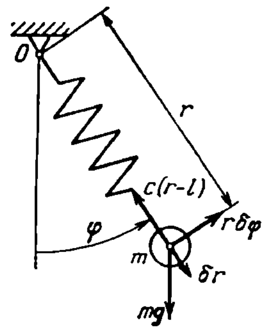

Fig. 18.5

freedom, and its polar coordinates can be taken as the independent coordinates q_1 and q_2; hence the generalized coordinates of the particle m are its polar radius $r = Om$ and the polar angle φ. The kinetic energy T of the particle m is

$$T = \frac{1}{2} mv^2 = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\varphi}^2)$$

(see the foregoing example). The pendulum is under the action of the force of weight mg directed vertically downward and the elastic force $c(r - l)$ of the spring acting along the axis of the spring toward the point O when $r > l$ and away from the point O when $r < l$. To compute the generalized forces $Q_1 = Q_r$ and $Q_2 = Q_\varphi$ we shall make use of formula (17.13). Suppose that the pendulum is given a virtual displacement δr along the spring (see Fig. 18.5); let us compute the corresponding elementary work $\delta_1 A$ of all the active forces applied to the pendulum:

$$\delta_1 A = Q_r\, \delta r = -c(r - l)\, \delta r + mg\, \delta r \cos\varphi$$

From this equality we find

$$Q_r = \frac{\delta_1 A}{\delta r} = -c(r-l) + mg\cos\varphi$$

Now let us impart to the pendulum a virtual displacement $r\,\delta\varphi$ perpendicular to the spring (see Fig. 18.5) and compute the corresponding work $\delta_2 A$:

$$\delta_2 A = Q_\varphi\, \delta\varphi = mgr\, \delta\varphi \cos\left(\frac{\pi}{2} + \varphi\right) + c(r-l)\, r\, \delta\varphi \cos\frac{\pi}{2} = -mgr \sin\varphi\, \delta\varphi$$

We see that

$$Q_\varphi = \frac{\delta_2 A}{\delta\varphi} = -mgr\sin\varphi$$

Lagrange's equations (18.11) are written in the form

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\varphi}^2 = -c(r-l) + mg\cos\varphi$$

$$\frac{d}{dt}(mr^2\dot{\varphi}) = -mgr\sin\varphi$$

Differentiating the first parentheses and cancelling both the equations by m and, additionally, the second equation by $r \neq 0$ we bring these equations to

the form

$$\frac{d^2r}{dt^2}-r\left(\frac{d\varphi}{dt}\right)^2=-\frac{c}{m}(r-l)+g\cos\varphi,\quad r\frac{d^2\varphi}{dt^2}+2\frac{dr}{dt}\frac{d\varphi}{dt}=-g\sin\varphi \tag{18.15}$$

It should be noted that if the spring is replaced by an absolutely rigid weightless rod ($r \equiv l =$ const) we obtain a simple pendulum with one degree of freedom. Consequently, for the simple pendulum only the last equation (18.15) remains:

$$l\frac{d^2\varphi}{dt^2}=-g\sin\varphi$$

For small vibrations ($\sin\varphi \approx \varphi$) this equation turns into (16.18):

$$\ddot{\varphi}+k^2\varphi=0 \quad \left(k^2=\frac{g}{l}\right)$$

Let us come back to a system of two nonlinear equations of the fourth order (18.15). Its general solution cannot be expressed in terms of either elementary functions or their quadratures. However this system admits of a particular solution of the form

$$\varphi\equiv 0,\quad \ddot{r}+\frac{c}{m}=\frac{c}{m}l+g$$

describing the vertical oscillation of the mass m suspended from the spring. The last equation is a nonhomogeneous linear differential equation of the second order with constant coefficients. Its particular solution is

$$r^*=l+\lambda_{\text{stat}} \quad \left(\lambda_{\text{stat}}=\frac{mg}{c}\right)$$

Therefore its general solution is written thus:

$$r=a\cos(\omega t+\alpha)+l+\lambda_{\text{stat}} \quad \left(\omega=\sqrt{\frac{c}{m}}\right)$$

This solution describes the vertical oscillation of the mass m, suspended from the spring, in the vicinity of the lower equilibrium position of the pendulum with a spring ($\varphi = 0$, $r = r^*$). The amplitude a and the initial phase α are specified by the initial conditions and the period τ is

$$\tau=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{c}}$$

Problems

PROBLEM 18.1. The bell-crank lever AOB shown in Fig. 18.6 has a hinged support at the point O about which it can rotate in the vertical plane; the lever can also rotate about the vertical axis Oy. The angle AOB is equal to 90°, the lengths of the rods are $OA = a$ and $OB = b$. To the points A and B of the lever two equal weights P are attached. Neglecting the weights of the rods OA and OB and friction in the hinge O determine the value of the angular velocity ω for which the straight line AB passing through the centres of the weights is horizontal.

Answer. $\omega^2=\dfrac{\sqrt{a^2+b^2}}{ab}g.$

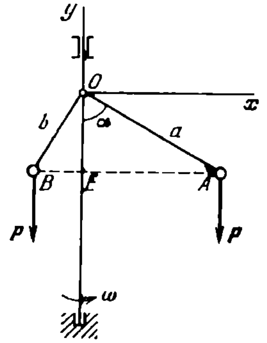

Fig. 18.6

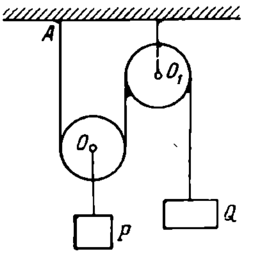

Fig. 18.7

PROBLEM 18.2. To the system of two pulleys shown in Fig. 18.7 (a movable pulley O and a fixed pulley O_1) two weights P and Q are suspended ($2Q > P$). Neglecting the masses of the pulleys determine the acceleration w of the weight Q.

Hint. The ratio of the moduli of the accelerations of the weights Q and P is equal to two.

Answer. $w = 2\dfrac{2Q-P}{4Q+P}g$.

Chapter 19 General Principles of Dynamics of a System of Particles

When solving mechanical problems it is very convenient to do without determining the constraint reactions or to determine only those reactions which are required in the condition of the problem. It is also very important to avoid the integration of the equations of motion or the solution of the general equation of dynamics. Some general consequences of the equations of motion are stated as the general principles (theorems) of dynamics of a system of particles. In some special cases these principles themselves and also the first integrals derived from them facilitate considerably the solution of the problems in comparison with the method of direct integration of the equations of motion.

Thus, the key question is how to find first integrals and how to use them for solving problems. One of the methods for determining first integrals is to study virtual displacements of the system and to investigate the first integrals connected with them; these methods were elaborated in the classical studies of L. Euler, J. L. Lagrange, N. E. Joukowski, S. A. Chaplygin, N. G. Chetayev and others.

N. E. Joukowski proceeded from the three general principles of dynamics of a system of particles the first of which is the principle of motion of the centre of mass of a system (the theorem on change of the momentum of the system).

Here we stress that the *constraints* imposed on the system of particles are supposed to *satisfy Conditions* (a), (b) and (c) enumerated in Sec. 1.1 of Chap. 18.

The principles (theorems) presented in §§ 1-3 of the present chapter deal with the absolute motion of a system, that is with its motion relative to the axes of an inertial coordinate system; the principles (theorems) in § 4 are stated for the relative motion of the system.

§ 1. Principle of Momentum for a System of Particles. Principle of the Motion of the Centre of Mass

1.1. Momentum of a System of Particles and Its Expression in Terms of the Mass of the System and the Velocity of the Centre of Mass. The *mass of a system of particles* consisting of n particles with masses $m_1, m_2, \ldots, m_n$ is equal to the sum of the masses M of all the particles:

$$M = m_1 + m_2 + \ldots + m_n$$

The *centre of mass of a system of particles* in a Cartesian coordinate system $Oxyz$ is the point C whose radius vector $\boldsymbol{r}_C$ is determined by the formula

$$\boldsymbol{r}_C = \frac{1}{M} \sum_{\nu=1}^{n} m_\nu \boldsymbol{r}_\nu \tag{19.1}$$

where $\boldsymbol{r}_\nu$ ($\nu = 1, 2, \ldots, n$) is the radius vector of the νth particle of the system whose coordinates are x_ν, y_ν and z_ν.

This definition implies the following expressions for the coordinates x_C, y_C and z_C of the centre of mass of the system

$$x_C = \frac{1}{M} \sum_{\nu=1}^{n} m_\nu x_\nu, \quad y_C = \frac{1}{M} \sum_{\nu=1}^{n} m_\nu y_\nu, \quad z_C = \frac{1}{M} \sum_{\nu=1}^{n} m_\nu z_\nu \tag{19.2}$$

If the acceleration of gravity $\boldsymbol{g}$ is constant for the particles of the system then

$$M = \frac{P}{g} \qquad m_\nu = \frac{1}{g} p_\nu \quad (\nu = 1, 2, \ldots, n)$$

and the centre of mass of the system coincides with its centre of gravity (see formulas (6.9) and (6.10)).

By the *total* (*linear*) *momentum of a system of particles* is meant a free vector $\boldsymbol{Q}$ equal to the vector sum of the momenta of the particles of the system (Fig. 19.1):

$$\boldsymbol{Q} = m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2 + \ldots + m_n \boldsymbol{v}_n = \sum_{\nu=1}^{n} m_\nu \boldsymbol{v}_\nu \tag{19.3}$$

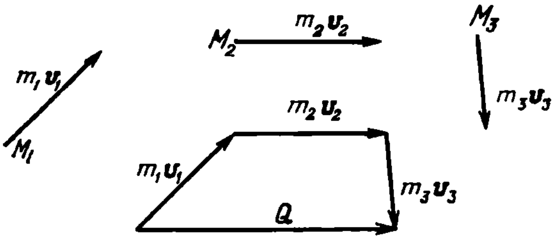

Fig. 19.1

In what follows we shall simply call the total (linear) momentum of a system of particles the *momentum of the system*. Geometrically, the momentum of the system is the closing line of the vector polygon constructed on the momenta of all the particles of the system (see Fig. 19.1).

LEMMA. *The momentum of a system of particles is equal to the momentum of the centre of mass of the system under the assumption that the total mass of the system is concentrated at that centre*:

$$Q = Mv_C. \tag{19.4}$$

Proof. Let us use formula (19.1) to transform formula (19.3):

$$Q = \sum m_\nu v_\nu = \sum m_\nu \frac{dr_\nu}{dt} = \frac{d}{dt} \sum m_\nu r_\nu = \frac{d}{dt}(Mr_C) = Mv_C$$

The lemma is proved.

1.2. Theorem on Change of the Projection of the Momentum of a System. As before, let us denote by F_ν the resultant of all the given active forces applied to the νth particle of the system; the projections of the resultant F_ν on the axes of an inertial coordinate system $Oxyz$ will be denoted by

$$X_\nu, Y_\nu, Z_\nu \quad (\nu = 1, 2, \ldots, n)$$

THEOREM ON CHANGE OF THE PROJECTION OF THE MOMENTUM OF A SYSTEM (*in differential form*): *If among the virtual displacements of a system of particles there is a translatory displacement in a certain direction* (under this displacement the system moves as a rigid body with the distances between its any two particles remaining invariable) *then the time derivative of the projection of the momentum of the system on that direction is equal to the sum of the projections of all the active forces on the same direction.*

Drawing the axis Ox along the direction of the translatory virtual displacement we can express the assertion of the theorem in the form

$$\frac{dQ_x}{dt} = \sum_{\nu=1}^{n} X_\nu \tag{19.5}$$

Proof. Suppose that the constraints imposed on the system are such that among the virtual displacements of the system there is a translatory displacement, say, along the axis Ox:

$$\delta x_\nu = \delta a \neq 0, \ \delta y_\nu = \delta z_\nu = 0 \quad (\nu = 1, 2, \ldots, n)$$

The character of the motion of the system is specified by general equation of dynamics (18.2) which in the case under consideration takes the form

$$\sum_{\nu=1}^{n} \left(X_\nu - m_\nu \frac{d^2 x_\nu}{dt^2} \right) \delta a = 0$$

Cancelling by δa we obtain

$$\frac{d}{dt} \sum_{\nu=1}^{n} m_\nu \frac{dx_\nu}{dt} = \sum_{\nu=1}^{n} X_\nu$$

Finally, using formula (19.3) we arrive at (19.5):

$$\frac{dQ_x}{dt} \equiv \frac{d}{dt} \sum_{\nu=1}^{n} m_\nu v_x^\nu = \sum_{\nu=1}^{n} X_\nu$$

The theorem is proved.

REMARK. Let us divide the given *active forces* acting on each of the particles of the system into two groups: the *external forces* and the *internal forces* (see Sec. 1.1 of Chap. 13). Here we abstract from the nature of internal forces and only assume that they are functions dependent solely on the distances between the particles, their directions are along the straight lines joining the particles and they obey Newton's third law. Then

$$\boldsymbol{F}_\nu = \boldsymbol{F}_\nu^{(\mathrm{ext})} + \boldsymbol{F}_\nu^{(\mathrm{int})} \quad (\nu = 1, 2, \ldots, n)$$

where $\boldsymbol{F}_\nu^{(\mathrm{ext})}$ and $\boldsymbol{F}_\nu^{(\mathrm{int})}$ are the resultants of the external and the internal active forces applied to the νth particle. As is known, the vector sum of the internal forces acting within the given system of particles is equal to zero (by virtue of Newton's third law; see Sec. 1.1 of Chap. 13):

$$\sum_{\nu=1}^{n} \boldsymbol{F}_\nu^{(\mathrm{int})} = 0$$

Hence, the resultant vector $\boldsymbol{R}$ of the active forces applied to the system is equal to the resultant vector $\boldsymbol{R}^{(\mathrm{ext})}$ of the *external active forces*:

$$\boldsymbol{R} = \sum_{\nu=1}^{n} \boldsymbol{F}_\nu = \sum_{\nu=1}^{n} \boldsymbol{F}_\nu^{(\mathrm{ext})} = \boldsymbol{R}^{\mathrm{ext}} \tag{19.6}$$

Therefore for the projection on any axis, for instance, on the axis Ox we have

$$\sum_{\nu=1}^{n} X_\nu = \sum_{\nu=1}^{n} X_\nu^{(\mathrm{ext})} = R_x^{(\mathrm{ext})} \tag{19.7}$$

COROLLARY. The theorem on change of the projection of the momentum of a system (formula (19.5)) can be written in the form

$$\frac{dQ_x}{dt} = R_x^{(\mathrm{ext})} \tag{19.5a}$$

Multiplying (19.5a) by dt and integrating from 0 to t we find

$$Q_x(t) - Q_x(0) = \int_0^t R_x^{(\mathrm{ext})}\, dt \tag{19.8}$$

The integral on the right-hand side is called the impulse (see Sec. 1.1 of Chap. 15).

We have thus proved the THEOREM ON CHANGE OF THE PROJECTION OF THE MOMENTUM OF A SYSTEM OF PARTICLES (*in finite form*): *If a system of particles can be in a translatory motion along an axis then the increment of the projection of the momentum of the system on that axis is equal to the impulse of the projection of the resultant vector of the external active forces on the same axis during the time under consideration.*

1.3. Principle of Motion of the Centre of Mass of a System. *If among the virtual displacements of a system of particles there is a translatory displacement parallel to the axis Ox then the motion of the centre of mass in the direction of this displacement coincides with the motion of a particle with mass $M = m_1 + m_2 + \ldots + m_n$ under the action of a force equal to the sum of the components of the external active forces along this direction:*

$$M\frac{dv_x^C}{dt} = R_x^{(\mathrm{ext})} \tag{19.9}$$

Proof. From formula (19.4) we have

$$Q_x = Mv_x^C$$

where $v_x^C = dx_C/dt$ is the projection of the velocity of the centre of mass on the axis Ox. Taking into account formula (19.7) we derive from formula (19.5) the equality

$$\frac{d}{dt}(Mv_x^C) = \sum_{\nu=1}^{n} X_\nu^{(\mathrm{ext})}, \quad \text{that is} \quad M\frac{dv_x^C}{dt} = R_x^{(\mathrm{ext})}$$

The principle is proved.

It should be stressed that all the three principles (theorems) we have stated and proved (they are in fact modifications of one of them) *do not involve* either the internal forces or (this is most important!) the *reactions of ideal constraints*. The last principle implies that the internal forces do not affect the motion of the centre of mass.

1.4. First Integrals. A *first integral* of motion (of a particle or a system of particles) is an equality connecting time, the coordinates of the particles, the projections of their velocities and also some arbitrary constants and possessing the property that it turns into identity when the coordinates of the particles and the projections of their velocities are substituted into it. It is meant that these coordinates and projections of velocities satisfy the differential equations of motion for any values of the arbitrary constants involved. Consequently, the general principles (theorems) of dynamics yield first integrals of motion in every case when it is possible to compute the integral on the right-hand side of the equation implied by the theorem.

In the special case when the right-hand side of the equation in question is identically equal to zero we obtain a *conservation law*. The condition that the right-hand side of the equation vanishes is referred to as the *conservation condition*. Galileo's law of inertia (Newton's first law) (see Sec. 1.3 of Chap. 13 and Sec. 1.3 of Chap. 15) is the simplest conservation law: it states that in every inertial motion of a particle the velocity vector of the particle remains invariable.

We shall investigate the conditions under which the principles (theorems) proved above yield first integrals.

Let us suppose that the sum of the projections of the external active forces on the axis Ox is identically equal to zero:

$$R_x^{(\mathrm{ext})} = \sum_{\nu=1}^{n} X_\nu^{(\mathrm{ext})} \equiv 0 \qquad (19.10)$$

Then equation (19.9) implies that

$$\frac{dv_x^C}{dt} \equiv 0, \quad \text{whence} \quad v_x^C(t) = v_x^C(0) = \mathrm{const}$$

It follows that

$$\frac{dx_C}{dt} = v_x^C(0), \quad \text{that is} \quad x_C(t) = v_x^C(0)\,t + x_C(0) \qquad (19.11)$$

Formula (19.11) expresses the *integral of motion of the centre of mass of a system*: if the system can be in a translatory motion as a rigid body along an axis and if the sum of the projections of the external

active forces on the same axis is identically equal to zero then the projection of the centre of mass of the system on that axis is in a uniform motion.

The integral of motion of the centre of mass of the system can be expressed in another form. Namely, under condition (19.10), formulas (19.5) and (19.7) imply that

$$\frac{dQ_x}{dt}=0, \quad \text{that is} \quad Q_x(t)=Q_x(0)=\text{const} \tag{19.12}$$

Therefore condition (19.10) is also the *conservation condition* for the projection of the momentum of the system on the axis Ox of the inertial coordinate system $Oxyz$. First integral (19.12) expresses the *law of conservation of the projection of the momentum of the system on the axis Ox.*

Let us suppose that the constraints imposed on a system of particles admit of translatory displacements of the system along all the three axes Ox, Oy and Oz (that is in any direction) as a rigid body. Then formula (19.9) implies that

$$M\frac{dv_x^C}{dt}=R_x^{(\mathrm{ext})}, \quad M\frac{dv_y^C}{dt}=R_y^{(\mathrm{ext})}, \quad M\frac{dv_z^C}{dt}=R_z^{(\mathrm{ext})}$$

This can be written in the vector form

$$M\frac{d\mathbf{v}_C}{dt}=\mathbf{R}^{(\mathrm{ext})} \tag{19.13}$$

The last equation justifies the fundamental idea of particle dynamics because in reality there exist physical bodies and there are no particles having infinitesimal dimensions. From the physical point of view the notion of a particle is justified as an idealization of phenomena in the case of a constrained body as well; however, in this case we must make use of the axiom of constraints.

Further, if the resultant vector of the external forces acting on a free system of particles is equal to zero then the velocity of the centre of mass is constant both in its magnitude and direction, that is

$$\text{if} \quad \mathbf{R}^{(\mathrm{ext})}\equiv 0, \quad \text{then} \quad \mathbf{v}_C(t)=\mathbf{v}_C(0) \tag{19.14}$$

Formula (19.14) expresses the law of *inertial motion* of the centre of mass of the system under the condition $\mathbf{R}^{(\mathrm{ext})}\equiv\mathbf{0}$. By virtue of formula (19.4), the last identity is the conservation condition for the momentum of a free system

EXAMPLE 19.1. A homogeneous rod AB of length $2l$ rests on its end A on a smooth horizontal plane; at the initial instant the rod is at rest and forms an angle α with the supporting plane. Determine the trajectory of the point B (Fig. 19.2).

Solution. Among the virtual displacements of the rod there is a translatory displacement in the vertical plane along the axis Ox. Since $\sum X_\nu^{(\mathrm{ext})}=0$ and

$v_x^C(0) = 0$ formula (19.11) implies that

$$x_C(t) = x_C(0) = 0$$

(the axis Oy passes through the centre of mass of the rod). For time $t > 0$ when the angle between the rod and the plane is $\varphi < \alpha$ we have the equalities

$$x = l \cos \varphi, \quad y = 2l \sin \varphi$$

for the coordinates of the point B. It follows that

$$\frac{x^2}{l^2} + \frac{y^2}{4l^2} = 1$$

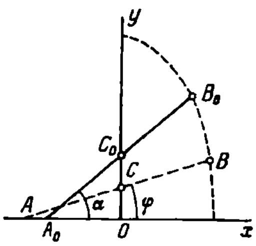

Fig. 19.2

and hence the trajectory of the point B is an arc of an ellipse with semiaxes l and $2l$ (in Fig. 19.2 this arc is shown by dash line).

EXAMPLE 19.2. A torpedo boat is in an inertial motion with the engine shut down, its velocity being v_0. At the initial instant a torpedo is shot from the torpedo tube with relative velocity u. The mass of the boat is M and the mass of the torpedo is m. Determine the velocity of motion of the boat after the torpedo is shot assuming that the resistance force of water is proportional to the first degree of the velocity.

Solution. The only external force acting along the direction of motion is the resistance force of water. During the time of the shot the action of the resistance force can be neglected, and according to the law of conservation of the projection of the momentum (see (19.12)) we have

$$(M + m) v_0 = M v_1 + m (v_0 - u)$$

From this equality we find the velocity v_1 of the boat immediately after the shot:

$$v_1 = v_0 + \frac{m}{M} u$$

The equation of motion of the centre of mass of the boat after the torpedo has been shot is written in form (19.9):

$$M \frac{dv}{dt} = -kv$$

The integration of this equation results in

$$v = v_1 e^{-\frac{k}{M} t}$$

§ 2. Principle of Angular Momentum for a System of Particles

2.1. **Definitions.** Let us construct the vectors representing the momenta $m_1 v_1, m_2 v_2, \ldots, m_n v_n$ (Fig. 19.3) for each of the particles of a system consisting of n particles and also the vectors $k_1, k_2, \ldots, k_n$ of angular momenta about a fixed centre O (see Sec. 2.1 of Chap. 15):

$$k_\nu = \mathrm{Mom}_O (m_\nu v_\nu) = [r_\nu, m_\nu v_\nu] \quad (\nu = 1, 2, \ldots, n)$$

The *total angular momentum K_O of the particles of the system about the centre O* is the vector sum of the angular momenta of all the par-

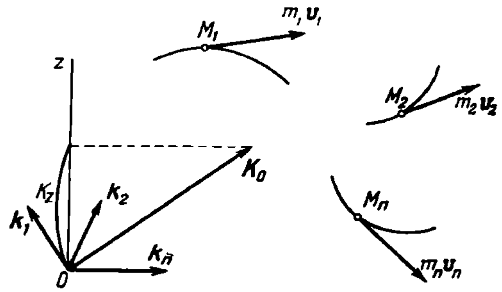

Fig. 19.3

ticles of the system about that centre:

$$K_O = k_1 + k_2 + \ldots + k_n = \sum_{\nu=1}^{n} [r_\nu, \; m_\nu v_\nu] \qquad (19.15)$$

The *total angular momentum K_z of the particles of the system about a fixed axis Oz* is the algebraic sum of the angular momenta of all the particles of the system about that axis:

$$K_z = \sum_{\nu=1}^{n} \operatorname{mom}_z (m_\nu v_\nu) \qquad (19.16)$$

Here the quantity $\operatorname{mom}_z (m_\nu v_\nu)$ ($\nu = 1, 2, \ldots, n$) is determined by formula (15.5).

Since the projection of a vector sum of vectors on an axis is equal to the algebraic sum of the corresponding projections (see (1.5)) we can also write, taking into account (15.6), the relation

$$K_z = \operatorname{proj}_{Oz} K_O \qquad (19.17)$$

Let us make use of formula (19.17) to derive an analytical expression for the (total) angular momentum of a system of particles about an axis. According to formula (1.16) for the vector product, we have

$$K_O = \sum_{\nu=1}^{n} [r_\nu, \; m_\nu v_\nu] = \sum_{\nu=1}^{n} \begin{vmatrix} i & j & k \\ x_\nu & y_\nu & z_\nu \\ m_\nu \dfrac{dx_\nu}{dt} & m_\nu \dfrac{dy_\nu}{dt} & m_\nu \dfrac{dz_\nu}{dt} \end{vmatrix}$$

Now, by formula (1.17), we obtain for K_z the expression

$$K_z = \sum_{\nu=1}^{n} m_\nu \left(x_\nu \frac{dy_\nu}{dt} - y_\nu \frac{dx_\nu}{dt} \right) \qquad (19.18)$$

2.2. Principle of Angular Momentum for a System of Particles about a Fixed Axis. *If among the virtual displacements of a system of*

particles there is a rotation about an axis Oz fixed relative to an inertial coordinate system (it is meant that the whole system can rotate as a rigid body) *then the time derivative of the (total) angular momentum of the system about the axis Oz is equal to the resultant moment of the external active forces about that axis:*

$$\frac{dK_z}{dt} = M_z^{(\text{ext})} \tag{19.19}$$

Proof. By analogy with the well-known formula (8.17) for the velocity of a particle of a body rotating about a fixed axis, we have

$$\delta \boldsymbol{r}_\nu = [\delta\varphi,\ \boldsymbol{r}_\nu] = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \delta\varphi \\ x_\nu & y_\nu & z_\nu \end{vmatrix}$$

$$= -y_\nu\,\delta\varphi\,\boldsymbol{i} + x_\nu\,\delta\varphi\,\boldsymbol{j} \qquad (\nu = 1,\ 2,\ \ldots,\ n)$$

where $\delta\varphi$ is an infinitesimal angle of rotation of the system of particles about the axis Oz. This yields the following expressions for change of the coordinates for the virtual displacement in question:

$$\delta x_\nu = -y_\nu\,\delta\varphi,\ \delta y_\nu = x_\nu\,\delta\varphi,\ \delta z_\nu = 0 \quad (\nu = 1,\ 2,\ \ldots,\ n)$$

The motion of the system is determined by general equation of dynamics (18.2) which, in the present conditions, takes the form

$$\sum_{\nu=1}^{n}\left[-\left(X_\nu - m_\nu\frac{d^2x_\nu}{dt^2}\right)y_\nu + \left(Y_\nu - m_\nu\frac{d^2y_\nu}{dt^2}\right)x_\nu\right]\delta\varphi = 0$$

Cancelling by $\delta\varphi$ we write

$$\sum_{\nu=1}^{n} m_\nu\left(x_\nu\frac{d^2y_\nu}{dt^2} - y_\nu\frac{d^2x_\nu}{dt^2}\right) = \sum_{\nu=1}^{n}(x_\nu Y_\nu - y_\nu X_\nu)$$

Let us transform the left-hand side of this formula using the identities

$$\frac{d}{dt}\left(x_\nu\frac{dy_\nu}{dt} - y_\nu\frac{dx_\nu}{dt}\right) = x_\nu\frac{d^2y_\nu}{dt^2} + \frac{dx_\nu}{dt}\frac{dy_\nu}{dt}$$

$$-y_\nu\frac{d^2x_\nu}{dt^2} - \frac{dy_\nu}{dt}\frac{dx_\nu}{dt} = x_\nu\frac{d^2y_\nu}{dt^2} - y_\nu\frac{d^2x_\nu}{dt^2} \qquad (\nu = 1,\ 2,\ \ldots,\ n)$$

After the transformation the general equation of dynamics takes the form

$$\frac{d}{dt}\sum_{\nu=1}^{n} m_\nu\left(x_\nu\frac{dy_\nu}{dt} - y_\nu\frac{dx_\nu}{dt}\right) = \sum_{\nu=1}^{n}(x_\nu Y_\nu - y_\nu X_\nu)$$

The right-hand member of the last equality is the resultant moment of the forces acting on the system of particles about the axis Oz (see formula (5.17)); the left-hand member is the time derivative

dK_z/dt (see (19.18)). Therefore the last equation implies that

$$\frac{dK_z}{dt} = M_z$$

The given active forces can be divided into external and internal and therefore

$$M_z = M_z^{(ext)} + M_z^{(int)}$$

Here $M_z^{(ext)}$ and $M_z^{(int)}$ are the resultant moments of the external and the internal active forces about the axis Oz. The internal forces acting within the system of particles split into pairs of opposite forces of the same magnitude (see Fig. 13.4) and therefore both their resultant force vector and their resultant moment about any axis are equal to zero; hence $M_z^{(int)} = 0$ and

$$M_z = M_z^{(ext)}$$

The principle is proved.

It should be stressed that the statement of this principle *does not involve the reactions of ideal constraints.*

2.3. **First Integrals.** Let us consider some conditions under which the principle proved above yields first integrals.

Suppose that the resultant moment of the external active forces about a fixed axis Oz is identically equal to zero:

$$M_z^{(ext)} \equiv 0 \tag{19.20}$$

Then from equation (19.19) it follows that

$$\frac{dK_z}{dt} = 0, \quad \text{that is} \quad K_z(t) = K_z(0) \tag{19.21}$$

Formula (19.21) expresses the law of conservation of the angular momentum of the system of particles about the axis Oz. Identity (19.20) is the conservation condition for the (total) angular momentum of the system of particles about the fixed axis Oz.

Now let us suppose that the constraints imposed on a system of particles admit of rotation of the system as a rigid body about three fixed (or inertial) axes Ox, Oy and Oz. For instance, this property holds for a free system or for a rigid body with one fixed point (see Sec. 3.3 of Chap. 5). In this case formula (19.19) yields

$$\frac{dK_x}{dt} = M_x^{(ext)}, \quad \frac{dK_y}{dt} = M_y^{(ext)}, \quad \frac{dK_z}{dt} = M_z^{(ext)} \tag{19.22}$$

These three scalar equations can be written as one vector equation

$$\frac{d\boldsymbol{K}_O}{dt} = \boldsymbol{M}_O^{(ext)} \tag{19.22a}$$

Here $\boldsymbol{K}_O$ is the total angular momentum of the system of particles about the fixed centre O (it is determined by formula (19.15)) and $\boldsymbol{M}_O^{(ext)}$ is the vector representing the resultant moment of the external active forces about that centre.

If in the case of the constraints described above the resultant moment of the external forces is identically equal to zero then the angular momentum of the system of particles is constant both in its modulus and direction, that is

$$\text{if } M_O^{(\text{ext})} \equiv 0 \text{ then } K_O(t) = K_O(0) \tag{19.23}$$

Formula (19.23) expresses the law of conservation of the total angular momentum of the system of particles about the centre O. The identity $M_O^{(\text{ext})} \equiv 0$ is the conservation condition for the angular momentum of the system of particles about the fixed point.

In the present section we do not consider examples; the application of the material of this section to dynamics of a rigid body will be demonstrated in § 2 of Chap. 21.

§ 3. Principle of Energy for a System of Particles

In this section we shall consider the principle of energy (the theorem on change of the kinetic energy of a system of particles). This principle is not connected with the type of virtual displacements of the system of particles (as was the case in §§ 1 and 2) but is based on some of their properties. Namely, we shall assume that real displacements belong to the class of virtual displacements. The assumption means that every real displacement of the system of particles coincides with one of its virtual displacements. This is the case when the constraints imposed on the system of particles do not depend on time explicitly, that is are *stationary* (see Sec. 1.1 of Chap. 17).

We remind the reader of the definition stated in Sec. 2.2 of Chap. 18: the quantity

$$T = \frac{1}{2}\sum_{\nu=1}^{n} m_\nu v_\nu^2 = \frac{1}{2}\sum_{\nu=1}^{n} m_\nu\left[\left(\frac{dx_\nu}{dt}\right)^2 + \left(\frac{dy_\nu}{dt}\right)^2 + \left(\frac{dz_\nu}{dt}\right)^2\right] \tag{19.24}$$

which is equal to the total sum of the kinetic energy of all the particles of the system is called the kinetic energy of the system in its absolute motion.

3.1. Differential Form of the Principle of Energy for a System of Particles. In this section we shall consider the differential form of the principle of energy for a system of particles which reads: *If the constraints imposed on a system of particles satisfy Conditions* (a), (b) *and* (c) *stated in Sec.* 1.1 *of Chap.* 18 *and are stationary then for any real displacement of the system the differential of the kinetic energy of the system is equal to the total sum of the elementary work of all the given active forces (both external and internal):*

$$dT = \sum_{\nu=1}^{n} (X_\nu\, dx_\nu + Y_\nu\, dy_\nu + Z_\nu\, dz_\nu) \tag{19.25}$$

Proof. For stationary constraints any real displacement of the system of particles coincides with one of its virtual displacements, that is

$$dx_\nu = \delta x_\nu,\ dy_\nu = \delta y_\nu,\ dz_\nu = \delta z_\nu \quad (\nu = 1, 2, \ldots, n)$$

In this case general equation of dynamics (18.2) can be represented in the following form:

$$\sum_{\nu=1}^{n} \left[\left(X_\nu - m_\nu \frac{d^2x_\nu}{dt^2} \right) dx_\nu + \left(Y_\nu - m_\nu \frac{d^2y_\nu}{dt^2} \right) dy_\nu + \left(Z_\nu - m_\nu \frac{d^2z_\nu}{dt^2} \right) dz_\nu \right] = 0$$

We rewrite this equation thus:

$$dt \sum_{\nu=1}^{n} m_\nu \left(\frac{d^2x_\nu}{dt^2} \frac{dx_\nu}{dt} + \frac{d^2y_\nu}{dt^2} \frac{dy_\nu}{dt} + \frac{d^2z_\nu}{dt^2} \frac{dz_\nu}{dt} \right) = \sum_{\nu=1}^{n} (X_\nu\, dx_\nu + Y_\nu\, dy_\nu + Z_\nu\, dz_\nu)$$

Taking into account that

$$dt \frac{dx_\nu}{dt} \frac{d^2x_\nu}{dt^2} = dt \cdot \frac{1}{2} \frac{d}{dt} \left[\left(\frac{dx_\nu}{dt} \right)^2 \right] = d \left[\frac{1}{2} \left(\frac{dx_\nu}{dt} \right)^2 \right] \quad (\nu = 1, 2, \ldots, n)$$

and that similar equalities hold for the other summands on the left-hand side we obtain

$$d \sum_{\nu=1}^{n} \frac{1}{2} m_\nu \left[\left(\frac{dx_\nu}{dt} \right)^2 + \left(\frac{dy_\nu}{dt} \right)^2 + \left(\frac{dz_\nu}{dt} \right)^2 \right] = \sum_{\nu=1}^{n} (X_\nu\, dx_\nu + Y_\nu\, dy_\nu + Z_\nu\, dz_\nu)$$

The expression of the left-hand side is the differential of the kinetic energy of the system (see (19.24)). The principle is proved.

The division of equality (19.25) by dt results in

$$\frac{dT}{dt} = N = \sum_{\nu=1}^{n} (X_\nu v_x^\nu + Y_\nu v_y^\nu + Z_\nu v_z^\nu) \tag{19.26}$$

We see that the time derivative of the kinetic energy of a system of particles is equal to the power N of all the given active forces (both external and internal) applied to the system.

Let us divide the given active forces applied to the νth particle ($\nu = 1, 2, \ldots, n$) into external forces $F_\nu^{(\text{ext})}$ and internal forces $F_\nu^{(\text{int})}$; for the projections of these forces we have

$$X_\nu = X_\nu^{(\text{ext})} + X_\nu^{(\text{int})}, \quad Y_\nu = Y_\nu^{(\text{ext})} + Y_\nu^{(\text{int})}, \quad Z_\nu = Z_\nu^{(\text{ext})} + Z_\nu^{(\text{int})}$$
$$(\nu = 1, 2, \ldots, n)$$

Now by the principle we have proved, we obtain the expression

$$dT = \sum_{\nu=1}^{n} (X_\nu^{(\text{ext})}\,dx_\nu + Y_\nu^{(\text{ext})}\,dy_\nu + Z_\nu^{(\text{ext})}\,dz_\nu) + \sum_{\nu=1}^{n} (X_\nu^{(\text{int})}\,dx_\nu + Y_\nu^{(\text{int})}\,dy_\nu + Z_\nu^{(\text{int})}\,dz_\nu)$$

3.2. Integral Form of the Principle of Energy for a System of Particles. In this section we shall derive the integral form of the principle of energy for a system of particles.

The integration of differential equality (19.25) results in

$$T - T_0 = A^{(\text{ext})} + A^{(\text{int})} \tag{19.27}$$

Hence, *the increment of the kinetic energy of a system of particles is equal to the total work of all the given active forces applied to the system.* Here the line integrals

$$A^{(\text{ext})} = \int_{\Gamma_0}^{\Gamma} \sum_{\nu=1}^{n} (X_\nu^{(\text{ext})}\,dx_\nu + Y_\nu^{(\text{ext})}\,dy_\nu + Z_\nu^{(\text{ext})}\,dz_\nu)$$

and

$$A^{(\text{int})} = \int_{\Gamma_0}^{\Gamma} \sum_{\nu=1}^{n} (X_\nu^{(\text{int})}\,dx_\nu + Y_\nu^{(\text{int})}\,dy_\nu + Z_\nu^{(\text{int})}\,dz_\nu)$$

express the work of the external and the internal active forces respectively when the system undergoes displacement from the initial configuration Γ_0 to the terminal configuration Γ.

As an example, let us compute the *work performed by the forces of gravity* for a displacement of a system of particles. For the projections of the forces of gravity we have

$$X_\nu^{(\text{ext})} = Y_\nu^{(\text{ext})} = 0, \quad Z_\nu^{(\text{ext})} = -m_\nu g$$

and therefore

$$dA^{(\text{ext})} = \sum_{\nu} (X_\nu^{(\text{ext})}\,dx_\nu + Y_\nu^{(\text{ext})}\,dy_\nu + Z_\nu^{(\text{ext})}\,dz_\nu) = -g\,d\sum_{\nu} m_\nu z_\nu$$

The sum under the sign of differential on the right-hand side is equal to Mz_C, where M is the mass of the system and z_C is the z-coordinate of the centre of mass of the system (see the third for-

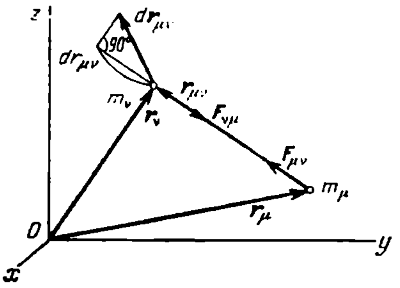

Fig. 19.4

mula (19.2)). Consequently

$$dA^{(\text{ext})} = -Mg\, dz_C$$

Integrating the last equation we find that the work corresponding to the displacement of the system of particles in the gravity field is equal to

$$A^{(\text{ext})} = -Mg\,[z_C - z_C(0)] \tag{19.28}$$

that is to the product of the weight of the system by the vertical downward displacement of the centre of mass.

The statements of the foregoing principles (theorems) of dynamics of a system of particles (§§ 1 and 2) involve neither the reactions of ideal constraints nor the internal forces. The principle of energy does not involve the reactions of ideal constraints either but, generally speaking, it involves the work of the internal forces.

3.3. Remark on the Work of Internal Forces. Here we shall indicate the cases when the principle of energy for a system of particles does not involve the work of the internal forces.

LEMMA. *For a system of particles with invariable configuration* (this means that the distances between any two particles remain constant during the whole time of motion) *the work of the internal forces is equal to zero for any real displacement of the system.*

Proof. Internal forces are produced by the interaction between the particles of the system; they depend solely on the distances between the particles, are directed along the straight lines joining the particles and obey Newton's third law. If two particles m_ν and m_μ of the system (Fig. 19.4) undergo elementary displacements $d\boldsymbol{r}_\nu$ and $d\boldsymbol{r}_\mu$ respectively the elementary work of the interaction forces between the particles is equal to

$$(\boldsymbol{F}_{\nu\mu}, d\boldsymbol{r}_\nu) + (\boldsymbol{F}_{\mu\nu}, d\boldsymbol{r}_\mu) = (\boldsymbol{F}_{\nu\mu}, d\boldsymbol{r}_\nu - d\boldsymbol{r}_\mu) = (\boldsymbol{F}_{\nu\mu}, d(\boldsymbol{r}_\nu - \boldsymbol{r}_\mu))$$

$$= (\boldsymbol{F}_{\nu\mu}, d\boldsymbol{r}_{\mu\nu})$$

Here $\boldsymbol{r}_{\mu\nu} = \boldsymbol{r}_\nu - \boldsymbol{r}_\mu$ is the vector joining the particles m_ν and m_μ. Since $F_{\nu\mu} = f(r_{\mu\nu})$ and since the vectors $\boldsymbol{F}_{\nu\mu}$ and $\boldsymbol{r}_{\mu\nu}$ are collinear we conclude (see Fig. 19.4) that

$$(\boldsymbol{F}_{\nu\mu}, d\boldsymbol{r}_{\mu\nu}) = f(r_{\mu\nu})\,|d\boldsymbol{r}_{\mu\nu}|\cos(\widehat{\boldsymbol{F}_{\nu\mu}, d\boldsymbol{r}_{\mu\nu}}) = -f(r_{\mu\nu})\,dr_{\mu\nu}$$

For the work of the internal forces of the system of particles we obtain

$$\sum_{\nu=1}^{n} (\boldsymbol{F}_\nu^{(\text{int})},\ d\boldsymbol{r}_\nu) = \sum_{\substack{\nu,\ \mu=1 \\ (\nu>\mu)}}^{n} [(\boldsymbol{F}_{\nu\mu},\ d\boldsymbol{r}_\nu) + (\boldsymbol{F}_{\mu\nu},\ d\boldsymbol{r}_\mu)] = -\sum_{\substack{\nu,\ \mu=1 \\ (\nu>\mu)}}^{n} f(r_{\mu\nu})\, dr_{\mu\nu}$$

For a system with invariable configuration of the particles we have $dr_{\mu\nu} = 0$ ($\nu,\ \mu = 1, 2, \ldots, n$) for all the particles (in contrast to Fig. 19.4!) and consequently

$$dA^{(\text{int})} = \sum_{\nu=1}^{n} (\boldsymbol{F}_\nu^{(\text{int})},\ d\boldsymbol{r}_\nu) = 0$$

which proves the lemma.

Since a perfectly rigid body can be regarded as a system of particles with invariable configuration the work of the internal forces corresponding to any displacement of a rigid body is always equal to zero. In this case the principle of energy is written in differential form as

$$dT = \sum_{\nu=1}^{n} (X_\nu^{(\text{ext})}\, dx_\nu + Y_\nu^{(\text{ext})}\, dy_\nu + Z_\nu^{(\text{ext})}\, dz_\nu) \qquad (19.25\text{a})$$

or in the equivalent form

$$\frac{dT}{dt} = N^{(\text{ext})} = \sum_{\nu=1}^{n} (X_\nu^{(\text{ext})} v_x^\nu + Y_\nu^{(\text{ext})} v_y^\nu + Z_\nu^{(\text{ext})} v_z^\nu) \qquad (19.26\text{a})$$

The right-hand members in formulas (19.25a) and (19.26a) express, respectively, the elementary work corresponding to the real displacement and the power of all the given active external forces.

Formula (19.27) is written in the form

$$T - T_0 = A^{(\text{ext})} = \int_{\Gamma_0}^{\Gamma} \sum_{\nu=1}^{n} (X_\nu^{(\text{ext})}\, dx_\nu + Y_\nu^{(\text{ext})}\, dy_\nu + Z_\nu^{(\text{ext})}\, dz_\nu) \qquad (19.27\text{a})$$

Thus, *the increment of the kinetic energy of a rigid body* (or *of a system of particles with invariable configuration*) *is equal to the total work of all the given active external forces applied to the body* (or *to the system of particles with invariable configuration*) *along the given path.* In Secs. 2.4 and 2.5 of Chap. 21 we shall come back to formula (19.27a).

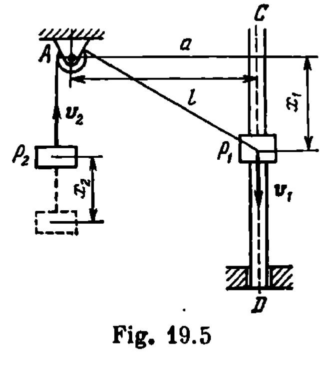

Fig. 19.5

EXAMPLE 19.3. From the ends of a flexible inextensible thread passed around a pulley A of infinitesimal radius two weights p_1 and p_2 are suspended (Fig. 19.5). The weight

p_1 can slide along a smooth vertical rod CD. Find the relationship between the velocity v_1 of the weight p_1 and its vertical downward path x_1 on condition that this weight starts moving downward without initial velocity and is on the same horizontal as the axis of the pulley at the initial instant $t = 0$.

Solution. Let the weight p_1 move downward and pass a distance x_1 and the weight p_2 move upward and pass a distance x_2. The constraints being ideal and stationary, the principle of energy for the system of particles is applicable to the displacement under consideration. By formula (19.24), the kinetic energy at instant t is equal to

$$T = \frac{p_1}{2g} v_1^2 + \frac{p_2}{2g} v_2^2$$

while the initial value of the kinetic energy is $T_0 = 0$. The work of the external active forces (of the forces of gravity) is determined by formula (19.28):

$$A^{(ext)} = p_1 x_1 - p_2 x_2$$

By formula (19.27), neglecting the work of the internal forces in the system (this system does not possess an invariable configuration!), we obtain

$$\frac{1}{2}\left(\frac{p_1}{g} v_1^2 + \frac{p_2}{g} v_2^2\right) = p_1 x_1 - p_2 x_2 \qquad (1)$$

Since the thread is inextensible simple geometric arguments imply that

$$x_2 = l - a = \sqrt{x_1^2 + a^2} - a$$

The differentiation of this constraint equation results in

$$\frac{dx_2}{dt} = \frac{x_1}{\sqrt{x_1^2 + a^2}} \frac{dx_1}{dt}$$

Further, we have $v_1 = dx_1/dt$ and $v_2 = dx_2/dt$ and consequently

$$v_2 = \frac{x_1}{\sqrt{x_1^2 + a^2}} v_1$$

After some simplification the substitution of the values of x_2 and v_2 we have found into (1) yields

$$v_1^2 = 2g\,(x_1^2 + a^2)\,\frac{p_1 x_1 - p_2\left(\sqrt{x_1^2 + a^2} - a\right)}{p_1 (x_1^2 + a^2) + p_2 x_1^2}$$

3.4. Energy Integral. We remind the reader of the definition stated in Sec. 2.3 of Chap. 18: if the active forces are such that there exists a function

$$U\,(x_1, y_1, z_1;\ x_2, y_2, z_2;\ \ldots;\ x_n, y_n, z_n)$$

satisfying the conditions

$$X_\nu = \frac{\partial U}{\partial x_\nu}, \quad Y_\nu = \frac{\partial U}{\partial y_\nu}, \quad Z_\nu = \frac{\partial U}{\partial z_\nu} \quad (\nu = 1, 2, \ldots, n)$$

then U is called the *force function* and the forces themselves are said to be *conservative**.

The function

$$V = -U$$

---

* In this case the system of particles itself is also said to be conservative.

is called the *potential energy of the system of particles.* Both the functions U and V are determined to within an additive constant. For a conservative system equality (19.25) can be written in the form

$$dT = \sum_{\nu=1}^{n} \left(\frac{\partial U}{\partial x_\nu} dx_\nu + \frac{\partial U}{\partial y_\nu} dy_\nu + \frac{\partial U}{\partial z_\nu} dz_\nu \right) = dU$$

where dU is the differential of the force function corresponding to the real motion of the system. Integrating we obtain

$$T = U + h \quad (h = T_0 - U_0 = \text{const}) \tag{19.27b}$$

Equality (19.27b) expresses a first integral of the equations of motion of a conservative system of particles; it is called the *energy integral.* The quantity $h = T - U = T + V$ is the total mechanical energy of the system.

The energy integral exists when the real displacements belong to the class of virtual displacements (see the beginning of § 3) and for the active forces there exists a time independent force function.

The energy integral expresses the property that the total mechanical energy $T + V$ remains constant during the whole time of motion.

We shall use the energy integral in the proof of Lagrange's theorem on stability of equilibrium (see Sec. 2.2 of Chap. 20).

3.5. Remarks on the Application of the General Principles of Dynamics of a System of Particles. In the principles (theorems) of §§ 1, 2 and 3 (the latter was applied to a system of particles with invariable configuration) the *external active forces* were assumed to be given. This assumption stressed that the corresponding formulas did not involve either internal forces or constraint reactions (which are external passive forces and are not given). Besides, in mechanics of a system of particles *we always considered ideal constraints*, that is constraints without friction.

1. However, the general principles of dynamics also make it possible to *determine the constraint reactions.* To this end all the constraints or their part (depending on whether all the reactions or only some of them must be determined) are replaced by the corresponding reactions. When discarding a part of the constraints or all of them we *add the corresponding reactions to the external active forces*, after which the general principles of dynamics are applied and the corresponding first integrals are used. In particular, it is also possible to consider the simplest constraints with friction; the forces of sliding friction appearing after the action of the constraints is replaced by the reactions are also added to the given external forces (see below Examples 21.1 and 21.4).

2. The general principles of dynamics can also be used when there are *constraints not satisfying Conditions* (a), (b) and (c) stated in

Sec. 1.1 of Chap. 18, in particular when there are constraints with friction. In these cases such constraints should be discarded and their action should be replaced by the *corresponding reactions which are added to the external active forces.*

3. All the general principles of dynamics of a system of particles also apply to systems for which all the constraints are discarded on condition that the corresponding reactions are added to the external active forces. In this case the condition that there must exist the corresponding virtual displacements (see §§ 1 and 2) can be discarded since a free system of particles possesses all the indicated virtual displacements.

§ 4. General Principles of Motion of a System of Particles Relative to the Centre of Mass

4.1. Angular Momentum and Kinetic Energy of a System Relative to König's Axes*. Besides the inertial coordinate system $Oxyz$ let us consider a coordinate system $Cx'y'z'$ with origin at the centre of mass C of the system of particles and with axes *parallel* to the fixed axes Ox, Oy and Oz (the axes Cx', Cy' and Cz' are called König's axes; Fig. 19.6). Thus, the coordinate system $Cx'y'z'$ is in a translatory motion (which, in the general ease, must not necessarily be rectilinear and uniform). The motion of the system of particles about the axes of the coordinate system $Cx'y'z'$ is called the *motion relative to the centre of mass.* The coordinates of the particle M_ν ($\nu = 1, 2, \ldots, n$) with mass m_ν with respect to the coordinate system $Oxyz$ will be denoted by x_ν, y_ν and z_ν and with respect to the coordinate system $Cx'y'z'$ by x'_ν, y'_ν and z'_ν; then

$$x_\nu = x_C + x'_\nu, \quad y_\nu = y_C + y'_\nu, \quad z_\nu = z_C + z'_\nu$$
$$(\nu = 1, 2, \ldots, n) \qquad (19.29)$$

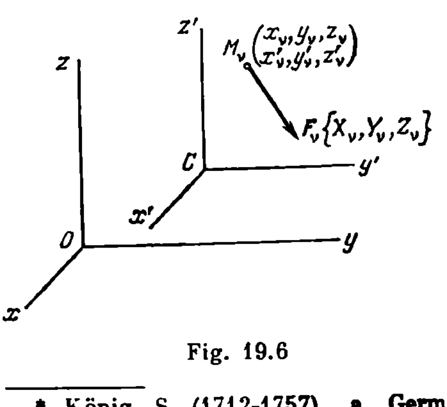

Fig. 19.6

The coordinates of the centre of mass C with respect to the coordinate system $Oxyz$ are

$$x_C = \frac{1}{M}\sum m_\nu x_\nu$$
$$y_C = \frac{1}{M}\sum m_\nu y_\nu$$
$$z_C = \frac{1}{M}\sum m_\nu z_\nu$$
$$\left(M = \sum m_\nu\right)$$

* König, S. (1712-1757), a German mathematician.

(see formulas (6.9) and (19.2)), where here and henceforth the summation extends over all the particles of the system. The coordinates of the centre of mass C with respect to the coordinate system $Cx'y'z'$ are naturally equal to zero, which means that there hold the equalities

$$\sum m_\nu x'_\nu = \sum m_\nu y'_\nu = \sum m_\nu z'_\nu = 0 \tag{19.30}$$

Let us compute the angular momentum of the system of particles about one of the fixed axes, for instance, about the axis Oz. Proceeding from formulas (19.18) and (19.29) and denoting the time derivatives by dots we find

$$\begin{aligned}K_z &= \sum m_\nu [(x_C + x'_\nu)(\dot{y}_C + \dot{y}'_\nu) - (y_C + y'_\nu)(\dot{x}_C + \dot{x}'_\nu)] \\ &= \sum m_\nu (x'_\nu \dot{y}'_\nu - y'_\nu \dot{x}'_\nu) + (x_C \dot{y}_C - y_C \dot{x}_C) \sum m_\nu \\ &\quad + x_C \sum m_\nu \dot{y}'_\nu + \dot{y}_C \sum m_\nu x'_\nu - y_C \sum m_\nu \dot{x}'_\nu - \dot{x}_C \sum m_\nu y'_\nu\end{aligned}$$

The second sum on the right-hand side is equal to the angular momentum $Q = Mv_C$ of the system about the axis Oz (see formula (19.4)). The last four sums are equal to zero during the whole time of motion, which follows from identities (19.30) after their differentiation. We have thus derived *König's first formula:*

$$K_z = K'_{z'} + M\left(x_C \frac{dy_C}{dt} - y_C \frac{dx_C}{dt}\right) = K'_{z'} + \mathrm{mom}_{Oz} Mv_C \tag{19.31}$$

where

$$K'_{z'} = \sum m_\nu \left(x'_\nu \frac{dy'_\nu}{dt} - y'_\nu \frac{dx'_\nu}{dt}\right) \tag{19.32}$$

Hence, *the angular momentum K_z of the system of particles about the fixed axis Oz is equal to the sum of the angular momentum $K'_{z'}$ of the system about the corresponding parallel moving axis passing through the centre of mass C and the angular momentum of the total mass of the system (applied to the centre of mass) about the fixed axis.* In other words, the total angular momentum of the system of particles in its absolute motion is equal to the angular momentum in its motion relative to König's axes plus the angular momentum of the centre of mass of the system in the absolute motion (on condition that the total mass of the system is placed at the centre of mass).

Now let us compute the kinetic energy of the system of particles. According to formula (19.24) and formulas (19.29) we have

$$\begin{aligned}T &= \frac{1}{2} \sum m_\nu [(\dot{x}_C + \dot{x}'_\nu)^2 + (\dot{y}_C + \dot{y}'_\nu)^2 + (\dot{z}_C + \dot{z}'_\nu)^2] \\ &= \frac{1}{2} \sum m_\nu (\dot{x}'^2_\nu + \dot{y}'^2_\nu + \dot{z}'^2_\nu) + \frac{1}{2} (\dot{x}^2_C + \dot{y}^2_C + \dot{z}^2_C) \sum m_\nu \\ &\quad + \dot{x}_C \sum m_\nu \dot{x}'_\nu + \dot{y}_C \sum m_\nu \dot{y}'_\nu + \dot{z}_C \sum m_\nu \dot{z}'_\nu\end{aligned}$$

The second sum on the right-hand side is equal to the kinetic energy of the centre of mass of the system of particles on condition that the total mass of the system is concentrated there. The last three sums are equal to zero during the whole time of motion, which follows from identities (19.30). We have thus derived *König's second formula*

$$T = T'_C + \frac{1}{2} M v_C^2 \tag{19.33}$$

where

$$T'_C = \frac{1}{2} \sum m_\nu \left[\left(\frac{dx'_\nu}{dt} \right)^2 + \left(\frac{dy'_\nu}{dt} \right)^2 + \left(\frac{dz'_\nu}{dt} \right)^2 \right] \tag{19.34}$$

Hence, *the kinetic energy T of the system of particles in its absolute motion is equal to the sum of the kinetic energy T'_C in the relative motion with respect to König's axes and the kinetic energy of the centre of mass of the system in the absolute motion (on condition that the total mass of the system is placed at the centre of mass).*

4.2. Résal's* Theorem on Angular Momentum of a System in Its Motion Relative to the Centre of Mass. Suppose that the system in question is subjected to *constraints satisfying Conditions* (a), (b) *and* (c) *stated in Sec.* 1.1 *of Chap.* 18; *let among the virtual displacements of the system of particles there be the following displacements:*

(a) *translatory displacements of the system as a rigid body along the axes Ox and Oy;*

(b) *rotation of the system as a rigid body about the fixed axis Oz.*

For these conditions Résal's theorem reads: the time derivative of the angular momentum $K'_{z'}$ of the system about König's axis Cz' is equal to the resultant moment of the external active forces about that axis:

$$\frac{dK'_{z'}}{dt} = M_{Cz'}^{(\text{ext})} = \sum_{\nu=1}^{n} \text{mom}_{Cz'} F_\nu^{(\text{ext})} \tag{19.35}$$

Proof. By virtue of Condition (a) of the theorem, there holds the principle of motion of the centre of mass of the system (see (19.9)):

$$M \frac{d^2 x_C}{dt^2} = \sum X_\nu^{(\text{ext})}, \quad M \frac{d^2 y_C}{dt^2} = \sum Y_\nu^{(\text{ext})} \tag{19.36}$$

Condition (b) means that there holds the principle of angular momentum for the system of particles about the fixed axis Oz (see formulas (19.19) and (5.17)):

$$\frac{dK_z}{dt} = M_z^{(\text{ext})} = \sum (x_\nu Y_\nu^{(\text{ext})} - y_\nu X_\nu^{(\text{ext})})$$

* Résal, H. (1828-1896), a French engineer.

Let us substitute the expressions specified by formulas (19.31) and (19.32) into the left-hand side and perform differentiation: also let us substitute the values of the coordinates determined by (19.29) into the right-hand side:

$$\frac{dK'_{z'}}{dt}+x_C M\frac{d^2y_C}{dt^2}-y_C M\frac{d^2x_C}{dt^2}$$
$$=\sum (x'_\nu Y_\nu^{(\mathrm{ext})}-y'_\nu X_\nu^{(\mathrm{ext})})+x_C\sum Y_\nu^{(\mathrm{ext})}-y_C\sum X_\nu^{(\mathrm{ext})}$$

The last two summands on the left-hand and the right-hand sides are equal (according to (19.36)); cancelling them we obtain

$$\frac{dK'_{z'}}{dt}=\sum (x'_\nu Y_\nu^{(\mathrm{ext})}-y'_\nu X_\nu^{(\mathrm{ext})})$$

The quantity on the right-hand side is nothing other than $M_{Cz'}^{(\mathrm{ext})}$ that is the resultant moment of the external active forces about König's axis Cz'. The theorem is proved.

4.3. König's Theorem on Change of the Kinetic Energy of a System in Its Motion Relative to the Centre of Mass. *If*

(a) *the constraints satisfy Conditions* (a), (b) *and* (c) *stated in Sec.* 1.1 of *Chap.* 18 *and are stationary* (do not depend on time explicitly),

(b) *and are such that among the virtual displacements of the system of particles in question there are translatory displacements in any direction* (in particular, in the directions of the axes Ox, Oy and Oz)

then the differential of the kinetic energy T_C of the system in its motion relative to König's axes is equal to the total elementary work of all the given active forces (both external and internal) along the real relative displacement of the system:

$$dT'_C=\sum_{\nu=1}^{n}(X_\nu\,dx'_\nu+Y_\nu\,dy'_\nu+Z_\nu\,dz'_\nu), \tag{19.37}$$

Proof. By virtue of Condition (b) of the theorem, there holds the principle of motion of the centre of mass of the system (see formulas (19.9) and (19.13)):

$$M\frac{d^2x_C}{dt^2}=\sum X_\nu,\quad M\frac{d^2y_C}{dt^2}=\sum Y_\nu,\quad M\frac{d^2z_C}{dt^2}=\sum Z_\nu. \tag{19.38}$$

By virtue of Condition (a), there holds the principle of energy in the absolute motion (see formula (19.25)):

$$dT=\sum (X_\nu\,dx_\nu+Y_\nu\,dy_\nu+Z_\nu\,dz_\nu)$$

Let us substitute the expressions given by formulas (19.33) and (19.34) into the left-hand side; also let us substitute the values of

the coordinates determined by (19.29) into the right-hand side:

$$dT'_C+M\frac{d^2x_C}{dt^2}dx_C+M\frac{d^2y_C}{dt^2}dy_C+M\frac{d^2z_C}{dt^2}dz_C$$

$$=\sum(X_\nu dx'_\nu+Y_\nu dy'_\nu+Z_\nu dz'_\nu)+dx_C\sum X_\nu+dy_C\sum Y_\nu+dz_C\sum Z_\nu$$

The last three summands on the left-hand and the right-hand sides are equal (by virtue of (19.38)); cancelling them we obtain (19.37). The theorem is proved.

We see that under certain conditions imposed on the constraints the principles (theorems) proved in §§ 1 and 2 can be extended to the case of motion of a system of particles relative to the centre of mass. The application of these principles (theorems) will be demonstrated in dynamics of a rigid body (see § 3 of Chap. 21).

Problems

PROBLEM 19.1. The trolley of a travelling crane (Fig. 19.7) moves relative to the crane with velocity v. The weight of the crane is P and the weight of the trolley is Q. Neglecting all the resistance forces determine the velocity u of the recoil of the crane.

Answer. $u=-\frac{Q}{P+Q}v$.

PROBLEM 19.2. Two weights A and B of masses m_1 and m_2 respectively are connected with an inextensible thread passed around the pulley D (Fig. 19.8). The weight B slides down the face of a triangular prism of mass m and imparts motion to the weight A. Find the displacement of the prism along the horizontal plane corresponding to the downward displacement of the weight B by a distance l. The masses of the pulley and the thread are negligibly small and at the initial instant the system is at rest.

Answer. $s=-\frac{m_1+m_2}{m+m_1+m_2}l\cos\alpha$.

PROBLEM 19.3. A man of mass m_1 starts walking along the platform of the car with constant relative velocity v_{rel} (Fig. 19.9). At the initial instant the car whose mass is m_2 is at rest. Find the absolute velocities v_1 and v_2 of the car and the man on condition that the car moves on a horizontal plane. The friction between the rails and the wheels of the car is negligibly small.

Answer. $v_1=-\frac{m_1}{m_1+m_2}v_{rel}$, $v_2=\frac{m_2}{m_1+m_2}v_{rel}$.

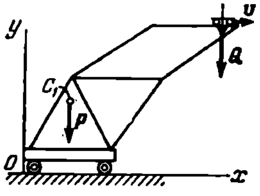

Fig. 19.7

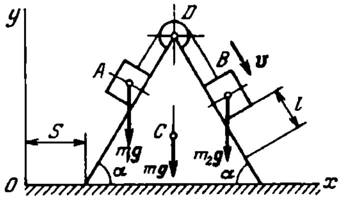

Fig. 19.8

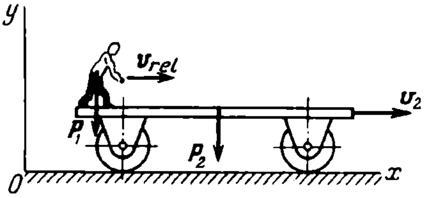

Fig. 19.9

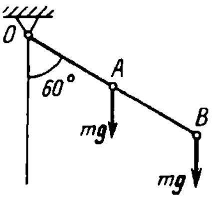

Fig. 19.10

PROBLEM 19.4. Two particles A and B of mass m each are attached to a rigid weightless rod of length $2a$; one of them is attached to the midpoint of the rod and the other to its end (Fig. 19.10). The other end O of the rod has a hinged support. At the initial instant the rod is given a deviation of 60° from the vertical and then let to move without initial velocity. Find the angular velocity ω of the rod at the instant when it passes through the position of stable equilibrium.

Answer. $|\omega| = \sqrt{\frac{3}{5}\frac{g}{a}}$ rad/s.

Chapter 20 D'Alembert's Principle. Stability of Equilibrium and Small Oscillation

Before passing to dynamics of a rigid body we shall consider two different problems concerning systems of particles.

§ 1. D'Alembert's Principle

1.1. D'Alembert's* Principle for a Particle. Let us come back to the motion of a constrained particle (see § 1 of Chap. 16). It is acted upon by the resultant $\boldsymbol{F}$ of the given active forces and the resultant $\boldsymbol{N}$ of the constraint reactions (the passive forces) (Fig. 20.1). According to Newton's second law, the acceleration $\boldsymbol{w}$ of the particle m moving under the action of the resultant

$$\boldsymbol{R} = \boldsymbol{F} + \boldsymbol{N}$$

of these forces is directed along the line of action of $\boldsymbol{R}$ and

$$\boldsymbol{R} = m\boldsymbol{w}$$

or, which is the same,

$$\boldsymbol{F} + \boldsymbol{N} + (-m\boldsymbol{w}) = 0 \tag{20.1}$$

The term $(-m\boldsymbol{w})$ has the dimension of force.

* D'Alembert, J. (1717-1783), a French philosopher and mathematician.

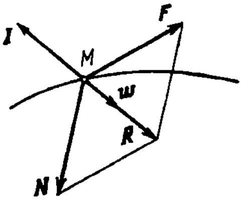

Fig. 20.1

By the *inertial force* I of the particle is meant the vector $(-mw)$. It is thought of as being "applied" to the moving particle; its modulus is equal to the product of the mass of the particle by its acceleration and its direction is opposite to the acceleration vector of the particle:

$$I = -mw \tag{20.2}$$

It should be stressed that the inertial force I which is thought of as a vector "applied" to the moving particle in its absolute motion is a fictitious force. An observer connected with the fixed coordinate system detects no other forces than those produced by the action of the material bodies, and all these forces are taken into account in the sum of the vectors $F + N$ in equation (20.1). Here the term "inertial force" should be understood as a conditional name. On the other hand, the inertial forces corresponding to the transportation and the Coriolis accelerations (see Sec. 2.1 of Chap. 16) are real because their magnitudes can be determined by comparing the data shown by the dynamometer in the fixed and in the moving coordinate systems.

Now let us state *D'Alembert's principle for a constrained particle*: *If inertial force* (20.2) *is added to the sum of the given forces acting on the particle and the constraint reactions then for each instant during the motion we obtain a balanced force system*:

$$F + N + I = 0 \tag{20.1a}$$

In particular, if the particle is free then we must put $N = 0$ in equation (20.1a).

It should be stressed that this statement has a conditional meaning since during the motion of the system there is no balance of forces because the active forces and the constraint reactions are really applied to the particle while the inertial force is fictitious (it is conditionally thought of as being "applied" to the moving particle).

D'Alembert's principle provides a convenient practical method for solving dynamic problems because it makes it possible to write equations of motion in the form of equilibrium equations. This however does not reduce dynamic problems to static ones because, although this method simplifies the setting of the equations of motion, in the general case the problem of integrating these equations remains unsolved.

Vector equation (20.1a) can be written in terms of the projections on the inertial coordinate axes:

$$F_x + N_x + I_x = 0, \quad F_y + N_y + I_y = 0, \quad F_z + N_z + I_z = 0 \tag{20.3}$$

The substitution of the values

$$I_x = -mw_x = -m\frac{d^2x}{dt^2}, \quad I_y = -mw_y = -m\frac{d^2y}{dt^2}$$
$$I_z = -mw_z = -m\frac{d^2z}{dt^2}$$

into these equations leads to equations (13.6).

Projecting both the sides of equation (20.1a) on the natural axes $M\tau$, Mn and Mb (see Sec. 3.3 of Chap. 7) we obtain

$$F_\tau + N_\tau + I_\tau = 0, \quad F_n + N_n + I_n = 0, \quad F_b + N_b = 0 \quad (20.4)$$

because $I_b = -mw_b \equiv 0$ (in the case of ideal constraints we have $N \perp M\tau$ and consequently $N_\tau = 0$). The substitution of the values

$$I_\tau = -mw_\tau = \quad m\frac{dv_\tau}{dt} \text{ and } I_n = -mw_n = -m\frac{v^2}{\rho}$$

into equations (20.4) (here ρ is the radius of curvature of the trajectory) results in equations (13.8). The vectors I_τ and I_n are called the *tangential inertial force* and the *normal inertial force* respectively.

EXAMPLE 20.1. Let a weightless rod OA of length l rotate about the ball-and-socket joint O with constant angular velocity ω (the rod describes a conic surface). To the other end of the rod a heavy particle A of mass m is attached (Fig. 20.2). Determine the angle γ between the rod and the vertical and the dynamic reaction N of the rod.

Solution. The acceleration of the particle in its uniform motion round the circle is

$$w = w_n = l \sin\gamma\, \omega^2$$

This acceleration is directed to the centre O_1 of the circle. The inertial force is directed away from the centre of the circle and its modulus is equal to

$$I_n = mw_n = ml\omega^2 \sin\gamma$$

Here the given force F is the force of weight mg, and the reaction N of the rod is directed along the rod towards the point O. Let us draw the axis Oy vertically downward, and let the plane Oxy pass through the point A at the given instant. Now we form equations (20.3):

$$-N\sin\gamma + ml\omega^2\sin\gamma = 0, \quad mg - N\cos\gamma = 0$$

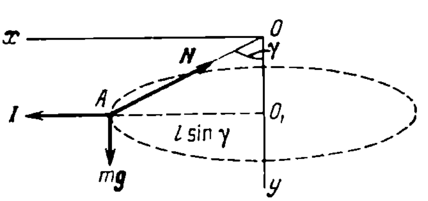

Fig. 20.2

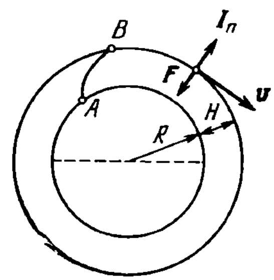

Fig. 20.3

From these equations we find

$$N = ml\omega^2, \quad \cos\gamma = \frac{mg}{N} = \frac{g}{l\omega^2}$$

The rod is acted upon by the centrifugal inertial force $ml\omega^2$ whose direction is opposite to that of the dynamic reaction N.

EXAMPLE 20.2. Determine the velocity of motion of an artificial satellite of the Earth in a circular orbit round the Earth at a distance H from the Earth's surface and also the period of rotation neglecting the air resistance (Fig. 20.3).

Solution. The attractive force F at the distance H from the Earth's surface is determined by formula (13.31) (see Example 13.7); the magnitudes of the tangential and the normal inertial forces are

$$I_\tau = mw_\tau = m\frac{dv}{dt}, \quad I_n = mw_n = m\frac{v^2}{R+H}$$

where m is the mass of the satellite. From equations (20.4) we find

$$m\frac{dv}{dt} = 0, \quad \frac{mgR^2}{(R+H)^2} - \frac{mv^2}{R+H} = 0$$

whence

$$v = \text{const}, \quad v = \sqrt{gR}\sqrt{\frac{R}{R+H}} = 7910\sqrt{\frac{R}{R+H}} \text{ m/s}$$

Knowing the arc length of the circle of radius $R + H$ we obtain the formula expressing the period of rotation of the satellite round the Earth:

$$T = \frac{2\pi(R+H)}{v} = 2\pi\sqrt{\frac{R}{g}}\left(\frac{R+H}{R}\right)^{3/2} = 84.43\left(1+\frac{H}{R}\right)^{3/2} \text{ min}$$

The point A in Fig. 20.3 indicates the location of the launch pad and the point B shows where the satellite achieves its circular orbit.

As an exercise, we recommend the reader to come back to the problem of a simple pendulum (see Sec. 3.3 of Chap. 16) and using equations (20.4) derive the differential equation of the oscillation of the pendulum and the expression for the modulus of the dynamic reaction S of the thread.

1.2. D'Alembert's Principle for a System of Particles. Let us consider a system of n particles $M_1, M_2, \ldots, M_n$ which is subjected to bilateral geometric constraints (see Sec. 1.1 of Chap. 17); here we do not suppose that the constraints are stationary and ideal. The masses of the particles will be denoted by $m_1, m_2, \ldots, m_n$. The resultant of the given active forces (both external and internal) applied to the νth particle ($\nu = 1, 2, \ldots, n$) will be denoted by $\boldsymbol{F}_\nu$, and the resultant of the constraint reactions applied to the νth particle will be denoted by $\boldsymbol{N}_\nu$ (Fig. 20.4). On the basis of Newton's second law we can write the equation

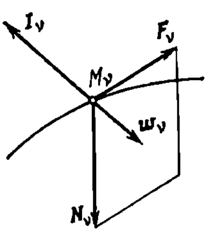

Fig. 20.4

$$\boldsymbol{F}_\nu + \boldsymbol{N}_\nu + (-m_\nu \boldsymbol{w}_\nu) = 0 \quad (\nu = 1, 2, \ldots, n) \tag{20.5}$$

for each of the particles of the system. The

vector

$$I_\nu = -m_\nu w_\nu \tag{20.6}$$

is called the inertial force of the νth particle and is thought of as being applied to that particle.

D'Alembert's principle for a system of particles reads: *if for each particle inertial forces* (20.6) *are added to all active forces and passive forces* (*the constraint reactions*) *applied to the particles of the system then a balanced force system is obtained* (see (20.5)).

As was indicated in Sec. 1.1, inertial forces (20.6) are fictitious. D'Alembert's principle can also be stated thus: if a moving system of particles is instantaneously stopped and to each particle of the system the active force F_ν, the passive force (the constraint reaction) N_ν and the fictitious inertial force I_ν acting on that particle at that instant are applied then the system remains in equilibrium. When saying that the system is instantaneously stopped we mean the following situation. Besides the given moving system imagine a replica of that system with the same constraints which is at rest. If we apply the same active forces and the same inertial forces to that imaginable system then this system will be in equilibrium, and the constraint reactions will be the same as in the given moving system.

D'Alembert's principle only formally reduces dynamic problems to problems of equilibrium of forces, that is to static problems. We have used the word "formally" in order to stress that equations of form (20.5) remain equations of motion, and, generally speaking, their complete solution requires integration.

However, in the general case when considering equations (20.5) as equilibrium equations for an arbitrary (spatial) force system applied to a rigid body we can use the *six equations of kinetostatics* which are:

(a) three equations for the sums of the projections of the external active forces, the constraint reactions and the fictitious inertial forces on the axes of an inertial coordinate system $Oxyz$:

$$\begin{aligned}
&\sum_{\nu=1}^{n} (X_\nu^{(\text{ext})} + N_x^\nu + I_x^\nu) = 0\\
&\sum_{\nu=1}^{n} (Y_\nu^{(\text{ext})} + N_y^\nu + I_y^\nu) = 0\\
&\sum_{\nu=1}^{n} (Z_\nu^{(\text{ext})} + N_z^\nu + I_z^\nu) = 0
\end{aligned} \tag{20.7}$$

(b) three equations for the sums of the moments of the external active forces, the constraint reactions and the fictitious inertial forces

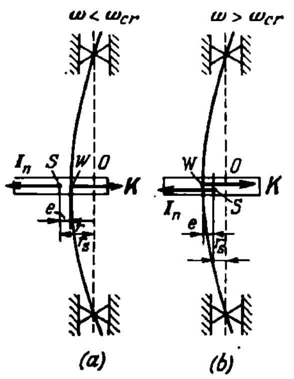

Fig. 20.5

about those three axes:

$$\sum_{\nu=1}^{n} (\mathrm{mom}_x F_\nu^{(ext)} + \mathrm{mom}_x N_\nu + \mathrm{mom}_x\ I_\nu) = 0$$

$$\sum_{\nu=1}^{n} (\mathrm{mom}_y F_\nu^{(ext)} + \mathrm{mom}_y N_\nu + \mathrm{mom}_y\ I_\nu) = 0$$

$$\sum_{\nu=1}^{n} (\mathrm{mom}_z F_\nu^{(ext)} + \mathrm{mom}_z N_\nu + \mathrm{mom}_z\ I_\nu) = 0 \qquad (20.8)$$

EXAMPLE 20.3. The simplest case of the determination of the critical angular velocity of the pulley on a flexible shaft.

Solution. Let us state the problem more precisely. We consider the system shown in Fig. 20.5. The vertical shaft is supposed to be rectilinear when it is not deformed; we also assume that it is untwistable and is elastic with respect to bending; it is also meant that the shaft can freely change its angle of inclination and freely move longitudinally in its bearings. At the midpoint of the shaft a cylindrical pulley is attached perpendicularly to the shaft; in the general case the point of attachment W of the pulley to the shaft does not coincide with the centre of gravity S of the pulley. The distance $e = WS$ is called the eccentricity of the pulley. The mass of the pulley is m. An engine imparts to the shaft rotation with constant angular velocity ω.

Let O be the point of intersection of the middle plane of the pulley with the rectilinear axis of the nondeformed shaft. We neglect the force of weight of the pulley and the friction forces in the bearings, and therefore the only active force K acting on the pulley is the elastic force of the shaft which is proportional to the deflection OW:

$$K = cOW$$

where the constant factor c depends on the material of the shaft, its cross-sectional area and the distance between the bearings. Since the shaft rotates with constant angular velocity ω, to every particle Δm of the pulley only the normal (centrifugal) inertial force is applied, its magnitude being $\Delta m \cdot r\omega^2$, where r is the distance from the particle to the axis of the nondeformed shaft. The modulus of the resultant I_n of the normal inertial forces (the resultant is applied at the centre of gravity S of the pulley) is equal to

$$I_n = mr_S\omega^2$$

where r_S is the distance from the centre of gravity S of the pulley to the axis of the nondeformed shaft.

According to D'Alembert's principle, the forces K and I_n are in balance at each instant during the motion of the system. This first of all means that the points O, W and S lie in one straight line; since there are two possible cases here (see Fig. 20.5 (a and b)), we have

$$OW = OS \mp WS = r_S \mp e$$

Besides, the equality $I_n = K$ must hold, that is

$$mr_S\omega^2 = c\,(r_S \mp e)$$

whence

$$r_S = \frac{\mp ce}{m\omega^2 - c} = \frac{\mp \omega_{cr}^2 e}{\omega^2 - \omega_{cr}^2} \quad \left(\omega_{cr} = \sqrt{\frac{c}{m}}\right)$$

Since the quantity r_S is positive, we must take the minus sign in the numerator when $\omega < \omega_{cr}$ (see Fig. 20.5*a*) and the plus sign when $\omega > \omega_{cr}$ (see Fig. 20.5*b*). The denominator vanishes for $\omega = \omega_{cr}$, which however, practically, does not lead to infinite deflections (because here we have made some simplifying assumptions); in reality this means that there can be strong perturbation when the shaft rotates with a velocity close to ω_{cr}: $\omega \approx \omega_{cr}$, which may lead to the fracture of the shaft. The angular velocity $\omega = \omega_{cr}$ is called *critical*. If the angular velocity ω is increased indefinitely then

$$r_S \to 0 \quad \text{for} \quad \omega \to \infty$$

This means that the centre of gravity S of the pulley tends to occupy the position O. This is known as self-centring of a flexible shaft; it was discovered by the Swedish inventor C. G. P. De Laval (1845-1913) in 1889.

D'Alembert's principle is particularly convenient for determining dynamic constraint reactions, that is reactions appearing in the motion of a system of particles. Below we demonstrate this by an example; later on (see § 2 of Chap. 22) we shall consider in greater detail the problem of determining dynamic reactions.

EXAMPLE 20.4. A chain is passed round a gear wheel of weight Q and radius R (Fig. 20.6). From the end of the chain a body A of weight P is suspended. The mass of the gear wheel is assumed to be distributed uniformly along its rim. Determine the acceleration w of the weight A, the tension T of the chain and the pressure on the bearing of the axis of the gear wheel.

Solution. Suppose that the weight A moves downward with an acceleration; then the inertial force of weight $I_A = -Pw/g$ is directed upward. The acceleration of any point of the gear wheel is the sum of the normal acceleration w_n and the tangential acceleration w_τ. Therefore applied to the particles Δm of the gear wheel are the normal inertial forces $I_n = -\Delta m w_n$ directed away from the axis Oz and the tangential inertial forces $I_\tau = -\Delta m w_\tau$ whose direction is opposite to that of w_τ. Let us apply to the point O the components R_x and R_y of the reaction of the bearing of the axis. Now we can write equations (20.7) and (20.8) for the system consisting of the gear wheel and the weight. The force system being plane, the last equation (20.7) and the first two equations (20.8) can be discarded; thus

$$\sum X = R_x = 0, \quad \sum Y = -P - Q + \frac{P}{g} w + R_y = 0$$

$$\sum \text{mom}_O = -PR + \frac{P}{g} wR + \sum \text{mom}_O I_\tau = 0 \tag{20.9}$$

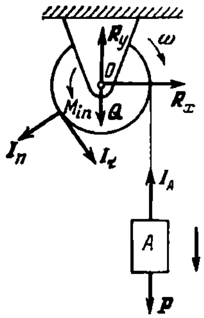

Fig. 20.6

Due to the symmetry, the vector sum of the inertial forces applied to the gear wheel is equal to zero. The normal inertial forces pass through the axis Oz and their moments about that axis are equal to zero. To compute the resultant moment of the tangential inertial forces we take into account the fact that

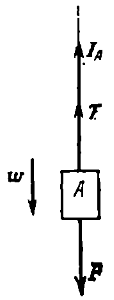

Fig. 20.7

since the mass $M = Q/g$ of the gear wheel is distributed along the rim we have

$$I_\tau = \Delta m w_\tau = \Delta m R\varepsilon, \quad \sum \mathrm{mom}_O\, I_\tau = \sum \Delta m R\varepsilon \cdot R = MR^2\varepsilon$$

On the other hand, $\varepsilon = w/R$ and therefore the moment of the inertial forces of the gear wheel about the point O (about the axis Oz) is

$$MR^2\varepsilon = \frac{Q}{g} Rw$$

The last equation (20.9) yields

$$w = \frac{P}{P+Q} g$$

and from the first two equations we find

$$R_x = 0, \quad R_y = P + Q - \frac{P}{g} w = P + Q - \frac{P^2}{P+Q} = \frac{(2P+Q)Q}{P+Q}$$

The pressure on the bearing of the axis is equal to $-R_y$.

In order to determine the tension of the chain we shall consider separately the motion of the weight (Fig. 20.7). Let us mentally cut the chain and replace its action by the tension T. Adding the fictitious inertial force $I_A = -Pw/g$ to the active force P and the passive force T we obtain, according to (20.1), the equation

$$-P + T + \frac{P}{g} w = 0$$

whence

$$T = P - \frac{P}{g} w = P - \frac{P^2}{P+Q} = \frac{PQ}{P+Q}$$

§ 2. Stability of Equilibrium and Small Oscillation

2.1. Statement of the Problem. We shall consider a system of n particles with masses $m_1, m_2, \ldots, m_n$ subjected to bilateral ideal geometric constraints (see Sec. 1.1 of Chap. 17) which do not depend explicitly on time. Let the state of the system be specified by generalized coordinates $q_1, q_2, \ldots, q_k$. We shall suppose that for the forces acting on the particles of the system there exists a force function $U(q_1, q_2, \ldots, q_k)$. The equilibrium position of such a system is determined by equations (17.15) which, by virtue of (18.12), take the form

$$\frac{\partial U}{\partial q_1} = 0, \quad \frac{\partial U}{\partial q_2} = 0, \ \ldots, \ \frac{\partial U}{\partial q_k} = 0 \tag{20.10}$$

Assuming that these equations determine an isolated equilibrium position we choose the generalized coordinates $q_1, q_2, \ldots, q_k$ in such a way that their values vanish for this equilibrium position. We shall also assume* that the kinetic energy T of the system is a

* Under the conditions stated this assumption can be proved.

homogeneous quadratic form with respect to the generalized velocities, that is

$$T = \frac{1}{2}\sum_{i,\, j=1}^{k} a_{ij}\dot{q}_i\dot{q}_j \quad (a_{ji} = a_{ij})$$

where the coefficients a_{ij} depend solely on the generalized coordinates:

$$a_{ij} = a_{ij}(q_1, q_2, \ldots, q_k) \quad (i, j = 1, 2, \ldots, k)$$

The *equilibrium position* $q_1 = q_2 = \ldots = q_k = 0$ of the system of particles is said to be *stable* if, given any small positive numbers α and α', there are two other positive numbers δ and δ' such that the inequalities

$$\sum_{\varkappa=1}^{k} q_{\varkappa 0}^2 \leqslant \delta, \qquad \sum_{\varkappa=1}^{k} \dot{q}_{\varkappa 0}^2 \leqslant \delta'$$

imply that the inequalities

$$\sum_{\varkappa=1}^{k} q_{\varkappa}^2 < \alpha, \qquad \sum_{\varkappa=1}^{k} \dot{q}_{\varkappa}^2 < \alpha'$$

hold for any instant $t \geqslant t_0$. Here $q_{\varkappa 0} = q_\varkappa(t_0)$ and $\dot{q}_{\varkappa 0} = \dot{q}_\varkappa(t_0)$ ($\varkappa = 1, 2, \ldots, k$) are the initial values of the generalized coordinates and velocities. In other words, the equilibrium position is stable if in the course of motion the values of the coordinates and of the velocities remain arbitrarily small provided that their initial values are sufficiently small.

2.2. Lagrange's Theorem on Stability of Equilibrium. This theorem states: *if the force function has an isolated maximum at the equilibrium position of the system of particles in question then this equilibrium position is stable.*

Proof. Since the force function is determined to within an additive constant we can assume that at the equilibrium position not only the kinetic energy of the system is equal to zero

$$T(0, 0, \ldots, 0;\ 0, 0, \ldots, 0) = 0$$

but also the function U vanishes:

$$U(0, 0, \ldots, 0) = 0$$

Since the maximum is isolated there is a neighbourhood

$$\sum_{\varkappa=1}^{k} q_\varkappa^2 \leqslant \lambda$$

of the equilibrium position within which the force function has no other stationary points than the equilibrium position itself. We shall consider the k-dimensional sphere of radius $\sqrt{\alpha}$ with centre

at the origin in the space $q_1, q_2, \ldots, q_k$:

$$\sum_{\varkappa=1}^{k} q_\varkappa^2 = \alpha \leqslant \lambda$$

Let us suppose that on this sphere the force function U satisfies the inequality

$$U \leqslant -\varepsilon$$

Also let the kinetic energy of the system satisfy the inequality

$$T \geqslant \varepsilon$$

on the k-dimensional sphere of radius $\sqrt{\alpha'}$ with centre at the origin in the space $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k$ when

$$\sum_{\varkappa=1}^{k} q_\varkappa^2 \leqslant \alpha, \qquad \sum_{\varkappa=1}^{k} \dot{q}_\varkappa^2 = \alpha' \tag{20.11}$$

The functions U and T being continuous, there are positive numbers δ and δ' such that in the domain

$$\sum_{\varkappa=1}^{k} q_\varkappa^2 \leqslant \delta, \qquad \sum_{\varkappa=1}^{k} \dot{q}_\varkappa^2 \leqslant \delta'$$

there hold the inequalities

$$U > -\frac{1}{2}\varepsilon, \qquad T < \frac{1}{2}\varepsilon$$

Now let us choose the initial values of the coordinates and of the velocities within this domain, that is

$$\sum_{\varkappa=1}^{k} q_{\varkappa 0}^2 \leqslant \delta, \qquad \sum_{\varkappa=1}^{k} \dot{q}_{\varkappa 0}^2 \leqslant \delta' \tag{20.12}$$

Then we obtain

$$U_0 > -\frac{1}{2}\varepsilon, \qquad T_0 < \frac{1}{2}\varepsilon$$

For the given conditions there holds energy integral (19.27b):

$$T - U = T_0 - U_0$$

Therefore during the whole time of motion the inequalities

$$T = U + T_0 - U_0 \leqslant T_0 - U_0 < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

are fulfilled. Since on sphere (20.11) the inequality $T \geqslant \varepsilon$ always holds the motion with initial conditions in domain (20.12) satisfies the inequalities

$$\sum_{\varkappa=1}^{k} q_\varkappa^2 < \alpha, \qquad \sum_{\varkappa=1}^{k} \dot{q}_\varkappa^2 < \alpha'$$

at any instant $t \geqslant t_0$; consequently, the equilibrium position is stable. The theorem is proved.

EXAMPLE 20.5. A homogeneous circular cylinder of radius r lies on a fixed circular cylinder of radius R, the axes of the cylinders being horizontal and mutually perpendicular at the initial instant (Fig. 20.8). Determine the conditions specifying the stability of the equilibrium position of that system.

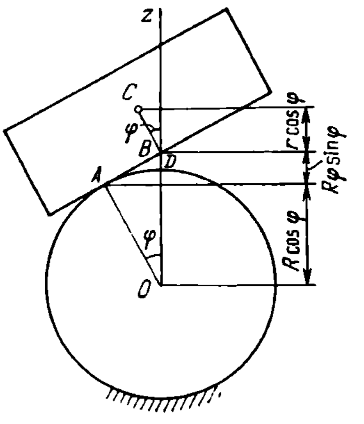

Fig. 20.8

Solution. Here the force function has the form

$$U = -mgz_C$$

(see (19.28)), where m is the mass of the upper cylinder and z_C is the z-coordinate of its centre of gravity C. Since $AB = \smile AD = R\varphi$ (see Fig. 20.8) we have

$$z_C = R\cos\varphi + R\varphi\sin\varphi + r\cos\varphi$$

where φ is the only generalized coordinate.

The equilibrium position is determined by single equation (20.10):

$$\frac{dU}{d\varphi} = -mg(R\varphi\cos\varphi - r\sin\varphi) = 0$$

Thus, for $\varphi = 0$ the system is in equilibrium. Let us compute the second derivative of the force function:

$$\frac{d^2U}{d\varphi^2} = -mg(R\cos\varphi - R\varphi\sin\varphi - r\cos\varphi)$$

Its value at the equilibrium position $\varphi = 0$ is

$$\left.\frac{d^2U}{d\varphi^2}\right|_{\varphi=0} = -mg(R-r)$$

If the condition

$$r < R$$

holds this value of the second derivative is negative, that is the force function has a maximum at the equilibrium position $\varphi = 0$. Hence, it is the last inequality that is sufficient for the stability of the equilibrium.

2.3. Small Free Oscillation of a Mechanical System with One Degree of Freedom about Its Stable Equilibrium Position. For a conservative system (see Sec. 3.4 of Chap. 19) with one degree of freedom we have

$$T = \frac{1}{2}a(q)\dot{q}^2, \quad U = U(q)$$

where q is the only generalized coordinate. Let us expand $a(q)$ and $U(q)$ into Maclaurin's series (see [1]) in the neighbourhood of the equilibrium position $q = 0$:

$$a(q) = a(0) + O(q)$$

$$U(q) = U(0) + \left(\frac{dU}{dq}\right)_0 q + \frac{1}{2}\left(\frac{d^2U}{dq^2}\right)_0 q^2 + O(q^3)$$

where $O(q)$ and $O(q^3)$ denote the terms of order of smallness not lower than the first and the third respectively. In the second expansion we can put $U(0) = 0$ (because the force function is determined to within an additive constant) and, by virtue of (20.10), the value of dU/dq is equal to zero for $q = 0$. We shall limit ourselves to small oscillation of the system and therefore we can assume that the quantities q and $\dot{q}$ remain so small that in the expansions of the kinetic energy and of the force function into power series it is possible to retain only the first terms. Then we can write

$$T = \frac{1}{2} a(0) \dot{q}^2, \quad U = -\frac{1}{2} cq^2 \quad \left(c = -\left(\frac{d^2U}{dq^2}\right)_{q=0}\right)$$

Let $c > 0$; this inequality means that the force function has a maximum at the equilibrium position and consequently, by the theorem proved in Sec. 2.2, this equilibrium position is stable.

In the case under consideration single Lagrange's equation (18.11a) takes the form

$$a(0)\ddot{q} = -cq$$

whence

$$\ddot{q} + \omega^2 q = 0 \quad \left(\omega = \sqrt{\frac{c}{a(0)}}\right) \tag{20.13}$$

We have thus arrived at the equation of harmonic oscillation of form (14.3) which was thoroughly investigated in § 1 of Chap. 14. It should be taken into account that equation (20.13) may describe oscillation of a complex construction about a stable equilibrium position provided that its number of degrees of freedom is equal to unity. If we wanted to take into account damping or some non-potential perturbing forces the situation would be analogous to the one considered in §§ 2 and 3 of Chap. 14. We shall demonstrate what has been said by an example.

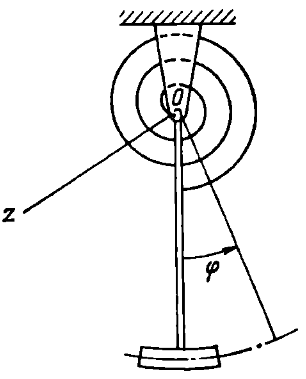

Fig. 20.9

EXAMPLE 20.6. The construction of a vibrograph meant for registering horizontal oscillation of the bases of machines (Fig. 20.9) involves a rod of length l m with a fixed axis of rotation Oz. Attached to the rod are a weight of mass m kg and a spiral spring with torsional stiffness factor γ N·m. Therefore when the spring is twisted by an angle φ there arises a restoring moment of magnitude $-\gamma\varphi$. When the rod is in the vertical position the spring is in the limp state. Determine the period of free oscillation of the vibrograph.

Solution. Let us consider the position of the vibrograph when the rod is given a deviation by an angle φ from the vertical; it is natural to take the angle φ as a generalized coordinate. The force function U is equal to the sum of two expressions: the force function $U_1 = mgl \times$

$\times(\cos\varphi - 1)$ (see Example 15.5) of the force of gravity and the force function $U_2 = -(1/2)\gamma\varphi^2$ of the elastic force of the spring twisted by the angle φ, that is

$$U = mgl(\cos\varphi - 1) - \frac{1}{2}\gamma\varphi^2 \tag{1}$$

The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\varphi}^2$$

Lagrange's equation (18.11a) takes the form

$$ml^2\ddot{\varphi} = -mgl\sin\varphi - \gamma\varphi$$

The last differential equation cannot be integrated in terms of elementary functions. However, limiting ourselves to the small oscillation and putting $\sin\varphi \approx \varphi$ we obtain, cancelling by ml^2, the equation

$$\ddot{\varphi} + \frac{\gamma + mgl}{ml^2}\varphi = 0 \tag{2}$$

Let us expand expression (1) into powers of φ:

$$U = -\frac{1}{2}(\gamma + mgl)\varphi^2 + O(\varphi^3)$$

This expansion shows that equation (2) can be derived directly from (20.13) because in the example under consideration we have

$$a(0) = ml^2, \qquad c = \gamma + mgl$$

The circular frequency ω of free oscillation is

$$\omega = \frac{1}{l}\sqrt{\frac{\gamma + mgl}{m}} \ 1/\mathrm{s}$$

and the period of oscillation is determined by formula (14.5):

$$T = 2\pi l\sqrt{\frac{m}{\gamma + mgl}} \ \mathrm{s}$$

If we wanted to take into account damping proportional to the angular velocity of the vibrograph it would be necessary to add a generalized force $\tilde{Q} = -\alpha\dot{\varphi}$ to Lagrange's equation (18.11a):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\varphi}}\right) - \frac{\partial T}{\partial\varphi} = \frac{\partial U}{\partial\varphi} + \tilde{Q}$$

This would lead to the equation

$$ml^2\ddot{\varphi} + \alpha\dot{\varphi} + (\gamma + mgl)\varphi = 0$$

which was considered in § 2 of Chap. 14.

Problems

PROBLEM 20.1. A body of weight G lies on the horizontal plane, the coefficient of friction between the body and the plane being f (Fig. 20.10). To the body a horizontal rope is attached which is passed round the pulley A and carries a weight P ($P > fG$) at its end. Find the acceleration w of the body

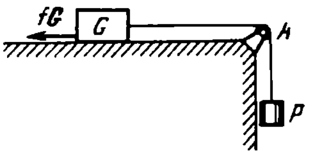

Fig. 20.10

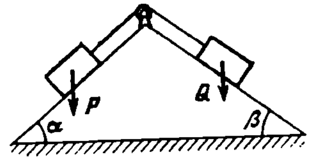

Fig. 20.11

and the tension T of the thread.

Answer. $w=\frac{P-fG}{P+G}g$, $T=\frac{(1+f)PG}{P+G}$.

PROBLEM 20.2. Two bodies of weights P and Q are connected by an inextensible thread and can slide without friction along two inclined planes forming angles α and β with the horizon (Fig. 20.11). Derive the condition under which the motion of the system is in the direction toward the weight P and determine the acceleration w of that motion.

Answer. $P>\frac{\sin\beta}{\sin\alpha}Q$, and the acceleration is $w=\frac{P\sin\alpha-Q\sin\beta}{P+Q}g$.

Chapter 21 Elements of Dynamics of a Rigid Body

§ 1. Moment of Inertia of a Rigid Body with Respect to an Axis

1.1. Definitions. The notion of the moment of inertia of a rigid body with respect to an axis invariably connected with that body plays an important role in dynamics of a rigid body. This notion provides an important characteristic of the configuration of the particles of the body with respect to the axis.

By the *moment of inertia of a particle with respect to an axis Oz* is meant the product of the mass m of the particle by the square of the distance h from the particle to that axis:

$$J_z = mh^2$$

The *moment of inertia of a rigid body with respect to an axis Oz* is equal to the sum of the products of the masses Δm of the elementary particles of the body by the squares of the distances h from the particles to that axis (Fig. 21.1):

$$J_z=\sum \Delta m\cdot h^2 \qquad (21.1)$$

where the summation extends over the whole volume of the body.

REMARK. Let $\beta = \gamma/g$ be the density of the body; then $\Delta m = \beta\,\Delta V$, where ΔV is the volume of an elementary particle. The

square of the distance from the particle to the axis Oz is equal to $h^2 = x^2 + y^2$, and formula (21.1) takes the form

$$J_z = \sum (x^2 + y^2)\,\Delta m = \sum \beta\,(x^2 + y^2)\,\Delta V \tag{21.1a}$$

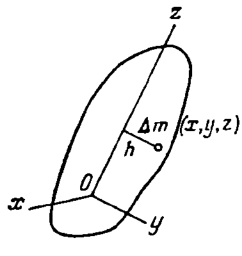

Fig. 21.1

If the mass is distributed continuously, by the moment of inertia is meant the limit of sum (21.1) or (21.1a) as the volume of each of the elementary particles tends to zero:

$$J_z = \lim_{\max \Delta V \to 0} \sum \beta h^2 \Delta V = \int_D \beta h^2\, dV \tag{21.2}$$

where D is the domain of integration (the volume of the body). In the Cartesian rectangular coordinates the last formula takes the form

$$J_z = \iiint_D \beta\,(x^2 + y^2)\,dx\,dy\,dz \tag{21.2a}$$

The dimension of the moment of inertia is equal to the dimension of mass multiplied by the square of unit length; in the International System of Units (SI) this unit is kg·m². The ratio J_z/M, where M is the mass of the body, has the dimension of the square of length. We denote this ratio by ρ^2, that is

$$\rho = \sqrt{\frac{J_z}{M}} \quad (J_z = M\rho^2) \tag{21.3}$$

The quantity ρ determined by formula (21.3) is called the *radius of inertia of the body with respect to the axis Oz.*

In other words, the radius of inertia ρ of a rigid body with respect to an axis Oz is the distance from that axis at which the total mass of the body can be placed without changing the moment of inertia. The introduction of the notion of the radius of inertia makes it possible to express the moment of inertia of a rigid body as the moment of inertia of a particle with mass equal to that of the body lying at the distance ρ from the axis Oz (see the second formula (21.3)).

In the general case, the moments of inertia of a body with respect to different axes assume different values. In particular, for parallel axes there exists a relationship between the moments of inertia with respect to these axes established by the Huygens-Steiner theorem.

1.2. The Huygens-Steiner Parallel Axis Theorem. *The moment of inertia of a body with respect to an axis Oz is equal to the sum of*

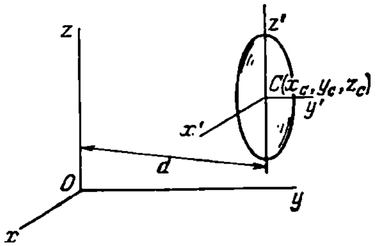

Fig. 21.2

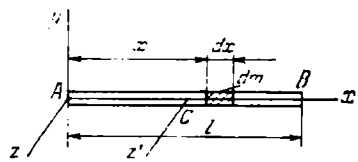

Fig. 21.3

the moment of inertia of that body with respect to the axis Cz' parallel to the given axis and passing through the centre of mass C of the body and the product of the mass M of the body by the square of the distance d between the axes (Fig. 21.2):

$$J_{Oz} = J_{Cz'} + Md^2 \tag{21.4}$$

Proof. Let us take a rectangular Cartesian coordinate system $Oxyz$ and another coordinate system $Cx'y'z'$ with origin at the centre of mass of the given body, the axes Cx' and Cy' being parallel to the axes Ox and Oy respectively. As usual, the coordinates of the centre of mass C in the coordinate system $Oxyz$ will be denoted by x_C, y_C and z_C; according to the condition of the theorem, we have

$$d^2 = x_C^2 + y_C^2$$

For any particle of the body there hold the equalities

$$x = x' + x_C, \quad y = y' + y_C$$

Therefore, by formula (21.1a),

$$J_z = \sum (x^2 + y^2)\,\Delta m = \sum [(x' + x_C)^2 + (y' + y_C)^2]\,\Delta m$$
$$= \sum (x'^2 + y'^2)\,\Delta m + (x_C^2 + y_C^2) \sum \Delta m + 2x_C \sum x'\,\Delta m + 2y_C \sum y'\,\Delta m$$

The last two sums on the right-hand side are equal to zero. Indeed, by formulas (19.2),

$$\sum x'\,\Delta m = Mx'_C = 0, \quad \sum y'\,\Delta m = My'_C = 0$$

because the coordinates of the centre of mass C of the body with respect to the coordinate system $Cx'y'z'$ are equal to zero. Finally, since

$$\sum (x'^2 + y'^2)\,\Delta m = J_{Cz'}, \quad \sum \Delta m = M, \quad x_C^2 + y_C^2 = d^2$$

we arrive at equality (21.4). The theorem is proved.

It follows from formula (21.4) that the moment of inertia of a body with respect to an axis passing through its centre of mass assumes the smallest value in comparison with all the moments of inertia with respect to the parallel axes.

1.3. Moments of Inertia of Simplest Bodies. In this section we shall compute the moments of inertia of some homogeneous bodies of simple form.

(1) Let us compute the *moment of inertia of a thin homogeneous rod of constant cross section with respect to an axis Az perpendicular to the rod and passing through its end A* (Fig. 21.3).

Let us denote by l the length of the rod, by M its mass and by $\varkappa = M/l$ the mass per unit length (the linear density). Then the mass of an element dx of the rod is $dm = \varkappa\, dx$ and hence

$$J_{Az} = \int_0^l x^2 \varkappa\, dx = \varkappa \frac{l^3}{3} = \varkappa l \frac{l^2}{3} = \frac{1}{3} Ml^2 \qquad (21.5)$$

(2) Let us determine the *moment of inertia of the same rod with respect to an axis Cz' perpendicular to the rod and passing through its midpoint* (see Fig. 21.3).

We shall apply the Huygens-Steiner theorem; to this end we put $d = l/2$ in formula (21.4):

$$\frac{1}{3} Ml^2 = J_{Cz'} + M\left(\frac{l}{2}\right)^2$$

It follows that

$$J_{Cz'} = \frac{1}{3} Ml^2 - \frac{1}{4} Ml^2 = \frac{1}{12} Ml^2 \qquad (21.6)$$

(3) Let us compute the *moment of inertia of a thin homogeneous tube of mass M and radius R with respect to its axis of symmetry Oz* (Fig. 21.4).

In Fig. 21.4 the cross section of the tube is shown. For any elementary particle of the tube the distance h from the particle to the axis Oz is equal to the radius R of the tube. Therefore

$$J_{Oz} = \sum \Delta m \cdot h^2 = R^2 \sum \Delta m = MR^2 \qquad (21.7)$$

(4) Let us determine the *moment of inertia of a continuous homogeneous solid right circular cylinder* (or *disk*) *of mass M and radius R with respect to its axis of symmetry* (Fig. 21.5).

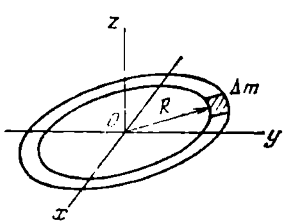

Fig. 21.4

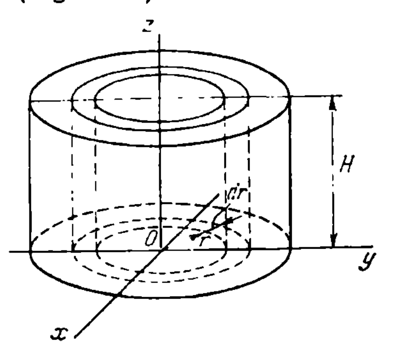

Fig. 21.5

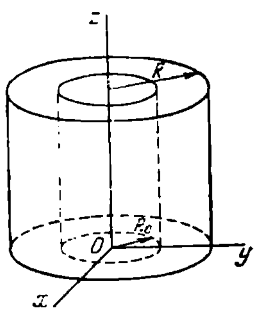

Fig. 21.6

We shall denote by H the altitude of the cylinder and by V its volume; then the density of the cylinder is

$$\beta = \frac{M}{V} = \frac{M}{\pi R^2 H}$$

Let us consider, within the cylinder, an elementary tube of radius r and width dr; for this tube

$$dV = 2\pi r H\, dr$$

and formula (21.2) implies that

$$J_{Oz} = \int_0^R \beta r^2 2\pi r H\, dr = 2\pi\beta H \frac{R^4}{4} = \frac{1}{2}\pi R^2 H \beta R^2$$

whence

$$J_{Oz} = \frac{1}{2} MR^2 \qquad (21.8)$$

The radius of inertia of the cylinder is determined by formula (21.3):

$$\rho = \sqrt{\frac{\frac{1}{2}MR^2}{M}} = \frac{\sqrt{2}}{2} R = 0.707R \qquad (21.9)$$

The comparison of formulas (21.7) and (21.9) shows that the moment of inertia of a cylinder with respect to its axis of symmetry is equal to the moment of inertia, with respect to the same axis, of a thin tube whose mass is equal to that of the cylinder and whose radius is $\sqrt{2}/2$ times as small as the radius of the cylinder.

(5) Now let us find the *moment of inertia of a homogeneous hollow cylinder with respect to its axis of symmetry Oz* (Fig. 21.6).

By analogy with (4),

$$J_z = \int_{R_0}^{R} \beta r^2 2\pi r H\, dr = \pi\beta H \frac{R^4 - R_0^4}{2} = \frac{1}{2}\pi\beta H (R^2 - R_0^2)(R^2 + R_0^2)$$

Finally

$$J_z = \frac{1}{2} M (R^2 + R_0^2) \qquad (21.10)$$

where $M = \pi (R^2 - R_0^2) H\beta$ is the mass of the hollow cylinder, R is its outer radius and R_0 is the inner radius.

We also presented without proof the formula for the *moment of inertia of a homogeneous ball about its diameter*:

$$J_x = J_y = J_z = \frac{2}{5} MR^2 \qquad (21.11)$$

and the formula for the *moment of inertia of a homogeneous right circular cone (not truncated) with respect to its axis of symmetry Oz*:

$$J_z = \frac{3}{10} MR^2 \qquad (21.12)$$

In these formulas M again denotes the mass of the body and R is the radius of the ball or of the base of the cone. It should be noted that the height of the body is not directly involved in formulas (21.7), (21.8), (21.10) and (21.12) (the mass M does of course depend on the height).

Now we pass to dynamics of a rigid body. In Chap. 8 we mentioned two simplest types of motion of a rigid body: translatory and rotational. The kinematic description of the translatory motion of a body reduces to the study of the motion of its arbitrary point, in particular, its centre of mass. According to the principle of motion of the centre of mass (see Sec. 1.3 of Chap. 19, formulas (19.9) and (19.13)), the dynamic study of the translatory motion of a body reduces to the corresponding problem of particle dynamics. Therefore only the second type of simplest motion of a rigid body, namely rotation about a fixed axis, must be studied independently; in the following sections we shall study this type of motion.

§ 2. Rotation of a Rigid Body about a Fixed Axis

2.1. Angular Momentum and Kinetic Energy of a Body in Rotational Motion. Let a perfectly rigid body rotate about a fixed axis Oz with angular velocity ω (in the general case ω may be variable) under the action of the given active external forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_m$ (Fig. 21.7). To begin with, we shall compute two quantities characterizing the rotational motion of the body: the angular momentum K_z about the axis Oz and the kinetic energy T.

For an elementary particle of the body of mass Δm lying at a distance h from the axis Oz we have

$$v = h\omega$$

The arm of the vector $\Delta m \cdot v$ with respect to the point O' is equal to h, and formula (19.16) yields

$$K_z = \sum \mathrm{mom}_z (\Delta m \cdot v) = \sum \Delta m \cdot h\omega h = \omega \sum \Delta m \cdot h^2$$

that is

$$K_z = J_z\omega \qquad (21.13)$$

Thus, *the angular momentum K_z (the moment of momentum) of a rigid body with respect to its axis of rotation Oz is equal to the product of the moment of inertia J_z of the body with respect to that axis by the algebraic value of the angular velocity ω.*

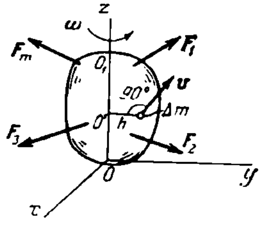

Fig. 21.7

The kinetic energy T of the rigid body in its rotational motion is determined by formula (19.24):

$$T=\frac{1}{2}\sum \Delta m\cdot v^2=\frac{1}{2}\sum \Delta m\,(\omega h)^2=\frac{1}{2}\,\omega^2\sum \Delta m\cdot h^2$$

Thus,

$$T=\frac{1}{2}J_z\omega^2 \tag{21.14}$$

The kinetic energy of a rigid body rotating about a fixed axis is equal to half the product of the moment of inertia of the body with respect to the axis of rotation by the square of the angular velocity.

REMARK. Formulas (21.13) and (21.14) also apply to the motion of a rigid body about a fixed point (see § 2 of Chap. 9). In this case instead of a fixed axis of rotation we shall mean an instantaneous axis of rotation passing through that fixed point O (the vector $\boldsymbol{\omega}$ of the instantaneous angular velocity is directed along that axis; see Sec. 2.1 of Chap. 9).

2.2. Differential Equation of Rotation of a Rigid Body about a Fixed Axis. For a body rotating about a fixed axis (see Fig. 21.7) the constraints can be realized by fixing two points O and O_1 of the body; for instance, there can be a bearing at O_1 and a thrust bearing at O. Since among the virtual displacements of the system there is a rotation about the fixed axis Oz (and there are no other displacements!) we can apply the principle of angular momentum for a system of particles about a fixed axis; here we use its coordinate form (19.19):

$$\frac{dK_z}{dt}=M_z^{(\mathrm{ext})}$$

Substituting $K_z=J_z\omega$ (see (21.13)) we obtain

$$J_z\frac{d\omega}{dt}=M_z^{(\mathrm{ext})} \tag{21.15}$$

Since the angular velocity of the body is $\omega=d\varphi/dt$, the angular acceleration ε of the body is expressed by

$$\varepsilon=\frac{d\omega}{dt}=\frac{d^2\varphi}{dt^2}$$

where φ is the angle of rotation of the body. The resultant moment of the given external active forces about the axis of rotation is equal to

$$M_z^{(\mathrm{ext})}=\mathrm{mom}_z\,\boldsymbol{F}_1+\mathrm{mom}_z\,\boldsymbol{F}_2+\ldots+\mathrm{mom}_z\,\boldsymbol{F}_m$$

Therefore the *differential equation of the rotational motion of a rigid body* (see (21.15)) can also be written in the form

$$J_z\varepsilon=M_z^{(\mathrm{ext})} \tag{21.15a}$$

and

$$J_z \frac{d^2\varphi}{dt^2} = \sum \mathrm{mom}_z \boldsymbol{F} \qquad (21.15b)$$

It should be noted that equation (21.15) does not involve the constraint reactions. By the way, since the reaction forces are applied at the points O and O_1 their moments about the axis Oz are equal to zero. In order to find the reaction forces one should discard the constraints and replace their action by the corresponding reactions; we shall come back to this question in § 2 of Chap. 22.

If the resultant moment $M_z^{(ext)}$ of the given external active forces about the axis of rotation is equal to zero ($M_z^{(ext)} \equiv 0$) the angular momentum K_z of the rigid body about that axis remains constant (see (19.21)). Indeed, equation (21.15) implies that if $M_z^{(ext)} \equiv 0$ then

$$J_z\omega(t) = J_z\omega(0), \quad \text{that is} \quad \omega(t) = \omega(0) = \text{const}$$

This is the case of the *inertial rotation* of the body. For instance, when a shaft of a machine is in a uniform rotation then the moments of the driving forces and of the resistance forces have equal magnitudes but opposite directions.

2.3. Compound Pendulum. A *compound pendulum* is a rigid body of arbitrary shape which can rotate about a fixed horizontal axis O, not passing through its centre of gravity, under the action of its own force of weight (Fig. 21.8). Let the line segment $OC = a$ be vertical when the compound pendulum is in equilibrium. Now we suppose that the pendulum is given a deviation from the equilibrium position; our aim is to investigate its motion neglecting the friction force in the axis of the suspension and the air resistance.

By Varignon's theorem (see Sec. 1.4 of Chap. 3, formula (3.5)), the sum of the moments of the given external active forces (that is the forces of gravity) is equal to the moment of the force of weight $\boldsymbol{P} = M\boldsymbol{g}$ of the pendulum which is applied at the centre of gravity C. The arm of this force relative to the point O is equal to $a \sin\varphi$. As usual, the positive direction of reckoning the angle φ is chosen to be counterclockwise. In the position shown in Fig. 21.8 the moment of the force of weight about the point O is negative and consequently

$$M_{Oz}^{(ext)} = -Pa\sin\varphi$$

For the compound pendulum differential equation of rotational motion (21.15b) is written in the form

$$J_{Oz}\frac{d^2\varphi}{dt^2} = -Pa\sin\varphi \qquad (21.16)$$

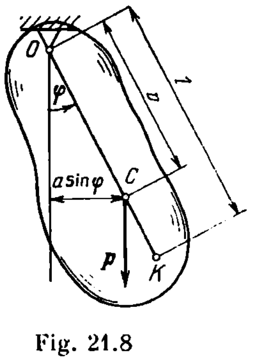

Fig. 21.8

where J_{Oz} is the moment of inertia of the compound pendulum about the axis of rotation. For small deviations we have $\sin\varphi \approx \varphi$, and dividing by J_{Oz} we bring the differential equation of small oscillation of the compound pendulum to the form

$$\ddot{\varphi}+\tilde{k}^2\varphi=0 \quad \left(\tilde{k}^2=\frac{Pa}{J_{Oz}}\right) \tag{21.17}$$

Let us compare equation (21.17) with the differential equation

$$\ddot{\varphi}+k^2\varphi=0 \quad \left(k^2=\frac{g}{l}\right)$$

describing small oscillation of a simple pendulum (see (16.18)). We see that these equations of harmonic oscillation (see Sec. 1.1 of Chap. 14) differ only in the coefficient in φ. This means that there exists a simple pendulum of length l which oscillates with the same period as the compound pendulum under consideration. In order to determine l we equate $\tilde{k}^2$ and k^2:

$$\frac{Mga}{J_{Oz}}=\frac{g}{l}$$

whence

$$l=\frac{J_{Oz}}{Ma} \tag{21.18}$$

The quantity $l = OK$ is called the *length of the equivalent compound pendulum*. The *period of oscillation of the compound pendulum* is determined by formula (14.5)

$$T=\frac{2\pi}{\tilde{k}}=2\pi\sqrt{\frac{J_{Oz}}{Mga}} \tag{21.19}$$

It can also be expressed in terms of the length of the equivalent compound pendulum:

$$T=2\pi\sqrt{\frac{l}{g}}$$

The *law of oscillatory motion* can be obtained from formula (16.19) in which k^2 should be replaced by Pa/J_{Oz}; the constants α and β are found from the initial conditions according to formulas (14.8). In particular, if

$$\varphi(0)=\varphi_0, \quad \dot{\varphi}(0)=0$$

then the oscillation law of the compound pendulum is

$$\varphi=\varphi_0\cos\sqrt{\frac{Mga}{J_{Oz}}}\,t \tag{21.20}$$

2.4. Work and Power of a System of Forces Applied to a Rigid Body Rotating about a Fixed Axis. Let us draw fixed axes Ox and

Oy perpendicular to the axis of rotation Oz (Fig. 21.9). Let there be a force $\boldsymbol{F} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ applied at a point A (x, y, z). The real displacement $d\boldsymbol{r}$ of the point A is equal to

$$d\boldsymbol{r} = \boldsymbol{v}\, dt$$

where the vector $\boldsymbol{v}$ (the velocity of the point A) is specified by formula (8.17): $\boldsymbol{v} = [\boldsymbol{\omega}, \boldsymbol{r}]$. Therefore

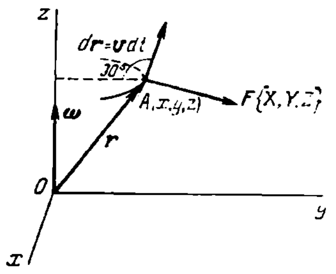

Fig. 21.9

$$d\boldsymbol{r} = [\boldsymbol{\omega}\, dt,\ \boldsymbol{r}] = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & d\varphi \\ x & y & z \end{vmatrix} = -y\, d\varphi\, \boldsymbol{i} + x\, d\varphi\, \boldsymbol{j}$$

The elementary work dA of the force $\boldsymbol{F}$ during the rotation of the body about the axis Oz through an angle $d\varphi$ is found from formula (15.14):

$$dA = (\boldsymbol{F},\ d\boldsymbol{r}) = (X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k})\,(-y\, d\varphi\, \boldsymbol{i} + x\, d\varphi\, \boldsymbol{j}) = (-Xy + Yx)\, d\varphi$$

The expression in the parentheses on the right-hand side is equal to $\mathrm{mom}_z \boldsymbol{F}$ (see (5.8)) and we have

$$dA = \mathrm{mom}_z \boldsymbol{F} \cdot d\varphi$$

For a system of forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_m$ applied to a rigid body (see Fig. 21.7) the elementary work corresponding to the rotation of the body about the fixed axis Oz through the angle $d\varphi$ is equal to

$$dA^{(\mathrm{ext})} = \sum_{\mu=1}^{m} \mathrm{mom}_z \boldsymbol{F}_\mu \cdot d\varphi = M_z^{(\mathrm{ext})}\, d\varphi \qquad (21.21)$$

Dividing by dt we obtain the formula *for the power*:

$$N = \frac{dA^{(\mathrm{ext})}}{dt} = M_z^{(\mathrm{ext})} \frac{d\varphi}{dt} = M_z^{(\mathrm{ext})} \omega \qquad (21.22)$$

Thus when a rigid body rotates about a fixed axis Oz the power is equal to the product of the resultant moment of the external forces about the axis Oz by the angular velocity (here for both the factors in the product their algebraic values are meant, that is their algebraic signs are taken into account).

Proceeding from (21.21), for the *work* of a force system (applied to a rigid body) corresponding to the rotation of the body through an angle $\varphi - \varphi_0$ we obtain the formula

$$A^{(\mathrm{ext})} = \int_{\varphi_0}^{\varphi} M_z^{(\mathrm{ext})}\, d\varphi \qquad (21.23)$$

In particular, if $M_z^{(ext)}$ is a constant quantity then

$$A^{(ext)} = M_z^{(ext)} (\varphi - \varphi_0) \tag{21.24}$$

Hence, for a constant resultant moment of the external forces about the axis of rotation the work is computed by multiplying that moment by the angle of rotation.

2.5. Principle of Energy in Rotational Motion. Since the constraints are stationary (the axis Oz is assumed to be fixed!) the real displacements belong to the class of virtual displacements; therefore the principle of energy (see Sec. 3.2 of Chap. 19) holds. According to formula (19.27a)

$$T - T_0 = A^{(ext)}$$

Now let us make use of formulas (21.14) and (21.23) and write the formula for the increment of the kinetic energy corresponding to the rotation of the rigid body about the fixed axis through an angle $\varphi - \varphi_0$

$$\frac{1}{2} J_z (\omega^2 - \omega_0^2) = \int_{\varphi_0}^{\varphi} M_z^{(ext)} d\varphi \tag{21.25}$$

In particular, if the resultant moment of the external forces about the axis Oz is constant we obtain from (21.25) the expression for the terminal value of the modulus of the angular velocity of the body:

$$|\omega| = \sqrt{\omega_0^2 + 2\frac{M_z^{(ext)}}{J_z}(\varphi - \varphi_0)}$$

EXAMPLE 21.1. A homogeneous disk of weight P and radius R (Fig. 21.10) rotates about the vertical axis Oz, the initial angular velocity being ω_0. During the rotation the brake block A is pressed to the disk with a radial force N, and t_1 seconds after the initial instant the disk stops rotating because of the friction. Determine the coefficient of friction f.

Solution. The radial force N has a zero moment about the axis Oz and therefore the resultant moment of the external forces about the axis Oz is equal to the moment of the friction force:

$$M_z^{(ext)} = -fNR \tag{1}$$

The resultant moment $M_z^{(ext)}$ being constant, equation (21.15a) implies that the angular acceleration ε is also constant, that is the rotation is uniformly decelerated. Therefore formula (8.8) yields

$$\omega(t_1) = 0 = \omega_0 + \varepsilon t_1, \text{ that is } \varepsilon = -\frac{\omega_0}{t_1}$$

Substituting this value into equation (21.15a) we obtain

$$-J_z\frac{\omega_0}{t_1} = -fNR \tag{2}$$

Finally, the application of formula (21.8) results in

$$f = \frac{J_z\omega_0}{NRt_1} = \frac{PR\omega_0}{2gNt_1}$$

REMARK. Equation (2) can also be derived without resorting to kinematic formula (8.8). Indeed, multiplying identity (21.15) by dt and integrating with

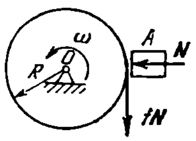

Fig. 21.10

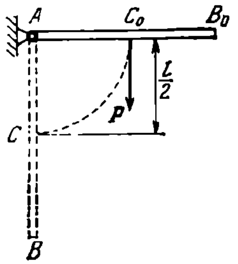

Fig. 21.11

respect to t from 0 to t_1 and with respect to ω from $\omega_0 = \omega(0)$ to $\omega_1 = \omega(t_1)$ we obtain

$$J_z\omega_1 - J_z\omega_0 = \int_0^{t_1} M_z^{(\text{ext})}\,dt \tag{3}$$

In the present example we have $\omega_1 = 0$, the quantity $M_z^{(\text{ext})}$ is expressed by formula (1), and equation (3) yields

$$-J_z\omega_0 = -fNRt_1$$

whence readily follows (2).

Generally, the principle of angular momentum about the axis of rotation and differential equation of rotational motion of a rigid body (21.15) resulting from that principle lead to a first integral provided that $M_z^{(\text{ext})}$ depends solely and explicitly on time or, in particular, is constant, which is the case in the example under consideration.

EXAMPLE 21.2. A horizontal axis about which a thin homogeneous rod of length l and weight P can rotate passes through the end A of the rod (Fig. 21.11). At the initial instant the rod occupies the horizontal position and then is let to move without initial velocity. Determine the velocity of the end B of the rod when it passes through the vertical position.

Solution. We can use the principles of angular momentum and of energy in the case of rotational motion. However the first of them does not yield a first integral (see the remark to the foregoing example) and therefore we shall use the second principle. Since $T_0 = 0$ we have

$$T = A^{(\text{ext})}$$

In order to compute the work of the force of weight P it is more convenient to use formula (15.26) instead of formula (21.25); according to the former formula

$$A^{(\text{ext})} = P\frac{l}{2}$$

Consequently

$$J_{Az}\frac{\omega^2}{2} = P\frac{l}{2}$$

Now, using formula (21.5), we find

$$|\omega| = \sqrt{\frac{Pl}{J_{Az}}} = \sqrt{\frac{Pl}{\frac{1}{3}\frac{P}{g}l^2}} = \sqrt{\frac{3g}{l}}$$

For the modulus of the velocity of the point B we obtain

$$v = l|\omega| = \sqrt{3gl}$$

§ 3. Plane Motion of a Rigid Body

3.1. Differential Equations of Plane Motion. As is known from kinematics (see Sec. 1.1 of Chap. 10), by a plane motion of a rigid body is meant one for which all the particles of the body move in planes parallel to a fixed plane. For a body in its plane motion let us draw through the centre of mass C a plane Oxy parallel to the fixed plane (Fig. 21.12). The study of the plane motion thus reduces to the investigation of the motion of a plane figure in its plane Oxy. Let the coordinate axes of the system $Cx'y'$ (König's axes: see Sec. 4.1 of Chap. 19) remain parallel to the corresponding axes of the coordinate system Oxy during the whole time of motion. Further, we take the coordinate system $Cx''y''$ with origin at the centre of mass which is rigidly connected with the plane figure. The axes Cx'' and Cx' form a variable angle φ. The derivative $d\varphi/dt$ is equal to ω, that is to the algebraic value of the angular velocity of the plane figure (see Sec. 1.1 of Chap. 10).

Suppose that the body is in a plane motion parallel to the xy-plane under the action of external forces (both the active forces and the constraint reactions) $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_m$. By virtue of formulas (19.32) and (21.13), the angular momentum $K'_{Cz'}$ of the rigid body about König's axis Cz' is equal to

$$K'_{Cz'} = J_{Cz'}\omega \tag{21.26}$$

Here $J_{Cz'}$ is the moment of inertia of the body with respect to the axis Cz' passing through the centre of mass of the body. In this case the conditions of the principle of angular momentum for the system in its relative motion hold (see Sec. 4.2 of Chap. 19) and therefore equations (19.36) and (19.35) take place:

$$M\frac{d^2x_C}{dt^2} = \sum_{\mu=1}^{m} X_\mu, \quad M\frac{d^2y_C}{dt^2} = \sum_{\mu=1}^{m} Y_\mu \tag{21.27}$$

and

$$J_{Cz'}\frac{d\omega}{dt} = \sum_{\mu=1}^{m} \mathrm{mom}_{Cz'}\boldsymbol{F}_\mu \tag{21.28}$$

where M is the mass of the body.

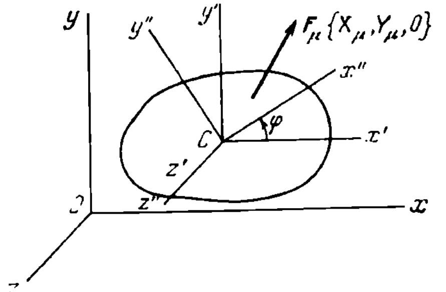

Fig. 21.12

Equations (21.27) and (21.28) are the *differential equations of the plane motion of a rigid body.* Equations (21.27) are the differen-

tial equations describing the motion of the centre of mass of the body, and equation (21.28) is the differential equation of the rotational motion of the body about the axis Cz' passing through the centre of mass of the body and perpendicular to the fixed plane. The resultant moment $M_{Cz'}^{(ext)}$ of the external active forces is also taken with respect to that axis.

EXAMPLE 21.3. Let us consider a cylinder sliding down an inclined plane. Shown in Fig. 21.13 is a homogeneous right circular cylinder of radius R and mass M moving down a fixed smooth inclined plane forming an angle α with the horizon. At the initial instant $t = 0$ the cylinder is at rest and its axis is horizontal. Determine the motion of the cylinder and the pressure it exerts on the inclined plane.

Solution. Let us replace the constraint imposed on the cylinder by the reaction force N. Differential equations (21.27) and (21.28) are written in the form

$$M\frac{d^2x_C}{dt^2} = Mg\sin\alpha, \quad M\frac{d^2y_C}{dt^2} = -Mg\cos\alpha + N, \quad J_{Cz'}\frac{d\omega}{dt} = 0$$

The constraint imposed on the cylinder indicates that $y_C = R$ during the whole time of motion, and the second equation yields

$$N = Mg\cos\alpha$$

This determines the magnitude of the pressure. The third equation shows that $\omega = \text{const}$, and since $\omega(0) = 0$ at the initial instant, we have $\omega = 0$ during the whole time of motion. This means that the cylinder slides down the smooth inclined plane so that the rotation imparted to it at the initial instant remains unchanged.

From the first equation we conclude that the acceleration of the rectilinear motion of the centre of mass is constant and is equal to $g\sin\alpha$. For the given initial conditions we have (see Example 13.3):

$$v_C = \frac{dx_C}{dt} = gt\sin\alpha$$

and

$$x_C = \frac{1}{2}gt^2\sin\alpha$$

EXAMPLE 21.4. Let us consider a cylinder rolling down a rough inclined plane. The conditions of this problem are the same as in the foregoing example but the plane is rough, the coefficient of sliding friction f of the cylinder on the plane being known (Fig. 21.14).

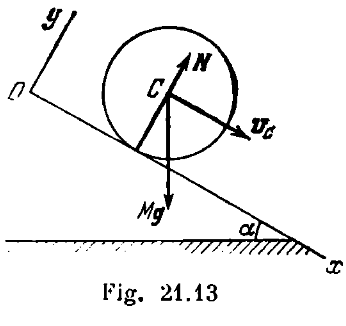

Fig. 21.13

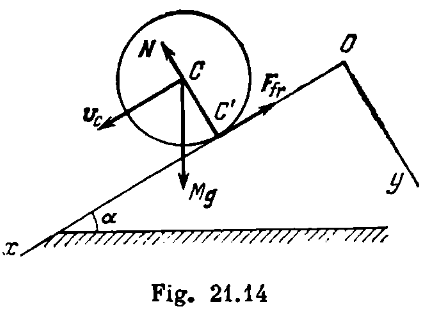

Fig. 21.14

Solution. Let us discard the constraint and replace its action on the cylinder by the normal reaction N and the friction force F_{fr}. The coordinate axes are chosen as shown in Fig. 21.14. Differential equations (21.27) and (21.28) are written in the form

$$M\frac{dv_x^C}{dt}=Mg\sin\alpha-F_{fr},\quad M\frac{d^2y_C}{dt^2}=Mg\cos\alpha-N,\quad \frac{1}{2}MR^2\frac{d\omega}{dt}=F_{fr}R \tag{1}$$

Since $y_C\equiv -R$ the second equation implies that $N=Mg\cos\alpha$.

(a) Let us begin with the case when the cylinder rolls down the plane without sliding. Then the velocity of the point C' of contact of the cylinder with the inclined plane is equal to zero. This point is the instantaneous velocity centre and we have (see (10.5))

$$v_x^C=R\omega \tag{2}$$

Here (as above) ω is understood as the algebraic value of the angular velocity of the cylinder. The first and the third differential equations (1) yield

$$MR\frac{d\omega}{dt}=Mg\sin\alpha-F_{fr},\quad MR\frac{d\omega}{dt}=2F_{fr}$$

The left-hand members of the equations being equal, the equality of their right-hand members implies that

$$F_{fr}=\frac{1}{3}Mg\sin\alpha$$

According to the Amonton-Coulomb law (see Sec. 1.1 of Chap. 4), in the absence of sliding we have $F_{fr}\leqslant fN$ and hence

$$\frac{1}{3}Mg\sin\alpha\leqslant fMg\cos\alpha,\quad \text{that is}\quad \tan\alpha\leqslant 3f$$

The last inequality expresses the condition under which the cylinder rolls down the rough plane without sliding. The substitution of (3) into the first equation (1) shows that the acceleration of the centre of gravity of the cylinder rolling without sliding is equal to

$$w_x^C=w_C=\frac{2}{3}g\sin\alpha$$

Consequently, for the zero initial conditions the law of motion of the centre of gravity is

$$x_C=\frac{1}{2}w_Ct^2=\frac{1}{3}gt^2\sin\alpha$$

The comparison of this result with that obtained in the foregoing example shows that during one and the same time t and for the same angle α the path travelled by the centre of gravity in rolling motion without sliding is 2/3 of the path travelled in sliding of the cylinder down the smooth inclined plane.

(b) Now let us suppose that $\tan\alpha>3f$. In this case the cylinder slides on the inclined plane during its rolling motion, equality (2) does not hold and

$$F_{fr}=fN=fMg\cos\alpha$$

Substituting this value into the first and the third equations (1) we obtain

$$\frac{d^2x_C}{dt^2}=a=g(\sin\alpha-f\cos\alpha),\quad \frac{1}{2}R\frac{d\omega}{dt}=fg\cos\alpha$$

The first of these equations expresses the acceleration of the centre of mass and specifies the law of motion of the centre of mass of the cylinder:

$$x_C=\frac{1}{2}at^2=\frac{1}{2}g(\sin\alpha-f\cos\alpha)t^2$$

From the second equation we find the angular acceleration of the rolling motion of the cylinder:

$$\varepsilon = \frac{d\omega}{dt} = \frac{2fg}{R} \cos \alpha$$

which implies the law of rotation of the cylinder about its axis (in this case the law of rotation does not depend on the motion of the centre of mass):

$$\varphi = \frac{1}{2} \varepsilon t^2 = \frac{fg}{R} t^2 \cos \alpha$$

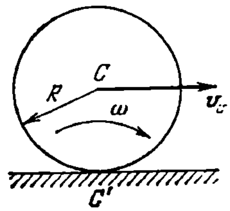

Fig. 21.15

3.2. Kinetic Energy of a Rigid Body in Plane Motion. On the basis of König's second formula (19.33) and formula (21.14), the kinetic energy of plane motion is equal to

$$T = \frac{1}{2} J_{Cz'} \omega^2 + \frac{1}{2} M v_C^2 \tag{21.29}$$

EXAMPLE 21.5. Compute the kinetic energy of a right circular cylinder in its rolling motion without sliding on condition that the velocity of its axis at the given instant is equal to v_C, the radius of the cylinder being R and the mass being M (Fig. 21.15).

Solution. From formulas (21.29) and (21.8) we have

$$T = \frac{1}{2} \frac{1}{2} MR^2\omega^2 + \frac{1}{2} Mv_C^2$$

The point C' is the instantaneous velocity centre (see Sec. 1.3 of Chap. 10) and therefore

$$|\omega| = \frac{v_C}{R}$$

Substituting this value we find

$$T = \frac{3}{4} Mv_C^2 \tag{21.30}$$

The same result is obtained if we consider the instantaneous motion of the cylinder which is the rotation about the axis $C'z$ passing through the instantaneous centre of zero velocity. By formula (21.14),

$$T = \frac{1}{2} J_{C'z}\omega^2$$

The moment of inertia $J_{C'z}$ is determined according to the Huygens-Steiner theorem (21.4):

$$J_{C'z} = J_{Cz} + MR^2 = \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2$$

Thus we again obtain (21.30):

$$T = \frac{1}{2} \frac{3}{2} MR^2 \frac{v_C^2}{R^2} = \frac{3}{4} Mv_C^2$$

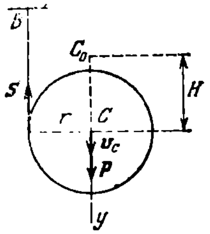

Fig. 21.16

EXAMPLE 21.6. A thread is wound around the middle of a right circular cylinder of weight P, the end B of the thread being fixed (Fig. 21.16). The cylinder

starts falling without initial velocity and the thread is unwound. Determine the velocity v_C and the acceleration w_C of the axis of the cylinder and also the tension of the thread at the instant when the axis of the cylinder is below its initial position at a distance H from it.

Solution. The kinetic energy of the cylinder is determined by formula (21.30):

$$T = \frac{3}{4} \frac{P}{g} v_C^2$$

Since the constraint imposed on the cylinder does not depend explicitly on time (it is stationary), the principle of energy in the absolute motion (see (19.27a)) applies. Here the only external active force is the force of weight of the cylinder whose work along the downward path $C_0C = H$ travelled by the centre of mass is equal to PH; therefore

$$T - T_0 = PH, \quad \text{that is} \quad \frac{3}{4} \frac{P}{g} v_C^2 = PH$$

because $T_0 = 0$, whence for the square of the velocity of the centre of mass of the cylinder at time t we obtain:

$$v_C^2 = \frac{4}{3} gH$$

Differentiating this identity with respect to t we find

$$2v_C \frac{dv_C}{dt} = \frac{4}{3} g \frac{dH}{dt}$$

Noting that $v_C = dH/dt$ we find from the last equality the constant acceleration of the centre of mass of the cylinder:

$$w_C = \frac{dv_C}{dt} = \frac{2}{3} g$$

Now let us determine the tension S of the thread; to this end we use the ordinary method of replacing the action of a constraint by its reaction force. Let us mentally cut the thread and add the reaction S to the external active force P; then the cylinder can be considered free. For that free cylinder among its virtual displacements there is a translatory displacement in the direction of the y-axis, and therefore the principle of motion of the centre of mass (see (19.9)) applies:

$$\frac{P}{g} \frac{dv_C}{dt} = P - S$$

From this equality it follows that

$$S = P - \frac{P}{g} \cdot \frac{2}{3} g = \frac{1}{3} P$$

and thus the tension of the thread is constant during the whole time of motion.

§ 4. Elementary Theory of Gyroscope

4.1. Precession of a Gyroscope. A *symmetrical gyroscope* is a homogeneous axially symmetric body which is in a fast rotation about a point on its axis of symmetry. We shall consider the motion of a gyroscope about a fixed point O on the axis (Fig. 21.17). Let ω_1

be the angular velocity vector of the gyroscope in its rotation about its own axis of symmetry, and let Ω be the instantaneous angular velocity vector of rotation about the instantaneous axis passing through the fixed point O (see Example 12.1). Then the vector

$$\omega_2 = \Omega - \omega_1$$

specifies the angular velocity of *precession*, that is of the rotation of the axis of the gyroscope about the vertical. The angular velocity of precession ω_2 is small in comparison with the angular velocity ω_1 of rotation of the gyroscope about its own axis of symmetry, and therefore the resultant vector of angular velocity

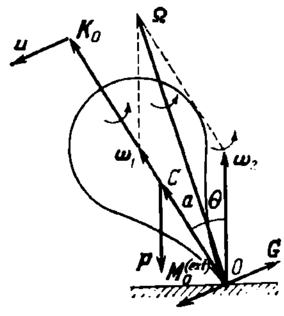

Fig. 21.17

$$\Omega = \omega_1 + \omega_2$$

has the modulus and the direction only slightly differing from those of the vector ω_1. Hence when computing the vector K_O representing the angular momentum of the gyroscope about the point O we can approximately write $\Omega \approx \omega_1$ and put

$$K_O \approx J\omega_1 \tag{21.31}$$

where J is the moment of inertia of the gyroscope with respect to its axis of symmetry. The quantity $J\omega_1$ is an approximate value of the angular momentum of the gyroscope.

4.2. Gyroscopic Moment. Foucault's Rule. Since the constraint imposed on the gyroscope (the condition that the point of support O is fixed) admits of rotation about any axis passing through the fixed point O, formula (19.22) implies that

$$\frac{dK_O}{dt} = M_O^{(ext)} = [a,\ P] \tag{21.32}$$

where a is the radius vector of the centre of gravity C of the gyroscope and P is its weight. The motion of the axis of the gyroscope can be specified by the motion of the point on that axis which, in the approximation we have assumed, coincides with the terminus of the vector K_O. Since the axis of the gyroscope rotates about the vertical with angular velocity ω_2 the velocity u of the terminus of the vector K_O is found with the aid of formula (9.13):

$$u = \frac{dK_O}{dt} = [\omega_2,\ K_O] \approx [\omega_2,\ J\omega_1] \tag{21.33}$$

Substituting this into (21.32) we find

$$[\omega_2,\ J\omega_1] = [a,\ P]$$

The last approximate equality can be written in the form

$$[J\boldsymbol{\omega}_1, \boldsymbol{\omega}_2] + [\boldsymbol{a}, \boldsymbol{P}] = 0, \quad \text{that is} \quad \boldsymbol{G} + \boldsymbol{M}_O^{(\mathrm{ext})} = 0 \qquad (21.34)$$

The vector

$$\boldsymbol{G} = [\boldsymbol{K}_O, \boldsymbol{\omega}_2] \approx [J\boldsymbol{\omega}_1, \boldsymbol{\omega}_2] \qquad (21.35)$$

which is equal to the vector product of the angular momentum of the gyroscope by the angular velocity of precession is called the *gyroscopic (restoring) moment*. As is seen from formula (21.35), the *gyroscopic moment is perpendicular to the plane containing the vectors* $\boldsymbol{\omega}_1$ *and* $\boldsymbol{\omega}_2$, *its direction being such that the couple of forces corresponding to it tends to make the angular velocity vector* $\boldsymbol{\omega}_1$ *(corresponding to the rotation of the gyroscope about its axis) coincident with the vector of angular velocity of precession* (this is known as *Foucault's* rule*). Identity (21.34) means that at each instant during the motion of the gyroscope the gyroscopic moment $\boldsymbol{G}$ balances the moment $\boldsymbol{M}_O^{(\mathrm{ext})}$ of the external forces (see Fig. 21.17). Formula (21.34) implies the equality of the moduli of both the vector products:

$$J\omega_1\omega_2 \sin\theta = aP \sin(180° - \theta)$$

It follows that

$$\omega_2 = \frac{Pa}{J\omega_1} \qquad (21.36)$$

The angular velocity of precession of the gyroscope is directly proportional to the product of its weight by the distance from its centre of gravity to the point of support and is inversely proportional to the angular momentum of the gyroscope.

4.3. Gyroscopic Effect. In engineering the action of a gyroscopic moment (the *gyroscopic effect*) appears in those cases when the axis of a rapidly rotating massive rotor is being turned. For instance, let a shaft O_1O_2 carrying a rotating rotor S be supported by two bearings O_1 and O_2 (Fig. 21.18). The rotor shown in the figure rotates so that the angular momentum vector $\boldsymbol{K}_O$ is directed from left to right. Now imagine that we try to turn the shaft O_1O_2 with the rotor in the plane of the figure in the clockwise direction. At first glance it seems that to perform the rotation of the shaft it is necessary to apply the vertical forces shown by dash line in Fig. 21.18. However in reality this is not the case. When the shaft is turned in the indicated way the terminus of the vector $\boldsymbol{K}_O$ gains a velocity directed downward in the plane of the figure. Hence according to formula (21.32), the vector representing the moment of the external forces must be directed downward. Conse-

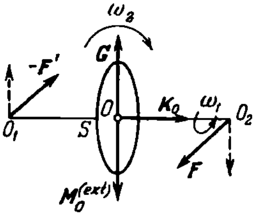

Fig. 21.18

* Foucault, J. B. L. (1819-1868), a distinguished French physicist.

quently, the forces F and $-F'$ must be perpendicular to the plane of the figure as is shown in Fig. 21.18.

On the other hand, the same result can be obtained if we construct the vector representing the gyroscopic moment (see (21.35)) which in the present case is directed upward in the plane of the figure. By formula (21.34), it follows that the vector representing the moment of the external forces must have the opposite direction, that is it must go downward. In order to determine the stresses in the bearings we make use of formula (21.34):

$$J\omega_1\omega_2 = F \cdot O_1O_2, \quad \text{that is} \quad F = \frac{J\omega_1\omega_2}{O_1O_2} \qquad (21.37)$$

EXAMPLE 21.7. The turbine of a ship is parallel to the axis of the hull of the ship, the mass of the rotor is 2500 kg, the radius of inertia of the rotor is $\rho = 0.9$ m, the angular velocity of rotation of the rotor is 1200 rpm and the distance between the bearings is $O_1O_2 = 1.9$ m (see Fig. 21.18). It is known that the greatest value of the angular velocity ω_2 of pitching motion is 0.11 rad/s. Determine the additional stresses appearing in the bearings in pitching motion.

Solution. According to formula (21.3),

$$J = M\rho^2 = 2500 \cdot 0.9^2 = 2025 \text{ kg} \cdot \text{m}^2$$

Formula (21.37) yields the following greatest value of the additional stresses in the bearings:

$$F = 2025 \frac{1200 \cdot 2\pi}{60} \cdot \frac{0.11}{1.9} = 14\,700 \text{ N} = 14.7 \text{ kN}$$

(the force F is expressed in kilonewtons; see Sec. 3.3 of Chap. 1).

The weight of the sea-going vessels is much greater than that of the rotating members of their engines and therefore the influence of the gyroscopic moments of the rotating members on the motion of the vessels can be neglected. On the contrary, for the propeller-driven aeroplanes the weight of the rotating members is a considerable part of the total weight. Therefore in general use are the constructions of two-motor aeroplanes whose propellers rotate in opposite directions in order to balance the gyroscopic moments.

Problems

PROBLEM 21.1. A flywheel of mass M makes N revolutions after the drive is switched off and stops rotating in t_1 seconds. Determine the moment M_{fr} of friction forces in the bearings assuming that it is constant and also find the initial angular velocity ω_0, the radius of inertia of the flywheel with respect to the axis of rotation being ρ.

Answer. $\omega_0 = \dfrac{4\pi N}{t_1}$, $M_{fr} = \dfrac{4\pi N M \rho^2}{t_1^2}$.

PROBLEM 21.2. Determine the equivalent length l and the period of oscillation T of a homogeneous rectilinear rod of length L suspended from a horizontal axis passing through its end.

Answer. $l = \dfrac{2}{3} L$, $T = 2\pi \sqrt{\dfrac{2L}{3g}}$.

PROBLEM 21.3. A homogeneous cylinder of radius R and weight P rolls without sliding under the action of a constant horizontal force F applied to the thread wound round the neck of the cylinder, the radius of the neck being r

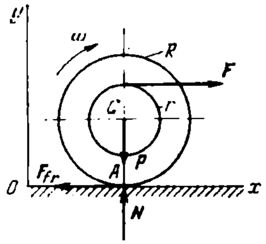

Fig. 21.19

(Fig. 21.19). Determine the acceleration of the centre of mass of the cylinder rolling on the horizontal plane on condition that the moment of inertia of the cylinder with respect to its axis is equal to J.

Hint. To the three differential equations of the plane motion of the cylinder add the corresponding kinematic equations. Namely, since $y_C = R$ during the whole time of motion of the cylinder we have $\ddot{y}_C \equiv \equiv 0$, and since the cylinder rolls without sliding the point A is the instantaneous velocity centre of the cylinder, that is $v_C = \dot{x}_C = R\omega = R\dot{\varphi}$, and consequently $\ddot{x}_C = R\ddot{\varphi}$.

Answer. $w_C = \ddot{x}_C = \dfrac{F(R+r)R}{Jg+PR^2}\,g.$

PROBLEM 21.4. To the axis of a wheel of radius R and weight P rolling on the horizontal rail (Fig. 21.20) a horizontal force F is applied. The radius of inertia of the wheel with respect to its axis is equal to ρ and the coefficient of sliding friction is equal to f. Under what condition imposed on the force F does the wheel roll without sliding?

Hint. See the hint to Problem 21.3. Since the wheel rolls without sliding the condition $F_{fr} \leqslant fN = fP$ must hold.

Answer. $F \leqslant f\,\dfrac{R^2+\rho^2}{\rho^2}\,P.$

PROBLEM 21.5. Solve the foregoing problem on condition that instead of the force F a rotational moment M_{rot} is applied to the wheel (Fig. 21.21).

Answer. $M_{rot} \leqslant f\,\dfrac{R^2+\rho^2}{\rho}\,P.$

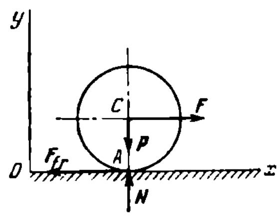

Fig. 21.20

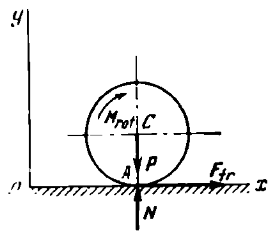

Fig. 21.21

Chapter 22 Dynamics of a Rigid Body (*Continued*)

§ 1. Moments and Products of Inertia

1.1. General Formula for the Moment of Inertia of a Rigid Body with Respect to an Arbitrary Axis. According to the Huygens-Steiner theorem (see Sec. 1.2 of Chap. 21) the moment of inertia of a body with respect to an arbitrary axis can be expressed in terms of the

moment of inertia of that body with respect to the parallel axis passing through its centre of mass. Therefore it is important to investigate the relationship between the moments of inertia with respect to the various axes passing through the centre of mass of a body. We shall extend the statement of the problem and consider an arbitrary axis Ol passing through a point O which must not necessarily coincide with the centre of mass of the body. Let us choose a rectangular Cartesian coordinate system $Oxyz$ with origin at the point O whose axes are rigidly connected with the body (Fig. 22.1). Let α, β and γ denote the angles between the axis Ol and the axes Ox, Oy and Oz respectively. We shall consider an elementary particle N of the body with mass Δm and coordinates x, y and z and denote by $h = NL$ the distance from that particle to the axis Ol. Let us compute the distance OL:

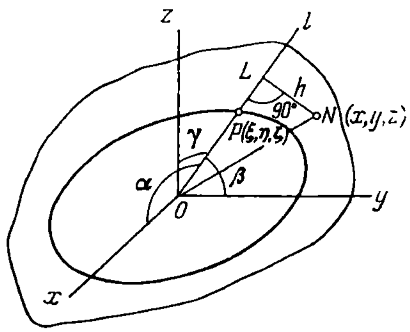

Fig. 22.1

$$OL = \text{proj}_l\, ON = (ON,\, \boldsymbol{l}) = (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})\,(\boldsymbol{i}\cos\alpha + \boldsymbol{j}\cos\beta + \boldsymbol{k}\cos\gamma) = x\cos\alpha + y\cos\beta + z\cos\gamma$$

Here $\boldsymbol{l}$ is the unit vector along the axis Ol, and $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the unit vectors along the axes Ox, Oy and Oz. Therefore taking into account that the sum of the squares of the direction cosines is equal to unity ($\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$) we obtain the following expression for h^2:

$$\begin{aligned} h^2 &= ON^2 - OL^2 = x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2 \\ &= (x^2 + y^2 + z^2)(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2 = (y^2 + z^2)\cos^2\alpha \\ &\quad + (z^2 + x^2)\cos^2\beta + (x^2 + y^2)\cos^2\gamma - 2yz\cos\beta\cos\gamma \\ &\quad - 2zx\cos\gamma\cos\alpha - 2xy\cos\alpha\cos\beta \end{aligned}$$

The moment of inertia J_l of the rigid body with respect to the axis Ol (see the definition in Sec. 1.1 of Chap. 21 and formula (21.1)) is equal to

$$\begin{aligned} J_l = \sum \Delta m \cdot h^2 &= \cos^2\alpha \sum (y^2 + z^2)\,\Delta m + \cos^2\beta \sum (z^2 + x^2)\,\Delta m \\ &\quad + \cos^2\gamma \sum (x^2 + y^2)\,\Delta m - 2\cos\beta\cos\gamma \sum yz\,\Delta m \\ &\quad - 2\cos\gamma\cos\alpha \sum zx\,\Delta m - 2\cos\alpha\cos\beta \sum xy\,\Delta m \end{aligned}$$

The third sum on the right-hand side is equal to J_z, that is to the moment of inertia of the body with respect to the axis Oz (see formula (21.1a)); similarly, the first two sums are equal to the moments of inertia J_x and J_y of the body with respect to the axes Ox and Oy respectively.

The *products of inertia* J_{yz}, J_{zx} and J_{xy} of the body are the quantities specified by the formulas

$$J_{yz}=\sum yz\,\Delta m,\quad J_{zx}=\sum zx\,\Delta m,\ J_{xy}=\sum xy\,\Delta m \qquad (22.1)$$

where x, y and z are the coordinates of an elementary particle of the body with mass Δm with respect to the axes Ox, Oy and Oz rigidly connected with the body, the summation extending over the whole body. The products of inertia (like the moments of inertia) depend only on the distribution of the mass within the body and also on the position of the axes; for the given rigid body with the coordinate axes fixed within it the products of inertia are constant quantities. In the general case the computation of the products of inertia (and of the moments of inertia) of a body reduces to the computation of triple integrals over the volume of the body.

Now we can write the formula for J_l in the form

$$J_l = J_x\cos^2\alpha + J_y\cos^2\beta + J_z\cos^2\gamma - 2J_{yz}\cos\beta\cos\gamma - 2J_{zx}\cos\gamma\cos\alpha - 2J_{xy}\cos\alpha\cos\beta \qquad (22.2)$$

1.2. Ellipsoid of Inertia. Let us take a point P on the axis Ol such that

$$OP=\frac{1}{\sqrt{J_l}}$$

For the coordinates ξ, η and ζ of the point P we have

$$\xi=OP\cos\alpha=\frac{\cos\alpha}{\sqrt{J_l}},\quad \eta=\frac{\cos\beta}{\sqrt{J_l}},\quad \zeta=\frac{\cos\gamma}{\sqrt{J_l}}$$

Substituting the expressions

$$\cos\alpha=\sqrt{J_l}\,\xi,\ \ \cos\beta=\sqrt{J_l}\,\eta,\quad \cos\gamma=\sqrt{J_l}\,\zeta$$

into (22.2) and cancelling by J_l we obtain the equation

$$J_x\xi^2 + J_y\eta^2 + J_z\zeta^2 - 2J_{yz}\eta\zeta - 2J_{zx}\zeta\xi - 2J_{xy}\xi\eta = 1 \qquad (22.3)$$

This is the equation of the locus of the points P whose distances from the origin are inversely proportional to the square roots of the moments of inertia with respect to the axis Ol. Since $J_l\neq\infty$ (because the body occupies a finite part of space) and $J_l\neq 0$ (because the particles of the body do not lie on one line) we have $OP\neq 0$ and $OP\neq\infty$. The only algebraic surface of the second order having no points at infinity is an ellipsoid. Therefore equation (22.3) describes an ellipsoid called the *ellipsoid of inertia of the body at the point O.*

The ellipsoid of inertia changes depending on the choice of the point O. The ellipsoid of inertia constructed for the centre of mass C of a body is called *central.*

If two products of inertia having one common subscript, say J_{yz} and J_{zx} (their common subscript is z), simultaneously vanish, that is

$$J_{yz} = J_{zx} = 0$$

then the axis Oz is called a *principal axis of inertia of the rigid body at the point O.*

A principal axis of inertia passing through the centre of mass C of the body is termed a *principal central axis of inertia of that body.*

If Cx, Cy and Cz are the principal central axes of inertia then

$$J_{yz} = J_{zx} = J_{xy} = 0$$

Equation (22.2) of the ellipsoid of inertia with respect to the principal central axes of inertia Cx, Cy and Cz takes the form

$$J_x\xi^2 + J_y\eta^2 + J_z\zeta^2 = 1 \tag{22.4}$$

1.3. Properties of the Ellipsoid of Inertia and of Principal Central Axes of Inertia.

(1) Suppose that we mentally stretch a body in a direction perpendicular to the axis Ol. Then h and, consequently, J_l will also increase, that is the point P will approach O. The ellipsoid of inertia at the point O will contract in the direction of that axis, and hence, in a certain sense, it characterizes the deformation of the body. For an absolutely rigid body it would be more precise to say that the ellipsoid of inertia characterizes the distribution "in the mean" of the mass of the body with respect to the point O.

(2) If two ellipsoids of inertia constructed for points O and O' have the axis OO' as a principal axis of inertia then this axis passes through the centre of mass C of the body (that is it is a principal central axis of inertia of the body) and is also a principal axis of inertia for any point belonging to it.

Proof. Let us take the axis OO' as the axis Oz (Fig. 22.2) and draw the axes Ox and Oy and the axes $O'x'$ and $O'y'$ parallel to them. Then $x = x'$, $y = y'$ and $z = z' + c$, where $c = OO'$. By the condition, we have $J_{yz} = J_{zx} = J_{y'z'} = J_{z'x'} = 0$ and consequently, according to formulas (19.2),

$$J_{yz} = \sum yz\,\Delta m = \sum y'(z'+c)\,\Delta m = J_{y'z'} + c\sum y'\,\Delta m = cMy'_C = 0$$

$$J_{zx} = \sum zx\,\Delta m = \sum (z'+c)\,x'\,\Delta m = J_{z'x'} + c\sum x'\,\Delta m = cMx'_C = 0$$

where $M = \sum \Delta m$. Therefore $x'_C = y'_C = 0$, that is the centre of mass of the body lies on the axis OO'. Let us take an arbitrary

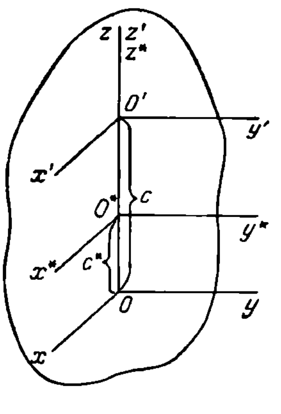

Fig. 22.2

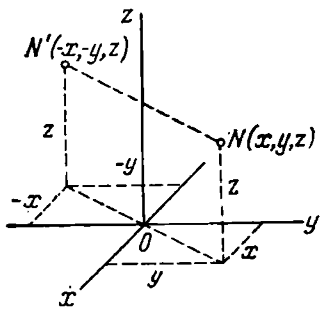

Fig. 22.3

point O^* on the axis OO' ($OO^* = c^*$) and compute $J_{y^*z^*}$ and $J_{z^*x^*}$:

$$J_{y^*z^*} = \sum y^*z^* \Delta m = \sum y(z - c^*) \Delta m = J_{yz} - c^* \sum y \Delta m = 0$$

$$J_{z^*x^*} = \sum z^*x^* \Delta m = \sum (z - c^*) x \Delta m = J_{zx} - c^* \sum x \Delta m = 0$$

This result shows that OO' is a principal axis of inertia for the point O^* as well. Property (2) is proved.

(3) If a homogeneous rigid body possesses an axis of symmetry then this axis is a principal central axis of inertia of that rigid body for any point lying on the axis.

Proof. The centre of mass C of the body always lies on its axis of symmetry (see Sec. 2.1 of Chap. 6) and therefore it only remains to prove that the axis of symmetry is a principal axis of inertia of the body for any point belonging to the axis. Let the axis Oz be an axis of symmetry (Fig. 22.3); then to every elementary particle N of mass Δm with coordinates x, y and z there corresponds a similar particle N' of the body with coordinates $-x$, $-y$ and z. Therefore

$$J_{yz} = \sum yz \Delta m = 0, \quad J_{zx} = \sum zx \Delta m = 0$$

which is what we had to prove.

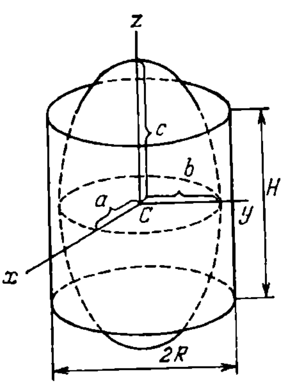

Fig. 22.4

EXAMPLE 22.1. Construct the central ellipsoid of inertia for a homogeneous circular cylinder of radius R, altitude H and density β (Fig. 22.4).

Solution. Let us place the origin at the centre of mass C of the cylinder and draw the axis Cz

along the axis of symmetry and the axes Cx and Cy in any way in the plane of symmetry. Since the axes Cx and Cy are also axes of symmetry Property (3) implies that

$$J_{yz} = J_{zx} = J_{xy} = 0$$

this means that the axes we have chosen are principal central axes of inertia. The moment of inertia of the cylinder with respect to the axis Cz is determined by formula (21.8):

$$J_z = \frac{1}{2} MR^2$$

where $M = \pi R^2 H\beta$ is the mass of the cylinder. The moment of inertia of the cylinder with respect to the axis Cx is found from a formula analogous to (21.2a):

$$J_x = \iiint\limits_D \beta (y^2 + z^2)\, dx\, dy\, dz = \beta \int\limits_{-H/2}^{H/2} dz \iint\limits_{D'} y^2\, dx\, dy + \beta \int\limits_{-H/2}^{H/2} z^2\, dz \iint\limits_{D'} dx\, dy$$

$$= \beta H \int\limits_0^{2\pi} \sin^2 \varphi\, d\varphi \int\limits_0^R r^3\, dr + \beta \frac{H^3}{12} \pi R^2 = \beta H \pi \frac{R^4}{4}$$

$$+ \frac{1}{12} H^2 \pi R^2 H \beta = \frac{1}{4} MR^2 + \frac{1}{12} MH^2$$

When computing the triple integral over the volume D of the cylinder we pass to polar coordinates in the domain D' (a circle of radius R with centre at the point C) lying in the plane Cxy (see [1]). It is evident that $J_y = J_x$. The equation of the central ellipsoid of inertia with respect to the principal axes $Cxyz$ is derived from (22.4):

$$\left(\frac{1}{4} MR^2 + \frac{1}{12} MH^2\right)(\xi^2 + \eta^2) + \frac{1}{2} MR^2 \zeta^2 = 1$$

This is an ellipsoid of revolution with semiaxes

$$a = b = \frac{2}{\sqrt{MR^2 + \frac{1}{3} MH^2}}, \quad c = \frac{2}{R\sqrt{2M}}$$

In Fig. 22.4 the ellipsoid is shown for the case $H = 2R$.

1.4. Computation of Products of Inertia. In addition to Sec. 1.3 we present below three more propositions.

1. If a body possesses a plane of symmetry then for each point in the plane of symmetry the straight line passing through that point perpendicularly to the plane is a principal axis of inertia.

Proof. Let the plane Oxy be a plane of symmetry of the body; then to each elementary particle Δm with coordinates x, y and z there corresponds a similar particle Δm with coordinates x, y and $-z$, and therefore

$$J_{yz} = \sum yz\, \Delta m = 0, \quad J_{zx} = \sum zx\, \Delta m = 0$$

which is what we had to prove.

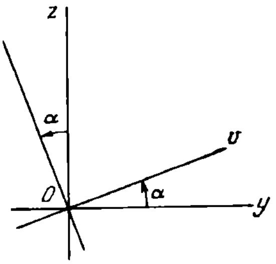

Fig. 22.5

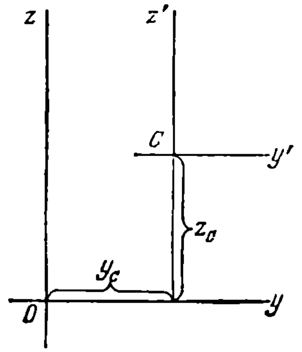

Fig. 22.6

2. Let us turn the axes Oy and Oz in the plane Oyz about the axis Ox through an angle α so that they become coincident with two axes Ov and Ow respectively which lie in the same plane Oyz and are mutually perpendicular (Fig. 22.5). If it turns out that at least one of the axes Ov or Ow is a principal axis of inertia of the body at the point O then the product of inertia J_{yz} can be computed using the formula

$$J_{yz} = \frac{1}{2}(J_w - J_v)\sin 2\alpha \tag{22.5}$$

Proof. According to the transformation formulas for the rotation of a rectangular Cartesian coordinate system (see [1]) we have

$$y = v\cos\alpha - w\sin\alpha, \quad z = v\sin\alpha + w\cos\alpha$$

Consequently

$$J_{yz} = \sum yz\,\Delta m = \sin\alpha\cos\alpha\left(\sum v^2\,\Delta m - \sum w^2\,\Delta m\right) + (\cos^2\alpha - \sin^2\alpha)\sum vw\,\Delta m$$

By the hypothesis the last sum on the right-hand side vanishes and we thus arrive at formula (22.5).

3. Let the centre of mass C of a body and coordinate axes $Cy'z'$ lie in the plane Oyz and be parallel to the corresponding axes Oyz (Fig. 22.6). Then the product of inertia J_{yz} of the body can be computed using the formula

$$J_{yz} = J_{y'z'} + My_Cz_C \tag{22.6}$$

where M is the mass of the body, and x_C, y_C and z_C are the coordinates of the centre of mass C of the body with respect to the coordinate system $Oxyz$.

Proof. It is evident that

$$y = y' + y_C, \quad z = z' + z_C$$

whence

$$J_{yz} = \sum yz\,\Delta m = \sum y'z'\,\Delta m + y_C z_C \sum \Delta m + z_C \sum y'\,\Delta m + y_C \sum z'\,\Delta m$$

The last two sums on the right-hand side are equal to My'_C and Mz'_C respectively (see formulas (19.2)), that is they vanish, and we thus obtain (22.6).

§ 2. Pressure Acting on the Axis of a Rotating Rigid Body

When a rotor of a machine is in its fast rotation about a fixed axis there arise inertial forces of great magnitude whose action on the supporting bearings leads to a considerable increase of the stresses in them. These undesirable phenomena disappear if the inertial forces of the rotating parts of the machine are balanced. For a designer of modern machines involving rapidly rotating members the problem of balancing the inertial forces is highly important.

The pressure exerted by a rotating body at the supporting points of the axis of rotation is equal in its modulus and opposite in the direction to the constraint reactions at those points. Therefore the problem of finding the pressure on the axis reduces to the determination of the constraint reactions. In order to find the reactions of the supports of a rotating rigid body let us use D'Alembert's principle; to this end we add the fictitious inertial forces to the real forces acting on the body.

2.1. Resultant Force Vector and Resultant Moment of Inertial Forces. Let a rigid body rotate about a fixed axis AB under the action of a given force system $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_m$. We shall take AB as the axis Az (Fig. 22.7); the axes Ax and Ay are meant to be rigidly connected with the body and thus rotate together with it.

Let us choose an arbitrary particle within the body, its mass being Δm and its distance from the axis Az being h. The coordinates of this particle relative to the axes $Axyz$ are

$$x = h\cos\alpha, \quad y = h\sin\alpha$$
$$z = \text{const}$$

Let us apply to this particle the normal and the tangential inertial forces (see Fig. 22.7) whose moduli are equal to

$$I_n = \Delta m \cdot w_n = h\omega^2\,\Delta m$$
$$I_\tau = \Delta m \cdot w_\tau = h\varepsilon\,\Delta m$$

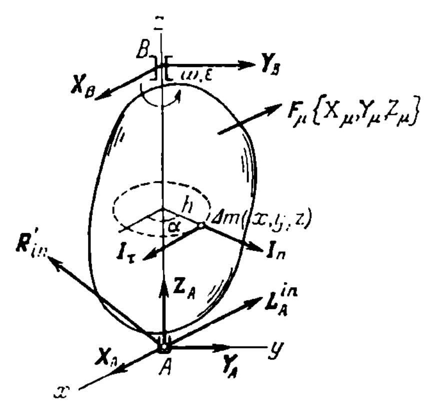

Fig. 22.7

respectively. Here ω and ε are the angular velocity and the angular acceleration of the body. Both the inertial forces are directed opposite to the corresponding accelerations. The normal inertial force is always directed away from the axis of rotation and therefore is sometimes called the centrifugal force (this inertial force must not be confused with the real force named centrifugal which is applied to the constraint). In Fig. 22.7 the rotating body is depicted under the assumption that it is in an accelerated rotation in the positive direction, that is $\omega > 0$ and $\varepsilon > 0$; therefore the tangential inertial force is directed along the tangent in the negative direction.

If all the inertial forces of all the particles of the body are reduced to an arbitrary centre, say to the point A (by analogy with what was done in § 2 of Chap. 5), then, in the general case, we obtain a resultant force vector applied at the reduction centre which is geometrically equal to the resultant vector of the inertial forces (we denote it by $\boldsymbol{R}'_{\text{in}}$) and a resultant couple of forces whose moment (we denote it by $\boldsymbol{L}_A^{\text{in}}$) is geometrically equal to the resultant moment of the inertial forces about the reduction centre:

$$\boldsymbol{R}' = \boldsymbol{R}'_{\text{in}} = \sum \boldsymbol{I}_\tau + \sum \boldsymbol{I}_n, \quad \boldsymbol{L}_A^{\text{in}} = \sum \mathbf{Mom}_A\, \boldsymbol{I}_\tau + \sum \mathbf{Mom}_A\, \boldsymbol{I}_n$$

Let us compute the projections of the resultant vector of the inertial forces on the axes Ax and Ay (see Fig. 22.7):

$$R'_x = \sum I_x^\tau + \sum I_x^n = \sum h\varepsilon\, \Delta m \sin\alpha + \sum h\omega^2\, \Delta m \cos\alpha$$
$$= \varepsilon \sum y\, \Delta m + \omega^2 \sum x\, \Delta m$$
$$R'_y = \sum I_y^\tau + \sum I_y^n = -\sum h\varepsilon\, \Delta m \cos\alpha + \sum h\omega^2\, \Delta m \sin\alpha$$
$$= -\varepsilon \sum x\, \Delta m + \omega^2 \sum y\, \Delta m$$

Using formulas (19.2) for the coordinates of the centre of mass of the body we obtain

$$R'_x = M\,(\varepsilon y_C + \omega^2 x_C), \quad R'_y = M\,(-\varepsilon x_C + \omega^2 y_C) \tag{22.7}$$

where M is the mass of the body. Since the inertial forces lie in the plane perpendicular to the axis of rotation we have

$$R'_z = 0$$

The modulus of the resultant vector of the inertial forces is found from (22.7):

$$R'_{\text{in}} = \sqrt{(R'_x)^2 + (R'_y)^2} = M\sqrt{(x_C^2 + y_C^2)(\omega^4 + \varepsilon^2)}$$

The resultant moment of the inertial forces is found using formulas (5.12) and (5.2):

$$\boldsymbol{L}_A^{\text{in}} = \sum \Delta m \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ \varepsilon y + \omega^2 x & -\varepsilon x + \omega^2 y & 0 \end{vmatrix}$$

It follows that the moments of the inertial forces about the axes Ax, Ay and Az (that is the projections of the resultant moment of the inertial forces on these axes) are expressed by the formulas

$$L_{Ax}^{\text{in}} = \varepsilon \sum zx\,\Delta m - \omega^2 \sum yz\,\Delta m$$

$$L_{Ay}^{\text{in}} = \varepsilon \sum yz\,\Delta m + \omega^2 \sum zx\,\Delta m$$

$$L_{Az}^{\text{in}} = -\varepsilon \sum (x^2 + y^2)\,\Delta m$$

Here the first four sums are the products of inertia (see (22.1)) and the last sum is the moment of inertia of the rigid body with respect to the axis Az (see (21.1a)), and we write

$$L_{Ax}^{\text{in}} = J_{zx}\varepsilon - J_{yz}\omega^2, \quad L_{Ay}^{\text{in}} = J_{yz}\varepsilon + J_{zx}\omega^2, \quad L_{Az}^{\text{in}} = -J_{Az}\varepsilon \tag{22.8}$$

2.2. Dynamic Reactions. Now let us set equations (20.7) and (20.8) which are often called the equations of kinetostatics. Suppose that the support at the point A is a thrust bearing and that at the point B there is a cylindrical bearing (in this case the system is statically determinate; see Sec. 3.3 of Chap. 5). We shall denote the components of the reactions by X_A, Y_A, Z_A, X_B and Y_B, and the distance AB by H. Then

$$\text{mom}_{Ax}\, Y_B = -Y_B H, \quad \text{mom}_{Ay}\, X_B = X_B H$$

All the other moments of the components of the reactions about the coordinate axes are equal to zero. Taking into account this fact and also formulas (22.7) and (22.8) we derive from (20.7) and (20.8) the equations

$$\sum_{\mu=1}^{m} X_\mu + X_A + X_B + M\varepsilon y_C + M\omega^2 x_C = 0 \tag{22.9}$$

$$\sum_{\mu=1}^{m} Y_\mu + Y_A + Y_B - M\varepsilon x_C + M\omega^2 y_C = 0 \tag{22.10}$$

$$\sum_{\mu=1}^{m} Z_\mu + Z_A = 0 \tag{22.11}$$

$$\sum_{\mu=1}^{m} \text{mom}_{Ax}\, \boldsymbol{F}_\mu - Y_B H + J_{zx}\varepsilon - J_{yz}\omega^2 = 0 \tag{22.12}$$

$$\sum_{\mu=1}^{m} \text{mom}_{Ay}\, \boldsymbol{F}_\mu + X_B H + J_{yz}\varepsilon + J_{zx}\omega^2 = 0 \tag{22.13}$$

$$\sum_{\mu=1}^{m} \text{mom}_{Az}\, \boldsymbol{F}_\mu - J_{Az}\varepsilon = 0 \tag{22.14}$$

The last of these equations does not involve constraint reactions and is a differential equation of form (21.15) describing the rotation

of a rigid body about the fixed axis. The first five equations serve for determining the five projections X_A, Y_A, Z_A, X_B and Y_B of the support reactions in rotation.

If the rigid body did not rotate ($\omega = \varepsilon \equiv 0$) then the last two terms in equations (22.9), (22.10), (22.12) and (22.13) would vanish and we would obtain the equilibrium equations of statics (see (5.40)). From these equations it would be possible to determine the projections of the support reactions at rest (of the *static reactions*) X_A^{stat}, Y_A^{stat}, Z_A^{stat}, X_B^{stat} and Y_B^{stat}. In this case equation (22.14) would express the equilibrium condition for the rigid body. By a *dynamic* (additional) *reaction* of a support is meant the difference between the reaction of that support in a rotational motion and the reaction at rest (the static reaction):

$$\boldsymbol{R}_A^{\text{dyn}} = \boldsymbol{R}_A - \boldsymbol{R}_A^{\text{stat}}, \quad \boldsymbol{R}_B^{\text{dyn}} = \boldsymbol{R}_B - \boldsymbol{R}_B^{\text{stat}}$$

For the corresponding projections this yields

$$X_A^{\text{dyn}} = X_A - X_A^{\text{stat}}, \quad Y_A^{\text{dyn}} = Y_A - Y_A^{\text{stat}}, \quad Z_A^{\text{dyn}} = Z_A - Z_A^{\text{stat}}$$
$$X_B^{\text{dyn}} = X_B - X_B^{\text{stat}}, \quad Y_B^{\text{dyn}} = Y_B - Y_B^{\text{stat}}$$

From equations (22.9)-(22.13) and (5.40) we obtain the equations for determining the projections of the dynamic reactions of the supports:

$$X_A^{\text{dyn}} + X_B^{\text{dyn}} = -M\varepsilon y_C - M\omega^2 x_C$$
$$Y_A^{\text{dyn}} + Y_B^{\text{dyn}} = M\varepsilon x_C - M\omega^2 y_C, \quad Z_A^{\text{dyn}} = 0$$
$$HY_B^{\text{dyn}} = J_{zx}\varepsilon - J_{yz}\omega^2, \quad HX_B^{\text{dyn}} = -J_{yz}\varepsilon - J_{zx}\omega^2$$

Solving these equations we derive the formulas for the projections of the dynamic reactions of the supports:

$$\begin{aligned} X_A^{\text{dyn}} &= -M\varepsilon y_C - M\omega^2 x_C + \frac{1}{H}(J_{yz}\varepsilon + J_{zx}\omega^2) \\ Y_A^{\text{dyn}} &= M\varepsilon x_C - M\omega^2 y_C + \frac{1}{H}(-J_{zx}\varepsilon + J_{yz}\omega^2), \quad Z_A^{\text{dyn}} = 0 \qquad (22.15) \\ X_B^{\text{dyn}} &= -\frac{1}{H}(J_{yz}\varepsilon + J_{zx}\omega^2), \quad Y_B^{\text{dyn}} = \frac{1}{H}(J_{zx}\varepsilon - J_{yz}\omega^2) \end{aligned}$$

2.3. Balancing Inertial Forces. There arises the problem of deriving conditions under which the reactions arising in rotation do not differ from the static reactions or, in other words, conditions under which the dynamic reactions are equal to zero.

From the last two equations (22.15) it follows that for the reaction $\boldsymbol{R}_B^{\text{dyn}}$ to vanish it is necessary and sufficient that the equalities $J_{yz} = J_{zx} = 0$ should hold; in other words, the axis Az must be a principal axis of inertia at the point A (see the definition in

Sec. 1.2). If this condition is fulfilled the first two equations (22.15) take the form

$$X_A^{\text{dyn}} = -M\varepsilon y_C - M\omega^2 x_C, \quad Y_A^{\text{dyn}} = M\varepsilon x_C - M\omega^2 y_C$$

When ω and ε are not identically equal to zero, for the reaction R_A^{dyn} to vanish it is necessary and sufficient that the relations $x_C = y_C = 0$ should hold, that is the centre of mass C of the body must lie on the axis of rotation. This means that the axis of rotation Az must coincide with one of the principal central axes of inertia of the rigid body (see the end of Sec. 1.2).

CONCLUSION. *The rotation of a rigid body does not produce additional pressure on the axis* (additional to the static reactions) *if and only if the fixed axis of rotation is one of the principal central axes of inertia of the body.* In other words, for the inertial forces of a rotating rigid body to balance it is necessary and sufficient that the axis of rotation should coincide with one of the principal central axes of inertia of the body.

As was established earlier, a rigid body can be in motion without external active forces and without constraints

(a) when it is in a uniform rectilinear translatory motion (see Sec. 1.4 of Chap. 19).

Now we can add the following two cases:

(b) when it is in a uniform rotation about any of its principal central axes of inertia;

(c) when it is in a compound motion consisting of any combination of motions (a) and (b).

EXAMPLE 22.2. A thin homogeneous rod of weight P and length l rotates with constant angular velocity ω about the vertical axis AB passing through the centre of mass C of the rod (Fig. 22.8). The axis of the rod forms a constant angle α with the axis AB. Determine the dynamic reactions of the supports.

Solution. We choose the axes $Axyz$ as shown in Fig. 22.8. Let us compute the products of inertia J_{zx} and J_{yz}. The rod lies in the plane Ayz and therefore

$$J_{zx} = \sum zx \, \Delta m = 0$$

because $x = 0$ for each particle of the rod. Let us draw the axis Cy' parallel to Ay through the centre of mass of the rod, and let the axis Cv go along the rod and the axis Cw be perpendicular to Cv. Then, by formula (22.5), we obtain

$$J_{y'z} = \frac{1}{2}(J_w - J_v) \sin [2(90° - \alpha)]$$

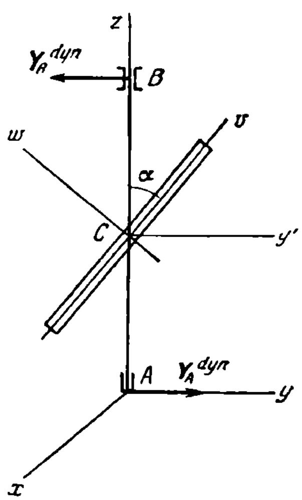

Fig. 22.8

The moment of inertia of the rod with respect to the axis Cv is equal to zero (because the rod is supposed to be sufficiently thin):

$$J_v = 0$$

Formula (21.6) yields

$$J_w = \frac{1}{12}\frac{P}{g} l^2$$

From formula (22.6) we find

$$J_{yz} = J_{y'z} = \frac{1}{24}\frac{P}{g} l^2 \sin 2\alpha$$

because $y_C = 0$. Now formulas (22.15) yield the following expressions for the projections of the dynamic reactions of the supports:

$$X_A^{\text{dyn}} = X_B^{\text{dyn}} = 0, \qquad Y_A^{\text{dyn}} = -Y_B^{\text{dyn}} = \frac{1}{24}\frac{P}{gH} l^2\omega^2 \sin 2\alpha \tag{1}$$

As is seen from Fig. 22.7, the dynamic reactions of the supports form a couple of forces whose (vector) moment has the magnitude

$$M_{\text{dyn}} = Y_A^{\text{dyn}} H = \frac{1}{24}\frac{P}{g} l^2\omega^2 \sin 2\alpha$$

and goes along the positive direction of the x-axis.

The same result can also be obtained without using formulas (22.15). Indeed, according to formulas (22.7), the resultant vector of the inertial forces is $\boldsymbol{R}' = 0$, and formulas (22.8) imply that the projections of the resultant moment of the inertial forces are

$$L_{Ax}^{\text{in}} = -J_{yz}\omega^2 = -\frac{1}{24}\frac{P}{g} l^2\omega^2 \sin 2\alpha, \qquad L_{Ay}^{\text{in}} = L_{Az}^{\text{in}} = 0$$

which means that the inertial forces of the rotating rod are balanced by the dynamic reactions of supports (1).

§ 3. Application of Lagrange's Equations to Dynamics of a Rigid Body (Examples)

In the present section we shall demonstrate by some examples the application of Lagrange's equations to dynamic problems concerning the motion of a rigid body.

EXAMPLE 22.3. There are two mutually perpendicular shafts with intersecting axes (Fig. 22.9); rotation is transmitted from one of them to the other with the aid of two bevel pinions having z_1 and z_2 teeth respectively. The moments of inertia of the shafts with the pinions put on them are J_1 and J_2 respectively. Shaft I is under the action of a torque M_1 while shaft II is subjected to the action of a resistance moment $-M_2$. Determine the angular acceleration of shaft I.

Solution. The system under consideration possesses one degree of freedom. Let us take the angle of rotation φ of the first pinion as an independent coordinate. The kinetic energy of the system is equal to

$$T = T_1 + T_2 = \frac{1}{2} J_1\omega_1^2 + \frac{1}{2} J_2\omega_2^2$$

The angular velocities ω_1 and ω_2 of the pinions are connected by the relation (see Example 8.2)

$$\omega_2 = k\omega_1$$

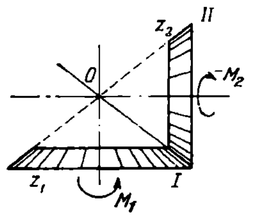

Fig. 22.9

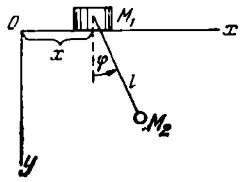

Fig. 22.10

where $k = z_1/z_2$ is the gear ratio. Therefore we finally obtain

$$T=\frac{1}{2}(J_1+k^2J_2)\,\omega_1^2=\frac{1}{2}(J_1+k^2J_2)\,\dot{\varphi}^2$$

Let us determine the generalized force Q_φ; to this end we give to the system a virtual displacement: let the first pinion turn through an angle $\delta\varphi$. Then the other pinion turns through the angle $\delta\varphi_2 = k\,\delta\varphi$. Now we use formula (21.24) to compute the elementary work (of the moments M_1 and M_2) corresponding to this virtual displacement:

$$\delta A = Q_\varphi\,\delta\varphi = M_1\,\delta\varphi - M_2\,\delta\varphi_2 = (M_1 - kM_2)\,\delta\varphi$$

From this relation we find

$$Q_\varphi=\frac{\delta A}{\delta\varphi}=M_1-kM_2$$

We see that the generalized force is the moment reduced to the first axis. Le us write Lagrange's equation (18.11) for the single independent coordinate φ

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\varphi}}\right)-\frac{\partial T}{\partial\varphi}=Q_\varphi$$

Substituting T and Q_φ into this equation we obtain

$$(J_1+k^2J_2)\,\ddot{\varphi}=M_1-kM_2$$

From the last equation we find the angular acceleration $\varepsilon \equiv \ddot{\varphi}$ of the first pinion:

$$\varepsilon=\frac{M_1-kM_2}{J_1+k^2J_2}$$

EXAMPLE 22.4. An elliptic pendulum is a system shown in Fig. 22.10; it consists of two bodies: the body M_1 of weight P_1 which can slide without friction along horizontal guides and the body M_2 of weight P_2 connected with the former by a weightless rod of length l. The body M_2 oscillates in the vertical plane. Write the equation of motion and determine the period of small oscillation of the elliptic pendulum.

Solution. The system has two degrees of freedom; let us take the abscissa x of the centre of gravity of the body M_1 and the angle φ of deviation of the rod from the vertical as two independent coordinates. The Cartesian coordinates of the body (particle) M_2 are

$$x_2 = x + l\sin\varphi, \quad y_2 = l\cos\varphi$$

Therefore for the projections of the velocity of the particle M_2 we obtain

$$\dot{x}_2 = \dot{x} + l\dot{\varphi}\cos\varphi, \quad \dot{y}_2 = -l\dot{\varphi}\sin\varphi$$

The kinetic energy of the system is

$$T=T_1+T_2=\frac{1}{2}\frac{P_1}{g}\dot{x}^2+\frac{1}{2}\frac{P_2}{g}(\dot{x}_2^2+\dot{y}_2^2)$$

$$=\frac{P_1+P_2}{2g}\dot{x}^2+\frac{P_2 l}{2g}(l\dot{\varphi}^2+2\dot{x}\dot{\varphi}\cos\varphi)$$

For any real displacement of the body M_1 along the horizontal guides the work of the forces of weight P_1 and P_2 is equal to zero because $P_1 \perp Ox$ and $P_2 \perp Ox$. Therefore the generalized force Q_x is equal to zero; the generalized force Q_φ is found as in Example 18.3:

$$Q_\varphi = -P_2 l \sin\varphi$$

The same expression can be obtained from the formula for the force function of the gravity field (see Example 15.6):

$$U = mgy = P_2 l \cos\varphi$$

The differentiation of U with respect to φ results in

$$Q_\varphi=\frac{dU}{d\varphi}=-P_2 l\sin\varphi$$

Let us form Lagrange's equations (18.11) for the system under consideration

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}}\right)-\frac{\partial T}{\partial x}=Q_x=0,\quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\varphi}}\right)-\frac{\partial T}{\partial\varphi}=Q_\varphi$$

In order to write these equations in full we compute the necessary derivatives:

$$\frac{\partial T}{\partial x}=0,\quad \frac{\partial T}{\partial\dot{x}}=\frac{P_1+P_2}{g}\dot{x}+\frac{P_2 l}{g}\dot{\varphi}\cos\varphi$$

$$\frac{\partial T}{\partial\varphi}=-\frac{P_2 l}{g}\dot{x}\dot{\varphi}\sin\varphi,\quad \frac{\partial T}{\partial\dot{\varphi}}=\frac{P_2 l}{g}(l\dot{\varphi}+\dot{x}\cos\varphi)$$

Thus, the equations of motion of the elliptic pendulum have the form

$$\frac{d}{dt}\left[\frac{P_1+P_2}{g}\dot{x}+\frac{P_2 l}{g}\dot{\varphi}\cos\varphi\right]=0 \tag{1}$$

$$\frac{P_2 l^2}{g}\ddot{\varphi}+\frac{P_2 l}{g}\ddot{x}\cos\varphi=-P_2 l\sin\varphi \tag{2}$$

From (1) we obtain a first integral:

$$(P_1+P_2)\dot{x}+P_2 l\dot{\varphi}\cos\varphi=c=\text{const}$$

Let the system be at rest at the initial instant $t=0$ and let the pendulum receive an angular deviation φ_0 at that instant; then $\dot{x}(0)=\dot{\varphi}(0)=0$ and $\varphi(0)=\varphi_0$. The substitution of these values into the first integral shows that $c=0$ and consequently

$$(P_1+P_2)\dot{x}+P_2 l\dot{\varphi}\cos\varphi=0 \tag{1a}$$

For small oscillation the angle φ is small and therefore $\sin\varphi\approx\varphi$ and $\cos\varphi\approx 1$. In this case equations (1a) and (2) take the form

$$(P_1+P_2)\dot{x}=-P_2 l\dot{\varphi},\quad \ddot{\varphi}+\frac{1}{l}\ddot{x}+\frac{g}{l}\varphi=0$$

The differentiation of the first equation yields

$$\frac{1}{l}\ddot{x}=-\frac{P_2}{P_1+P_2}\ddot{\varphi}$$

Substituting this value into the second equation we obtain

$$\ddot{\varphi}+\frac{P_1+P_2}{P_1}\frac{g}{l}\varphi=0$$

We have arrived at the differential equation of harmonic oscillation (see Sec. 1.1 of Chap. 14) with circular frequency

$$k=\sqrt{\frac{P_1+P_2}{P_1}\frac{g}{l}}$$

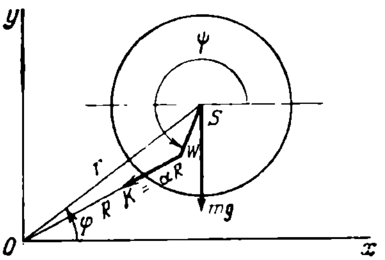

Fig. 22.11

and period

$$\tau=\frac{2\pi}{k}=2\pi\sqrt{\frac{P_1}{P_1+P_2}\frac{l}{g}}$$

EXAMPLE 22.5. A rotating flexible shaft carries a heavy pulley attached symmetrically to it (Fig. 22.11; see also Example 20.3). Investigate the rotation of the shaft.

Solution. Let us take the point of intersection of the axis of the nondeformed shaft with the middle plane of the pulley as the origin. The system under consideration has three degrees of freedom; as three independent coordinates we shall take the polar coordinates r and φ of the centre of gravity S of the pulley and the angle of rotation of the pulley ψ. The pulley is in a plane motion, and its kinetic energy T is determined by formula (21.29):

$$T=\frac{1}{2}mv_S^2+\frac{1}{2}J_{Sz}\omega^2 \quad \left(\omega=\frac{d\psi}{dt}\right)$$

Let us make use of formula (11.3) expressing the velocity of a particle in polar coordinates; it implies that

$$v_S^2=\dot{r}^2+r^2\dot{\varphi}^2$$

Now we put

$$J_{Sz}=m\rho^2$$

where m is the mass of the pulley and ρ is its radius of inertia (see Sec. 1.1 of Chap. 21). Then the kinetic energy of the pulley is written in the form

$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\varphi}^2)+\frac{1}{2}m\rho^2\dot{\psi}^2$$

There are two forces acting on the pulley: the force of weight mg and the elastic force K ($K=cR$, where $R=OW$) which is applied at the point W where the pulley is attached to the shaft. The work of the elastic force corresponding to the variation of R from R_0 to R is

$$A_{el}=\int_{R_0}^{R} cR\,dR=\frac{1}{2}cR^2-\frac{1}{2}cR_0^2$$

The potential energy V of the pulley is equal to the resultant work produced by the acting forces:

$$V=A_{el}+mgy=\frac{1}{2}cR^2-\frac{1}{2}cR_0^2+mgr\sin\varphi$$

From the triangle OWS we find

$$R^2 = r^2 + e^2 + 2er\cos(\psi - \varphi) \qquad (e = WS)$$

and for the force function $U = -V$ we obtain the expression

$$U = -\frac{1}{2}c\,[r^2 + e^2 + 2er\cos(\psi - \varphi)] - mgr\sin\varphi + \frac{1}{2}cR_0^2$$

Lagrange's equations of motion of the pulley (see (18.11a)) have the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot r}\right) - \frac{\partial T}{\partial r} = \frac{\partial U}{\partial r}, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\varphi}\right) - \frac{\partial T}{\partial \varphi} = \frac{\partial U}{\partial \varphi}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\psi}\right) - \frac{\partial T}{\partial \psi} = \frac{\partial U}{\partial \psi}$$

Let us compute the required derivatives:

$$\frac{\partial T}{\partial r} = mr\dot\varphi^2, \quad \frac{\partial T}{\partial \varphi} = \frac{\partial T}{\partial \psi} = 0, \quad \frac{\partial T}{\partial \dot r} = m\dot r, \quad \frac{\partial T}{\partial \dot\varphi} = mr^2\dot\varphi$$

$$\frac{\partial T}{\partial \dot\psi} = m\rho^2\dot\psi, \qquad \frac{\partial U}{\partial r} = -cr - ce\cos(\psi - \varphi) - mg\sin\varphi$$

$$\frac{\partial U}{\partial \varphi} = -cer\sin(\psi - \varphi) - mgr\cos\varphi, \qquad \frac{\partial U}{\partial \psi} = cer\sin(\psi - \varphi)$$

Now we write Lagrange's equations in the form

$$\ddot r - r\dot\varphi^2 = -\omega_{cr}^2\,[r + e\cos(\psi - \varphi)] - g\sin\varphi$$

$$r\ddot\varphi + 2\dot r\dot\varphi = -\omega_{cr}^2 e\sin(\psi - \varphi) - g\cos\varphi$$

$$\rho^2\ddot\psi = \omega_{cr}^2 er\sin(\psi - \varphi)$$

where

$$\omega_{cr} = \sqrt{\frac{c}{m}}$$

is the critical angular velocity of the pulley (see Example 20.3) These equations admit of the solution

$$r = \frac{4}{3}e - \frac{2g}{\omega_{cr}^2}\sin\frac{\omega_{cr}t}{2}, \quad \varphi = \frac{1}{2}\omega_{cr}t$$

$$\psi = \pi + \frac{1}{2}\omega_{cr}t$$

which can be checked by the direct substitution into the equations. Here an essential role is played by the fact that the particular solution of this type exists only for $\dot\psi = \omega_{cr}/2$, that is for the single value of the angular velocity of rotation of the shaft equal to half the critical velocity. The trajectory of the centre of gravity S of the pulley is Pascal's limaçon whose shape depends on the values of the eccentricity e and of the critical angular velocity ω_{cr}. In Fig. 22.12 this curve is constructed for $\omega_{cr}^2 = 3g/e$.

Problems

PROBLEM 22.1. A right rod triangle ABD with joints at the points A, B and D rotates about the vertical axis AD with constant angular velocity ω (Fig. 22.13). The rod AB of weight P and length l forms an angle φ with the axis of rotation. Determine the total reaction of the joint A and the stress to

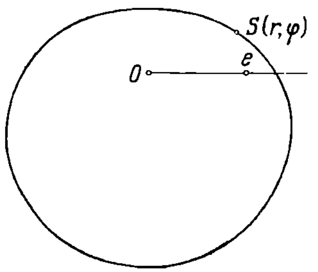

Fig. 22.12

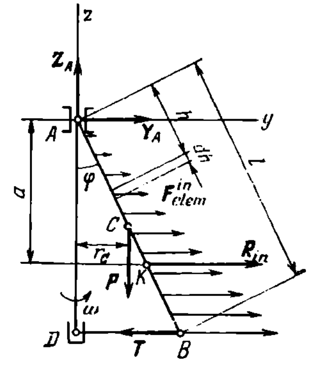

Fig. 22.13

which the rod BD is subjected on condition that the rods AD and BD are considered weightless.

Hint. To find the force T partition the triangle at the point B and consider the motion of the rod BD. To compute the inertial forces separate out an element of the rod of length dh lying at a distance h from the point A. The system of the inertial forces $F^{\text{in}}_{\text{elem}}$ of the elementary particles of the rod is a plane system of parallel forces. The point of application K of the resultant of this system (the centre of parallel forces) lies on the same horizontal as the centre of gravity of the area of the corresponding triangle, that is $a = (2/3)\, l \cos\varphi$. The modulus of the resultant R_{in} of the inertial forces is found using formula (22.7): $R_{\text{in}} = \dfrac{P}{g}\dfrac{l}{2}\,\omega^2 \sin\varphi$. Finally, write the three equations of kinetostatics (see Sec. 1.2 of Chap. 20).

Answer. $Y_A = -\dfrac{1}{2}P\left(\tan\varphi + \dfrac{1}{3g}\, l\omega^2 \sin\varphi\right)$, $Z_A = P$ and $T = \left(\dfrac{1}{3}\dfrac{l\omega^2}{g}\sin\varphi - \dfrac{1}{2}\tan\varphi\right)P$.

PROBLEM 22.2. An inextensible thread is passed round a pulley rotating about a fixed horizontal axis O (Fig. 22.14). From one end of the thread a weight of mass m is suspended, and the other end A of the thread is attached to the vertical spring with stiffness factor c whose lower end B is fixed. The mass M of the pulley is distributed along its rim, and the thread cannot slide on the pulley; the mass of the spring is negligibly small. Determine the period of oscillation of the weight.

Hint. This system has one degree of freedom, and the deviation x of the weight from its equilibrium position can be taken as a generalized coordinate. The kinetic energy of the system is $T = \dfrac{1}{2}m\dot{x}^2 + \dfrac{1}{2}J\omega^2$, where $J = MR^2$ is the moment of inertia of the pulley with respect to the axis O and $\omega = \dot{x}/R$ is the angular velocity of rotation of the pulley. The force function is

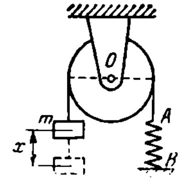

Fig. 22.14

$U = -\frac{1}{2}c(\lambda + x)^2 + mgx$, where $\lambda = mg/c$ is the static elongation of the spring.

Answer. $T = 2\pi\sqrt{\frac{M+m}{c}}$.

Chapter 23 Fundamentals of the Theory of Impact and of Dynamics of a Particle with Variable Mass

In the present chapter we investigate two different problems. However, their common feature is that the absolute rigidity of the body under consideration is violated. Namely, in § 1 where the theory of impact is considered we include the case of inelastic impact, and in § 2 we consider the motion of a body with variable mass.

§ 1. Fundamentals of the Theory of Impact

1.1. Impact. Let a particle M of mass m be in motion relative to an inertial coordinate system $Oxyz$ under the action of the resultant F of the forces applied to it (both the active and the passive forces, that is the constraint reactions). According to the principle of momentum for a particle (formula (15.2)), for the time interval from t to $t_1 = t + \tau$ we have

$$mv_1 - mv = \int_t^{t+\tau} F\,dt \tag{23.1}$$

The change of the momentum of the particle is equal to the (vector) impulse of the resultant of the applied forces. Suppose that the duration τ of the interval $(t, t+\tau)$ is infinitely small. In all the cases we considered before the change of the momentum of the particle during infinitesimal time was also an infinitesimal, that is the momentum changed continuously with time. However, there are such cases of motion of a particle or a body for which the impulse of the force and, consequently, the momentum and the velocity receive finite increments during infinitesimal time (that is these quantities change in jump-like manner). This is the case for impact.

Impact is a phenomenon of an interaction of bodies which, although the time of its duration is infinitesimal, results in a finite change of the velocities of the bodies. The time τ of impact is of the order of one thousandth of a second and even less. Since the increment of the velocities of the particles of the body during impact takes place during infinitesimal time, the accelerations of the particles assume

very large values. Therefore the impact forces producing these accelerations are very large. This makes it very difficult to measure these forces statically, that is by means of the dynamometer, or dynamically by considering the change of the magnitudes of the accelerations. A more convenient way is to characterize the impact force E_{imp} by the *impact momentum*:

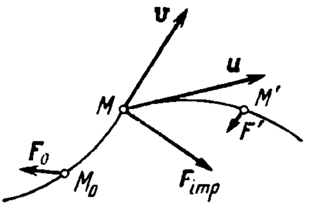

Fig. 23.1

$$S = \int_{t}^{t+\tau} F_{imp}\, dt \tag{23.2}$$

Let a particle M of mass m move under the action of an ordinary force $\boldsymbol{F}$ and describe a curve M_0M (Fig. 23.1). Let impact take place at instant t when the particle is at the point M of the trajectory and has velocity $\boldsymbol{v}$. Under the action of the impact force $\boldsymbol{F}_{imp}$ both the modulus and the direction of the velocity of the particle change instantaneously. Neglecting the impulses of the ordinary forces during the time of impact we write, on the basis of (23.1), the equality

$$m\boldsymbol{u} - m\boldsymbol{v} = \boldsymbol{S} \tag{23.3}$$

where $\boldsymbol{u} = \boldsymbol{v}(t+\tau)$ is the velocity of the particle after the impact. The difference $m\boldsymbol{u} - m\boldsymbol{v}$ of the vectors expressing the momenta of the particle before and after the impact is the momentum gained during the impact.

Equation (23.3) describes the action of the impact force on the particle; it reads: *the momentum gained by the particle during the impact is equal to the impact momentum.*

This equation plays a fundamental role in the theory of impact. Knowing the mass and the velocity of the particle at the beginning of the impact and the impact momentum we can find from this equation the velocity of the particle after the impact:

$$\boldsymbol{u} = \boldsymbol{v} + \frac{1}{m}\boldsymbol{S} \tag{23.3a}$$

In terms of the projections on a fixed axis Ox we can write

$$mu_x - mv_x = S_x \tag{23.4}$$

where S_x is the projection of the impact momentum on the axis Ox. Similar formulas hold for the projections on the axes Oy and Oz of a fixed (or inertial) coordinate system $Oxyz$.

Since the time of impact is negligibly small the displacements of the particles of the body during the impact are also negligibly small, and therefore we assume that the coordinates of the particles of the body remain constant during the impact.

We thus arrive at the following *practical conclusions:*

(1) we can neglect the action of the ordinary forces (for instance, of the force of gravity) during the time of impact;

(2) we can also neglect the displacements of the particles of the body during the time of impact.

In impact of bodies an essential role is played by the physical properties of the bodies. Usually we distinguish between two periods of impact: during the first period the bodies undergo deformation (compression) until their relative velocity vanishes. During this period (the *compression period*) the kinetic energy of the relative motion of the bodies is transformed into the potential energy of deformation, thermal energy, energy of sound oscillation, etc. During the second period of impact the shape of the bodies undergoes restitution due to the action of the elastic forces. During this period (the *restitution period*) the potential energy of deformation is again transformed into the kinetic energy, and at the end of the second period the bodies stop contacting.

For perfectly inelastic bodies impact ends after the first period (the so-called *perfectly inelastic* or *perfectly plastic impact*). For perfectly elastic bodies the total kinetic energy of the bodies at the end of the impact assumes the same value as at the beginning of the impact (the so-called *perfectly elastic impact*).

1.2. Direct Central Impact of a Body against a Fixed Surface. Let us consider impact of a body (for simplicity, we take a ball) against a fixed surface. Let the velocity vector v of the centre of mass at the beginning of the impact coincide with the normal to the surface at the point of contact (Fig. 23.2). Such impact is called *direct* (*head-on* or *straight-line*). We shall suppose that before and after the impact the ball is in a translatory motion and therefore regard it as a particle.

After the impact the ball gains a velocity u along the normal in the opposite direction. Experiments show that the modulus of the velocity at the end of the impact is directly proportional to the modulus of the velocity at the beginning of the impact:

$$u = kv \tag{23.5}$$

The proportionality factor k is called the *coefficient of restitution*. It depends solely on the materials of the ball and the fixed surface. The magnitude of k characterizes physical properties of the colliding bodies. There are the following *three basic cases:*

(1) $k = 0$ and consequently the velocity after the impact is $u = 0$. In this case the impact consists only of the first period, and the shape of the ball is not restored. This is *perfectly inelastic impact* (or *perfectly plastic impact*).

(2) $k = 1$ and consequently the moduli of the velocities at the end and at the beginning of the impact coincide: $u = v$. This is

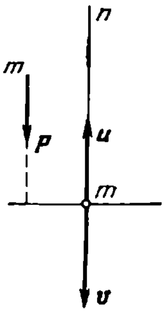

Fig. 23.2

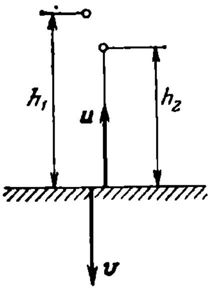

Fig. 23.3

the case of *perfectly elastic impact*; after the impact the shape of the ball is completely restored.

(3) If $0 < k < 1$ then $u < v$ and consequently the modulus of the velocity after the impact is less than the modulus of the velocity at the beginning of the impact. Such impact is called *inelastic*; after such impact the shape of the ball is not restored completely.

1.3. Experimental Determination of the Coefficient of Restitution. The ball is let to fall freely on a fixed horizontal plane from a height h_1 (Fig. 23.3). After the impact against the plane the ball is reflected and achieves a maximum height h_2. Neglecting the air resistance we can apply the law of free fall, which makes it possible to express the modulus v of the velocity of the ball at the beginning of the impact and the modulus u of its velocity at the end of the impact:

$$v = \sqrt{2gh_1}, \quad u = \sqrt{2gh_2}$$

Now using formula (23.5) we find

$$k = \frac{u}{v} = \sqrt{\frac{h_2}{h_1}} \tag{23.6}$$

Coefficients of restitution for some bodies are given in the table below:

Colliding bodies	k
Wood on gutta-percha	0.26
Wood on wood	0.50
Steel on steel	0.56
Ivory on ivory	0.89
Glass on glass	0.94

EXAMPLE 23.1. A ball of mass $m = 0.4$ kg falls on a horizontal plane; its initial height is $h_1 = 2$ m, and after the reflection it achieves the height $h_2 = 1.28$ m. Determine the coefficient of restitution and the impact momentum (Fig. 23.4).

Solution. The coefficient of restitution is found from formula (23.6):

$$k=\sqrt{\frac{h_2}{h_1}}=\sqrt{\frac{1.28}{2}}=0.8$$

Consequently, this is inelastic impact. The impact momentum can be found from formula (23.3). Let us draw the axis Ox as is shown in Fig. 23.4 and construct the vectors v, u and S. Now we project equation (23.3) on the axis Ox:

$$mu-(-mv)=S, \text{ that is } S=m(u+v)$$

Since $u = kv$ it follows that

$$S=m(1+k)v$$

The velocity of the ball at the beginning of the impact is found using the law of free fall (Galileo's formula):

$$v=\sqrt{2gh_1}$$

Finally,

$$S=(1+k)m\sqrt{2gh_1}$$

Substituting the numerical values we find the modulus of the impact momentum:

$$S=(1+0.8)\cdot 4\sqrt{39.2}=4.5 \text{ N}\cdot\text{s}$$

1.4. Direct Central Impact of Two Bodies. *Direct central impact of two bodies* is such impact for which the point of contact of the bodies lies on the straight line joining their centres of mass and the velocities of the centres of mass are directed along that line.

We shall consider the general case, that is inelastic impact of two bodies; from this case we can derive as special cases the laws of perfectly plastic impact and perfectly elastic impact. Let the axis Cx go along the line of impact; all the formulas below are written in terms of the projections on the axis Cx, that is we use the algebraic values of the velocities and the momenta (Fig. 23.5).

Let $v_1 > v_2$ at the beginning of the impact. By formula (23.3),

$$m_1(u_1-v_1)=S, \quad m_2(u_2-v_2)=-S \tag{23.7}$$

where v_1, v_2 and u_1, u_2 are the absolute velocities of the colliding bodies at the beginning and at the end of the impact respectively,

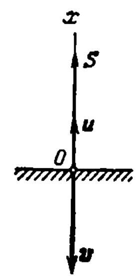

Fig. 23.4

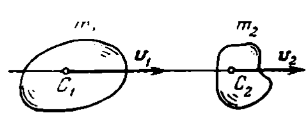

Fig. 23.5

and S is the impact momentum exerted by the second body on the first. The impact momentum exerted by the first body on the second is equal to $-S$ (this follows from Newton's third law).

The relative velocity of the colliding bodies at the beginning of the impact is equal to $v_1 - v_2$, and at the end of the impact it is equal to $u_1 - u_2$. Let us write formula (23.5) for the algebraic values of the velocities:

$$u_1 - u_2 = -k\,(v_1 - v_2) \tag{23.8}$$

Now we add together equations (23.7), which results in

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \tag{23.9}$$

We can regard the two colliding bodies as one complex system; then the impact forces between these bodies are internal within that system, and we can apply the law of conservation of the momentum of the system. It is this fact that is expressed by equation (23.9): *the sum of the momenta of the bodies before the impact and after the impact remains invariable.*

Solving algebraic equations (23.8) and (23.9) with respect to u_1 and u_2 we find the algebraic values (that is the values taken with the corresponding signs!) of the absolute velocities of the bodies at the end of the impact:

$$\left.\begin{aligned} u_1 &= v_1 - \frac{m_2}{m_1+m_2}(1+k)(v_1 - v_2) \\ u_2 &= v_2 + \frac{m_1}{m_1+m_2}(1+k)(v_1 - v_2) \end{aligned}\right\} \tag{23.10}$$

EXAMPLE 23.2. Two balls move towards each other, their velocities having equal moduli (Fig. 23.6). Assuming that the impact of the balls is direct, central and perfectly elastic determine the ratio of the masses of the balls on condition that after the impact one of the balls remains at rest; also determine the velocity of the other ball after the impact.

Solution. Let the velocities of the balls before the impact be $v_1 > 0$ and $v_2 = -v_1$, and after the impact $u_1 = 0$ and u_2. Using formulas (23.10) with $k = 1$ we obtain

$$0 = v_1 - 2\,\frac{m_2}{m_1+m_2}\,[v_1 - (-v_1)], \quad u_2 = -v_1 + 2\,\frac{m_1}{m_1+m_2}\,[v_1 - (-v_1)]$$

The first of these formulas yields

$$m_1 + m_2 = 4m_2, \quad \frac{m_1}{m_2} = 3$$

and hence the mass of the ball remaining at rest after the impact is three times as great as the mass of the other ball. From the second formula we obtain

$$u_2 = -v_1 + 4\,\frac{m_1/m_2}{(m_1/m_2)+1}\,v_1 = -v_1 + 4\,\frac{3}{3+1}\,v_1 = 2v_1$$

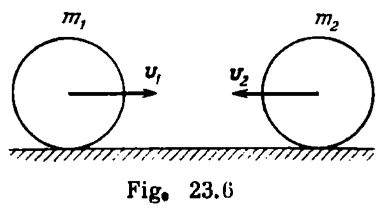

Fig. 23.6

This means that the direction of the velocity of the second ball changes to the opposite after the impact, and the modulus of its velocity increases twice. It can readily be verified that equations (23.8) and (23.9) are fulfilled. This is quite natural because we proceed from formulas (23.10) expressing the solution of equations (23.8) and (23.9).

1.5. Carnot's Theorem*. It is convenient to characterize impact by the loss of kinetic energy; to this end the kinetic energy T of bodies after the impact is subtracted from the initial kinetic energy T_0 of both bodies:

$$T_0 - T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \left(\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 \right)$$

Substituting the values of u_1 and u_2 expressed by (23.10) we obtain after some simple algebraic transformations the formula

$$T_0 - T = \frac{1}{2} (1 - k^2) \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2)^2 \qquad (23.11)$$

We have derived the *formula for the loss of the kinetic energy of two colliding bodies* taking part in arbitrary direct central impact.

Special cases.

(1) If one of the bodies is at rest before the impact, for instance, if $v_2 = 0$, then formula (23.11) yields

$$T_0 - T = (1 - k^2) \frac{m_2}{m_1 + m_2} \frac{m_1 v_1^2}{2} = (1 - k^2) \frac{m_2}{m_1 + m_2} T_0 \quad (23.12)$$

(2) In the case of perfectly elastic impact ($k = 1$) formula (23.11) implies that

$$T_0 - T = 0, \quad \text{that is} \quad T = T_0$$

Thus, in perfectly elastic impact the kinetic energy of the colliding bodies is not lost.

CARNOT'S THEOREM. *In perfectly plastic direct central impact of two bodies the loss of the kinetic energy of the colliding bodies is equal to the kinetic energy corresponding to the loss of the velocities*:

$$T_0 - T = \frac{1}{2} m_1 (v_1 - u)^2 + \frac{1}{2} m_2 (v_2 - u)^2$$

Proof. By virtue of equation (23.8), for $k = 0$ we have $u_1 = u_2 = u$ and equation (23.9) yields

$$(m_1 + m_2) u = m_1 v_1 + m_2 v_2, \quad \text{that is} \quad u_1 = u_2 = u = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \qquad (23.13)$$

The same formula for the common velocity of both the bodies at the end of perfectly plastic impact can be derived from (23.10).

* Carnot, L. N. M. (1753-1823), a French mathematician and military engineer.

The expression for the loss of the kinetic energy of the colliding bodies can be transformed thus:

$$T_0-T=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2-\frac{1}{2}(m_1+m_2)u^2$$
$$=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2-(m_1+m_2)u\cdot u+\frac{1}{2}(m_1+m_2)u^2$$
$$=\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2-(m_1v_1+m_2v_2)u+\frac{1}{2}(m_1+m_2)u^2$$
$$=\frac{1}{2}m_1(v_1-u)^2+\frac{1}{2}m_2(v_2-u)^2$$

Here we have used the equality $(m_1+m_2)u=m_1v_1+m_2v_2$ (see (23.13)). The theorem is proved.

EXAMPLE 23.3. Determine the efficiency η_h of a hammer of mass m_1 striking an anvil of mass m_2.

Solution. For the hammer the useful (effective) energy is equal to the loss of the kinetic energy T_0-T which is expended on the deformation of the forged piece of metal. The total expended energy is the kinetic energy T_0 at the beginning of the impact. Proceeding from formula (23.12) (it applies because the anvil is at rest!) we find

$$\eta_h=\frac{T_0-T}{T_0}=(1-k^2)\frac{m_2}{m_1+m_2}=\frac{1-k^2}{1+(m_1/m_2)} \qquad (23.14)$$

The smaller the mass m_1 of the hammer in comparison with the mass m_2 of the anvil the greater is the efficiency of the hammer. The kinetic energy T remaining after the impact is expended on the shaking of the foundation and is not useful.

For instance, let us take the value $k=0.56$ (see the table in Sec. 1.3); then if, for instance, $m_1=0.05m_2$, we obtain

$$\eta_h=\frac{1-0.56^2}{1+0.05}=0.65$$

EXAMPLE 23.4. Determine the efficiency η_d of a pile driver hammering in a pile of mass m_2. The mass of the hammer head is equal to m_1.

Solution. For the pile driver the effective energy is the kinetic energy T retained after the impact; it is expended on the motion of the pile driver and the pile. The total expended energy is the kinetic energy T_0 at the beginning of the impact. From formula (23.12) we obtain

$$T=T_0-(1-k^2)\frac{m_2}{m_1+m_2}T_0=\frac{m_1+m_2k^2}{m_1+m_2}T_0$$

The efficiency η_d is equal to

$$\eta_d=\frac{T}{T_0}=\frac{m_1+m_2k^2}{m_1+m_2}=1-\frac{m_2(1-k^2)}{m_1+m_2}=1-\frac{1-k^2}{1+(m_1/m_2)} \qquad (23.15)$$

Since the loss of the kinetic energy T_0-T is mainly expended on the deformation of the pile it must be made as small as possible. From formula (23.15) it follows that the greater the weight of the pile driver in comparison with the weight of the pile the higher is the efficiency of the pile driver.

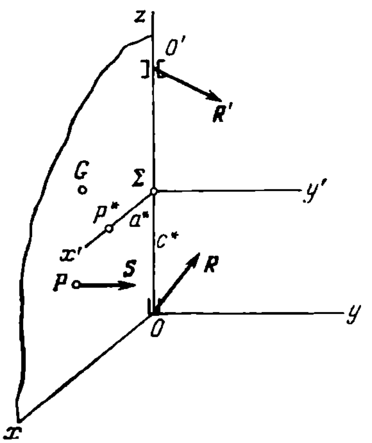

Fig. 23.7

For instance, if $m_1 = 0.05m_2$ and if the hammer head and the pile are made of steel then, taking $k = 0.56$ (see the table in Sec. 1.3), we find

$$\eta_d = 1 - \frac{1-0.56^2}{1+0.05} = 0.35$$

The comparison of this result with the one obtained in Example 23.3 shows that, for the statement of the problem we have considered, in the case of identical materials and the same ratio of the masses we have

$$\eta_d = 1 - \eta_h$$

1.6. Impact against a Rigid Body Rotating about a Fixed Axis. Let us consider a rigid body which is at rest at the initial instant $t = 0$ and which can rotate freely about a fixed axis supported by a thrust bearing O and a bearing O', the distance between the bearings being $OO' = h$ (Fig. 23.7). Let us choose a fixed coordinate system $Oxyz$ such that the centre of mass G of the body is in the plane Ozx at the initial instant and has the coordinates G (ξ, 0, ζ). Suppose that an impact momentum S {0, S, 0} is exerted on the body at its point P (a, 0, c) in the direction of the axis Oy, the weight of the body being negligibly small.

During the time of impact τ the position of the body does not change; however the body gains an angular velocity ω {0, 0, ω}, and therefore the projections of the velocity v_ν of the νth particle of the body with mass m_ν and coordinates x_ν, y_ν and z_ν on the coordinate axes $Oxyz$ are determined by formulas (9.14):

$$v_x^\nu = -\omega y_\nu, \quad v_y^\nu = \omega x_\nu, \quad v_z^\nu = 0 \tag{23.16}$$

Removing the constraints at the points O and O' and replacing their action on the body during the time of impact by reactive impact momenta R {R_x, R_y, R_z} and R' {R'_x, R'_y, 0} (see Fig. 23.7) we can regard the body as being free and apply the general principles of dynamics of a system. The principle of momentum applied to this system (see (19.8)) results in the equations

$$\begin{gathered} \sum m_\nu v_x^\nu = R_x + R'_x, \quad \sum m_\nu v_y^\nu = S + R_y + R'_y \\ \sum m_\nu v_z^\nu = R_z \end{gathered} \tag{23.17}$$

which hold during the time of impact. Let us use formulas (23.16) and (19.2) for the coordinates of the centre of mass of the body and

rewrite the left-hand sides of equations (23.17) in the form

$$\sum m_\nu v_x^\nu = -\omega \sum m_\nu y_\nu = 0$$

$$\sum m_\nu v_y^\nu = \omega \sum m_\nu x_\nu = \omega M \xi, \quad \sum m_\nu v_z^\nu = 0$$

where M is the mass of the body. Hence, equations (23.17) take the form

$$R_x + R'_x = 0, \quad R_y + R'_y - \omega M \xi = -S, \quad R_z = 0 \qquad (23.18)$$

Now let us apply the principle of angular momentum for a body (for the time of impact). By virtue of formulas (19.15) and (23.16), the angular momentum of the body about the centre O is

$$K_O - \sum [r_\nu, m_\nu v_\nu] = \sum \begin{vmatrix} i & j & k \\ x_\nu & y_\nu & z_\nu \\ -\omega m_\nu y_\nu & \omega m_\nu x_\nu & 0 \end{vmatrix}$$

From this expression we find the angular momenta of the body about the axes $Oxyz$ at instant $t = \tau$:

$$\begin{aligned} K_x &= -\omega \sum m_\nu z_\nu x_\nu = -J_{zx}\omega \\ K_y &= -\omega \sum m_\nu y_\nu z_\nu = -J_{yz}\omega, \quad K_z = J_z \omega \end{aligned} \qquad (23.19)$$

where J_{zx} and J_{yz} are the products of inertia of the body (see formula (22.1)); here we have used formula (21.13). Now we multiply equalities (19.22) by dt and integrate them with respect to t from $t = 0$ to $t = \tau$, which yields

$$K_x = \int_0^\tau M_x^{(\text{ext})}\, dt, \quad K_y = \int_0^\tau M_y^{(\text{ext})} \; d \quad K_z = \int_0^\tau M_z^{(\text{ext})}\, dt \quad (23.20)$$

because the angular momentum of the body before the impact is equal to zero. Neglecting the impulses of the momenta of ordinary forces during the time of impact τ we obtain for the right-hand sides of equations (23.20) the expressions

$$\int_0^\tau M_x^{(\text{ext})}\, dt = -Sc - hR'_y, \quad \int_0^\tau M_y^{(\text{ext})}\, dt = hR'_x, \quad \int_0^\tau M_z^{(\text{ext})}\, dt = Sa$$

Therefore, taking into account (23.19), we write equations (23.20) in the form

$$J_{zx}\omega - hR'_y = Sc, \quad J_{yz}\omega + hR'_x = 0, \quad J_z\omega = Sa \qquad (23.21)$$

Equations (23.18) and (23.21) form a system of six algebraic linear equations with respect to six unknowns: the five projections R_x, R_y, R_z, R'_x and R'_y of the reactive impact momenta and the angular velocity ω of the body after the impact. The solution of the system encounters no difficulties. We shall consider this system in an aspect important for applications.

1.7. Centre of Percussion. If impact at the point P exerted on the body as shown in Fig. 23.7 does not affect the thrust bearing and the bearing, this point of impact $P = P^*$ $(a^*, 0, c^*)$ is called the *centre of percussion*. In such a case not only the reaction R_z vanishes (this is implied by the third equation (23.18)) but also

$$R_x = R_y = R'_x = R'_y = 0$$

The remaining four equations (23.18), (23.21) take the form

$$\omega M\xi = S, \quad J_{zx} = Sc^* \qquad (23.22)$$
$$J_{yz}\omega = 0, \quad J_z\omega = Sa^*$$

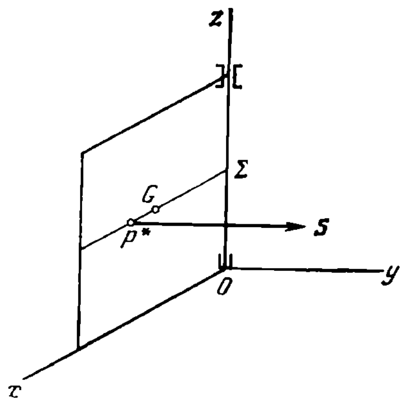

Fig. 23.8

The division of the fourth and the second equations by the first results in

$$a^* = \frac{J_z}{M\xi}, \qquad c^* = \frac{J_{zx}}{M\xi} \qquad (23.23)$$

It is these formulas that specify the abscissa and the z-coordinate of the centre of percussion P^* (in the statement of the problem presented here the ordinate of P^* is equal to zero). The second formula (23.23) can be interpreted in the following way. Let us transfer the origin to the point Σ $(0, 0, c^*)$ on the axis Oz. By virtue of formulas (19.2), taking into account the relation $z'_\nu = z_\nu - c^*$, we find the expressions of the products of inertia $J_{zx'}$ and $J_{y'z}$:

$$J_{zx'} = \sum m_\nu x_\nu (z_\nu - c^*) = J_{zx} - M\xi c^* = 0$$
$$J_{y'z} = \sum m_\nu y_\nu (z_\nu - c^*) = J_{yz}$$

Now taking into account that according to the third equation (23.22) the product of inertia J_{yz} must be equal to zero, we see that for the point Σ the axis Oz is a principal axis of inertia (see the definition in Sec. 1.2 of Chap. 22). Now we can state the *rule for constructing the centre of percussion P^*: It is necessary to find the point on the axis of rotation for which this axis is a principal axis of inertia and then to set off from this point a perpendicular of length a^* to the axis of rotation towards the centre of mass.*

EXAMPLE 23.5. Find the centre of percussion for a homogeneous square plate rotating about its side, the impact momentum being perpendicular to the plane of the plate (Fig. 23.8).

Solution. The axis of rotation is a principal axis of inertia for the midpoint Σ of the side of the plate about which it rotates because the products of inertia

$J_{y'z}$ and $J_{zx'}$ are obviously equal to zero. Let us compute a^* using the first formula (23.23). We denote the length of the side of the square by b, and the area density of the plate by $\varkappa$. The moment of inertia of the plate with respect to the axis of rotation is computed by formula (21.5):

$$J_z = \frac{1}{3} Mb^2$$

where $M = b^2\varkappa$ is the mass of the plate. The abscissa of the centre of mass is obviously equal to $\xi = b/2$ and from (23.23) we obtain

$$a^* = \frac{Mb^2 \cdot 2}{3Mb} = \frac{2}{3} b$$

Finally, using the rule stated at the end of Sec. 1.7, we construct the centre of percussion P^*.

§ 2. Fundamentals of Dynamics of a Particle with Variable Mass

One of the first creators of the project of a rocket flying vehicle was the great Russian scientist and inventor K. E. Tsiolkovsky (1857-1935). In the fundamental works by K. E. Tsiolkovsky and I. V. Meshchersky (1859-1935) the foundation of dynamics of a translatory motion of a rocket (dynamics of a particle with variable mass) was developed.

2.1. Meshchersky's Equation. Let a material system moving relative to an inertial coordinate system $Oxyz$ be bounded by a definite surface S, for instance, by the shell of a rocket and the cross sections of its nozzles (Fig. 23.9). Let the mass inside the surface S vary according to a given law (this means that $m = m(t)$ is a given function of time). The resultant vector of the external active forces $F_\nu^{(ext)}$ applied to the system will be denoted by F:

$$F = \sum F_\nu^{(ext)}$$

We shall suppose that the system of particles enveloped in S at the initial instant t_0 has a mass $M = m(t_0) = \text{const}$; our aim is

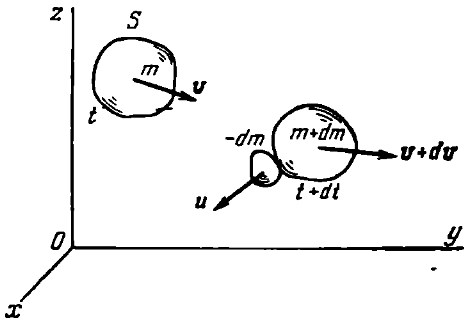

Fig. 23.9

to investigate the motion of this system of particles for which a part of the particles will leave the surface S in the course of the motion. Let the constraints imposed on the system be such that translatory displacements of the system are possible in any direction. Under these conditions the principle of momentum in form (19.13) is applicable to the system:

$$d\left(M\frac{d\boldsymbol{r}_C}{dt}\right)=\boldsymbol{F}\,dt$$

that is

$$\left(M\frac{d\boldsymbol{r}_C}{dt}\right)_{t_0+dt}-\left(M\frac{d\boldsymbol{r}_C}{dt}\right)_{t_0}=\boldsymbol{F}\,dt \tag{23.24}$$

Let us denote by $\boldsymbol{v}$ the absolute velocity of the centre of mass for the particles which are inside S at instant t_0:

$$\boldsymbol{v}=\left(\frac{d\boldsymbol{r}_C}{dt}\right)_{t=t_0}$$

By $\boldsymbol{v}+d\boldsymbol{v}$ we shall denote the absolute velocity of the centre of mass of the part of the particles of the system under consideration which remains inside S at instant $t+dt$. Finally, let us denote by $\boldsymbol{u}$ the absolute velocity of the particles with total mass $(-dm)$ $(dm < 0)$ leaving the volume enclosed in the surface S during time dt. Then, according to the conditions stated,

$$\left(M\frac{d\boldsymbol{r}_C}{dt}\right)_{t=t_0}=m(t_0)\,\boldsymbol{v};\quad \left(M\frac{d\boldsymbol{r}_C}{dt}\right)_{t=t_0+dt}=[m(t_0)+dm](\boldsymbol{v}+d\boldsymbol{v})$$
$$+(-dm)\,\boldsymbol{u}=m(t_0)\,\boldsymbol{v}+m(t_0)\,d\boldsymbol{v}+dm\cdot\boldsymbol{v}+dm\cdot d\boldsymbol{v}-dm\cdot\boldsymbol{u}$$

Now we substitute these expressions into (23.24) and divide by dt:

$$m(t_0)\frac{d\boldsymbol{v}}{dt}-(\boldsymbol{u}-\boldsymbol{v})\frac{dm}{dt}+\frac{dm}{dt}\,d\boldsymbol{v}=\boldsymbol{F}$$

Passing to the limit we obtain *Meshchersky's equation of motion of a particle with variable mass*:

$$m\frac{d\boldsymbol{v}}{dt}=\boldsymbol{F}+\frac{dm}{dt}(\boldsymbol{u}-\boldsymbol{v}) \tag{23.25}$$

Here m is the variable mass of the particle (the whole body in a translatory motion is regarded as a particle); $\boldsymbol{v}$ is the absolute velocity of the particle; $\boldsymbol{u}$ is the absolute velocity of the particles leaving the system and $\boldsymbol{F}$ is the resultant of the external forces acting on the particle. Now we note that $\boldsymbol{u}-\boldsymbol{v}=\boldsymbol{v}_{rel}$ is the relative velocity (with respect to the moving system) of the particles leaving the system; therefore Meshchersky's equation can be written in the form

$$m\frac{d\boldsymbol{v}}{dt}=\boldsymbol{F}+\frac{dm}{dt}\boldsymbol{v}_{rel} \tag{23.25a}$$

The vector

$$\frac{dm}{dt}\,\boldsymbol{v}_{\mathrm{rel}}$$

is called the *reactive force* (we denote it by $\boldsymbol{R}$).

If the relative velocity $\boldsymbol{v}_{\mathrm{rel}}$ of the particles leaving the system is equal to zero the reactive force $\boldsymbol{R}$ vanishes and the equation of motion of a body with variable mass (23.25a) takes the ordinary form of the equation of motion of a body with constant mass implied by Newton's second law.

In case the absolute velocity $\boldsymbol{u}$ of the particles leaving the system is equal to zero, Meshchersky's equation (23.25) takes the form

$$m\frac{d\boldsymbol{v}}{dt}=\boldsymbol{F}-\frac{dm}{dt}\boldsymbol{v},\quad \text{that is}\quad \frac{d}{dt}(m\boldsymbol{v})=\boldsymbol{F}$$

which coincides with that of Newton's second law.

2.2. **Tsiolkovsky's First Problem.** Let a rocket with continuous gas discharge fly vertically upward. It is required to determine the velocity v of motion of the rocket on condition that the relative velocity v_{rel} of gas discharge has constant modulus and is directed opposite to the motion of the rocket. The air resistance and the force of gravity are neglected.

Solution. We shall regard the rocket as a particle with variable mass and write Meshchersky's equation (23.25a) for the case under consideration ($F = 0$):

$$m\frac{dv}{dt}=-v_{\mathrm{rel}}\frac{dm}{dt}$$

Let us separate the variables:

$$dv=-v_{\mathrm{rel}}\frac{dm}{m}$$

Next we integrate the equation, which yields

$$\int\limits_{v_0}^{v} dv=-v_{\mathrm{rel}}\int\limits_{m_0}^{m}\frac{dm}{m}$$

Computing the integrals we find

$$v-v_0=-v_{\mathrm{rel}}\,(\ln m-\ln m_0)$$

where v_0 and m_0 are the initial velocity and the initial mass of the rocket respectively (at the instant when the gas discharge starts). Consequently, under the assumptions stated above, the velocity of the rocket is

$$v=v_0+v_{\mathrm{rel}}\ln\frac{m_0}{m} \tag{23.26}$$

This is known as *Tsiolkovsky's formula*.

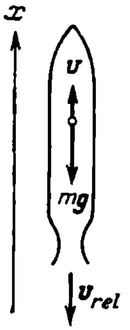

Fig. 23.10

Tsiolkovsky's formula can be used for estimating approximately the velocity of the rocket in those cases when the force of air resistance and the force of gravity are small in comparison with the reactive thrust force.

EXAMPLE 23.6. The relative velocity of gas discharge from a rocket is constant and is equal to 3000 m/s. What must be the ratio of the initial mass of the rocket to its mass at the end of the combustion period for the rocket to achieve the circular velocity (see Example 20.2)?

Solution. By Tsiolkovsky's formula (23.26), we obtain

$$7910 = 3000 \ln \frac{m_0}{m}$$

whence

$$\ln \frac{m_0}{m} = \frac{7910}{3000} = 2.64, \qquad \frac{m_0}{m} = e^{2.64} = 14.0$$

We see that the launching weight of the rocket must be more than 14 times as great as the final stage weight.

2.3. Tsiolkovsky's Second Problem. In Tsiolkovsky's second problem the vertical upward flight of a rocket is investigated taking into account the action of the force of gravity (Fig. 23.10). All the other conditions of the problem are the same as in the first problem.

Solution. Let us suppose that the launching part of the trajectory is small in comparison with the radius of the Earth; then the acceleration of gravity can be considered constant and we shall put it equal to its value on the Earth's surface. Meshchersky's equation (23.25a) takes the form

$$m \frac{dv}{dt} = -mg - \frac{dm}{dt} v_{\text{rel}}$$

whence

$$dv = -g\,dt - v_{\text{rel}} \frac{dm(t)}{m(t)}$$

Now we integrate the equation:

$$\int_{v_0}^{v} dv = -\int_{0}^{t} \left[g + v_{\text{rel}} \frac{\dot{m}(t)}{m(t)} \right] dt$$

Computing the integrals we find the solution of Tsiolkovsky's second problem:

$$v - v_0 = -gt - v_{\text{rel}} (\ln m - \ln m_0)$$

that is

$$v = v_0 - gt + v_{\text{rel}} \ln \frac{m_0}{m} \tag{23.27}$$

In particular, if not only the relative velocity $v_{\rm rel}$ of gas discharge is constant but the mass flow rate is also constant ($\dot{m} = -\mu = $ $= \text{const}$) then

$$m = m_0 - \mu t$$

and hence

$$v = v_0 - gt + v_{\rm rel} \ln \frac{m_0}{m_0 - \mu t}$$

Here μ is the mass flow rate.

EXAMPLE 23.7. A body moves vertically upward with constant velocity v_0 in the homogeneous gravitational field. The relative velocity $v_{\rm rel}$ of the particles leaving the body is also constant. The resistance of the medium is negligibly small, and the acceleration of gravity is equal to g. Find the law of change of the mass of the body.

Solution. On the basis of formula (23.27), for $v = v_0$ we have

$$gt = v_{\rm rel} \ln \frac{m_0}{m}$$

whence

$$\frac{m_0}{m} = e^{\frac{}{v_{\rm rel}} t}, \quad \text{that is} \quad m = m_0 e^{-\frac{g}{v_{\rm rel}} t}$$

which means that the mass of the body must decrease exponentially.

Problems

PROBLEM 23.1. A ball A of mass m_1 falls freely from height h on a plate B of mass m_2 supported by a spring (Fig. 23.11). Assuming that the impact is perfectly plastic determine the velocity u of the plate after the impact.

Answer. $u = \dfrac{m_1}{m_1 + m_2} \sqrt{2gh}$.

PROBLEM 23.2. Two balls of masses m_1 and m_2 with coefficient of restitution k are in a translatory motion in one direction (Fig. 23.12*a*). The velocity of the first ball exceeds that of the second, and after the former strikes the latter the first ball stops while the second ball gains velocity u_2 (Fig. 23.12*b*). Find the velocities v_1 and v_2 of the balls before the impact.

Answer. $v_1 = \dfrac{1+k}{k} \dfrac{m_2}{m_1 + m_2} u_2$, $v_2 = \dfrac{km_2 - m_1}{k(m_1 + m_2)} u_2$.

PROBLEM 23.3. The driving head of a pile driver of weight 3000 N falls from a height $h = 3$ m on a pile of weight 500 N. The coefficient of restitution is $k = 0.52$. Determine the efficiency η of the pile driver, the work A_1 expended on the deformation of the pile and the work A_2 expended on driving the pile into the ground.

Answer. $\eta = 0.896$, $A_1 = 940$ J and $A_2 = 8060$ J.

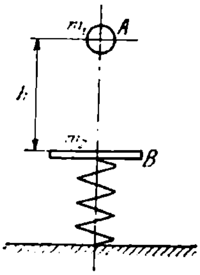

Fig. 23.11

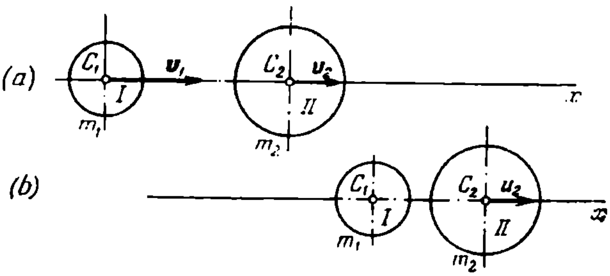

Fig. 23.12

PROBLEM 23.4. Three perfectly elastic balls of masses m_1, m_2 and m_3 lie in one straight line. The ball m_1 moving with a known velocity v strikes the ball m_2 (which is at rest before the impact). The ball m_2 starts moving and strikes and makes to move the ball m_3 which is also initially at rest. Determine the value of the mass m_2 for which the velocity of the ball m_3 assumes the greatest possible value (this is Huygens' problem).

Answer. $m_2 = \sqrt{m_1 m_3}$.

Chapter 24 Motion of a Particle in a Field of Central Force

§ 1. Binet's Formulas

1.1. Field of Central Force. A force acting on a particle is said to be *central* if its line of action constantly passes through a fixed point O called the *centre*. If the origin is placed at the centre then the general expression for a central force $\boldsymbol{F}$ has the form

$$\boldsymbol{F} = F \frac{\boldsymbol{r}}{r}$$

where $\boldsymbol{r}$ is the radius vector of the particle m to which the force is applied (Fig. 24.1), r, the modulus of the radius vector $\boldsymbol{r}$, is equal to the polar radius of m, and F is the projection of the force $\boldsymbol{F}$ on the direction of the radius vector $\boldsymbol{r}$ (the algebraic value of the force $\boldsymbol{F}$). For a repulsive force we have $F > 0$ and, on the contrary, for an attractive force $F < 0$.

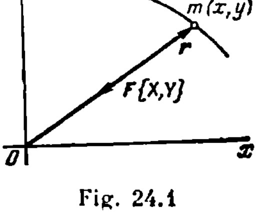

Fig. 24.1

A part of space in which central forces act is called a *field of central force*. Examples of such fields are the gravitational field generated by a particle or by a homogeneous ball and also the electrostatic field generated by a point charge.

Of great practical importance is the case when F^* depends solely on the distance r, that is $F = F(r)$ and $\boldsymbol{F} = F(r)\,\boldsymbol{r}^0$, where $\boldsymbol{r}^0 = \boldsymbol{r}/r$ is the unit vector along the radius vector of the point of application of the force. Such force fields are said to be *spherically symmetric*. The above two examples of fields of central force belong to this class. In what follows we shall always assume that the field of central force in question is spherically symmetric. Let us prove that under this assumption the field is potential (see Sec. 3.3 of Chap. 15).

According to Corollary (1) in Sec. 2.3 of Chap. 15 (also see Example 15.3) the trajectory of a particle moving under the action of a central force is a plane curve; namely, it lies in the plane passing through the initial velocity vector of the particle and the centre O. Let us place the coordinate system Oxy in that plane (see Fig. 24.1). Then the projections X and Y of the central force $\boldsymbol{F}$ are

$$X = F\cos\varphi = F\frac{x}{r}, \quad Y = F\sin\varphi = F\frac{y}{r}$$

Now, proceeding from Sec. 3.3 of Chap. 15, we compute the elementary work of the force $\boldsymbol{F}$ along the real displacement $d\boldsymbol{r} = \boldsymbol{i}\,dx + \boldsymbol{j}\,dy$:

$$\delta A = (\boldsymbol{F},\ d\boldsymbol{r}) = X\,dx + Y\,dy = F\frac{x\,dx + y\,dy}{r} = F(r)\,dr \qquad (24.1)$$

because the differentiation of the equality $x^2 + y^2 = r^2$ results in

$$x\,dx + y\,dy = r\,dr$$

The expression $F(r)\,dr$ is the differential of the function $U(r)$ determined by the formula

$$U(r) = \int F(r)\,dr \qquad (24.2)$$

In other words, $F(r)\,dr$ can be regarded as the total differential of the function $U(r) = U(x, y)$ of two variables x and y which is the force function for the force $F(r)$.

1.2. Binet's First Formula.** We begin with stating a kinematic definition: the *areal* (or *sector*) *velocity* $d\sigma/dt$ of a point M is the limit of the ratio of the area $\Delta\sigma$ swept over by the radius vector $\boldsymbol{r}$ (Fig. 24.2) of the point M to the corresponding time

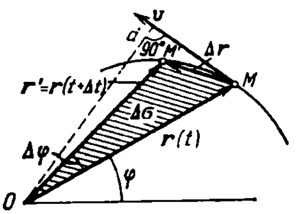

Fig. 24.2

* We again stress that throughout the present chapter F denotes the algebraic value of the force.

** Binet, J. P. M. (1786-1856), a French mathematician and astronomer.

Δt for $\Delta t \to 0$, that is

$$\frac{d\sigma}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \sigma}{\Delta t}$$

As we know, the modulus of the vector product of two vectors is equal to twice the area of the triangle constructed on the given vectors (see Sec. 2.2 of Chap. 1), whence

$$2\Delta\sigma = |[\boldsymbol{r}, \Delta \boldsymbol{r}]|$$

We thus conclude that the duplicated sector velocity is

$$2\frac{d\sigma}{dt} = \lim_{\Delta t \to 0} \left| \left[\boldsymbol{r}, \frac{\Delta \boldsymbol{r}}{\Delta t} \right] \right| = |[\boldsymbol{r}, \boldsymbol{v}]| = \mathrm{Mom}_O\, v = v \cdot d \qquad (24.3)$$

that is it is equal to the modulus of the moment of the velocity of the particle about the centre O.

Let us derive the expression for the sector velocity in polar coordinates for the case of plane motion. To within infinitesimals of the second order the magnitude of the element of area is equal to

$$\Delta\sigma = \frac{1}{2} r^2 |\Delta\varphi|$$

(see Fig. 24.2); here $\Delta\varphi$ is the increment of the polar angle φ during time Δt. According to the definition,

$$\frac{d\sigma}{dt} = \lim_{\Delta t \to 0} \frac{1}{2} r^2 \left| \frac{\Delta\varphi}{\Delta t} \right| = \frac{1}{2} r^2 \left| \frac{d\varphi}{dt} \right| \qquad (24.4)$$

Now we proceed to the derivation of Binet's first formula. Let a particle m move under the action of a central force. As was shown in Corollary (1) of Sec. 2.3 of Chap. 15, in this case the principle of angular momentum (in its vector form (15.10)) leads to the first integral

$$\mathbf{Mom}_O\, m\boldsymbol{v}(t) = \mathbf{Mom}_O\, m\boldsymbol{v}(0)$$

The invariability of the direction of the vector representing the angular momentum of the particle about the centre O implies that the trajectory of the particle m lies in a fixed plane perpendicular to the vector $\mathbf{Mom}_O\, m\boldsymbol{v}(0)$. According to (24.3), the constancy of the modulus of the angular momentum of the particle about O implies the constancy of the sector velocity:

$$2\frac{d\sigma}{dt} = \mathrm{Mom}_O\, \boldsymbol{v}(0) = \mathrm{const} \qquad (24.5)$$

Therefore integral (24.5) is said to express the *law of areas*. Taking into account (24.4) we can write the law of areas in polar coordinates in the form

$$r^2 \frac{d\varphi}{dt} = c \qquad (c \equiv \mathrm{mom}_O\, v(0) = \mathrm{const}) \qquad (24.6)$$

where the sign plus or minus of c is taken depending on whether the polar angle φ of the moving particle m increases or decreases.

By formula (11.3), the square of the velocity of the particle in polar coordinates is expressed by the formula

$$v^2=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\varphi}{dt}\right)^2=\left(\frac{dr}{d\varphi}\right)^2\left(\frac{d\varphi}{dt}\right)^2+\frac{1}{r^2}\left(r^2\frac{d\varphi}{dt}\right)^2$$

Now we make use of the law of areas (24.6); it implies that $d\varphi/dt = c/r^2$ and we write

$$v^2=c^2\left[\left(\frac{1}{r^2}\frac{dr}{d\varphi}\right)^2+\frac{1}{r^2}\right], \quad \text{that is} \quad v^2=c^2\left\{\left[\frac{d\left(\frac{1}{r}\right)}{d\varphi}\right]^2+\frac{1}{r^2}\right\} \tag{24.7}$$

We have derived *Binet's first formula* expressing the square of the velocity of a particle in terms of the reciprocal of the polar radius and the derivative of that reciprocal with respect to the polar angle.

1.3. Binet's Second Formula. By virtue of (24.1), for a particle moving in a field of central force the principle of energy (see (15.21)) is written in the form

$$d\left(\frac{1}{2}mv^2\right)=F\,dr$$

Let us divide this relation by $d\varphi$, substitute into it the expression for v^2 given by Binet's first formula and then perform differentiation:

$$F\frac{dr}{d\varphi}=\frac{d}{d\varphi}\frac{mc^2}{2}\left\{\left[\frac{d\left(\frac{1}{r}\right)}{d\varphi}\right]^2+\left(\frac{1}{r}\right)^2\right\}$$

$$=\frac{mc^2}{2}\left[2\frac{d\left(\frac{1}{r}\right)}{d\varphi}\frac{d^2\left(\frac{1}{r}\right)}{d\varphi^2}+2\frac{1}{r}\frac{d\left(\frac{1}{r}\right)}{d\varphi}\right]$$

$$=mc^2\left[-\frac{1}{r^2}\frac{dr}{d\varphi}\frac{d^2\left(\frac{1}{r}\right)}{d\varphi^2}-\frac{1}{r}\frac{1}{r^2}\frac{dr}{d\varphi}\right]$$

Cancelling by $dr/d\varphi$ we obtain *Binet's second formula*:

$$F=-\frac{mc^2}{r^2}\left[\frac{d^2\left(\frac{1}{r}\right)}{d\varphi^2}+\frac{1}{r}\right] \tag{24.8}$$

Binet's second formula makes it possible to determine the force for the given trajectory of motion $r = r(\varphi)$, that is to solve a problem analogous to the first basic problem of particle dynamics.

§ 2. Elements of the Theory of Planetary Motion. Earth's Artificial Satellites

2.1. Derivation of Newton's Law of Gravitation from Kepler's Laws. In the foundation of celestial mechanics *Kepler's three laws* lie:

(1) *The trajectories of the planets (and of the comets) are conic sections at one of whose foci the Sun is.*

(2) *The planets (and the comets) move along plane trajectories round the Sun, and their motion along the trajectories obeys the law of areas* (Sec. 1.2).

(3) *The squares of the periods of revolution of the planets are in the ratio of the cubes of the semimajor axes of their orbits* (here the sidereal time is meant).

These laws were established by Kepler on the basis of experimental data (see Sec. 3 of Introduction to Dynamics). On the basis of these laws I. Newton found the law of forces acting on the planets and then established the law of gravitation.

Kepler's second law holds if and only if the force is central. From Kepler's first law we can determine the force using Binet's second formula (24.8).

If we place the pole at the right focus of an ellipse then the equation of this conic section in polar coordinates r and φ has the form

$$\frac{1}{r}=\frac{1+e\cos\varphi}{p} \tag{24.9}$$

where e is the eccentricity and p is the latus rectum (focal parameter) (for an ellipse or a hyperbola it is equal to b^2/a; see [7]). Using formula (24.8) we find

$$F=-\frac{mc^2}{r^2}\left(-\frac{e\cos\varphi}{p}+\frac{1+e\cos\varphi}{p}\right)=-m\mu\frac{1}{r^2}\quad\left(\mu=\frac{c^2}{p}\right) \tag{24.10}$$

We have thus arrived at a central attractive force whose magnitude is inversely proportional to the distance from the centre. It now remains to show that the constant μ is one and the same for all the planets and comets; at present we only know that the quantity c involved in the law of areas is constant for the given motion. Let T be the period of revolution of a planet round its orbit. Since c is equal to the duplicated sector velocity and since the area of an ellipse is equal to πab we have $c = 2\pi ab/T$ and, consequently, by Kepler's third law,

$$\mu=\frac{c^2}{p}=\frac{4\pi^2a^2b^2\cdot a}{T^2b^2}=4\pi^2\frac{a^3}{T^2}=\text{const}$$

Hence, the coefficient μ (the Gaussian constant*) is one and the

* Gauss, K. F. (1777-1855), the great German mathematician.

same for all the planets and is equal to GM, where G is the gravitational constant and M is the mass of the Sun. Formula (24.10) takes the form

$$F = -\frac{GmM}{r^2}$$

and expresses *Newton's law of gravitation.*

I. Newton established this law geometrically. Binet's formulas were derived later. The derivation presented here as well as Newton's derivation are not completely rigorous because in reality not the Sun itself is fixed but the centre of mass of the system consisting of the Sun and the planet (or the comet). Kepler's laws are also approximate. We have considered the inverse problem; now we pass to the direct problem which will be investigated under the assumption that the central body occupies a fixed position.

2.2. **The Kepler-Newton Problem.** Let us consider the motion of a particle of mass m in the field of central force generated by a fixed body O (see Fig. 24.1) in accordance with the law

$$F = -\frac{m\mu}{r^2} \tag{24.11}$$

It is possible to integrate directly the differential quations of motion (see Sec. 3.4 of Chap. 13); however it is preferable to make use of first integrals because we know two of them and this allows us to avoid two integrations. The law of areas yields $r^2\, d\varphi/dt = c$. Another first integral is the energy integral. Let us compute the force function by substituting (24.11) into (24.2):

$$U = -m\mu \int \frac{dr}{r^2} = \frac{m\mu}{r}$$

Formula (15.27) expressing the energy integral is written in the form

$$\frac{1}{2} mv^2 - \frac{m\mu}{r} = \frac{1}{2} mv_0^2 - \frac{m\mu}{r_0}$$

whence

$$v^2 = \frac{2\mu}{r} + h \qquad \left(h = v_0^2 - \frac{2\mu}{r_0}\right) \tag{24.12}$$

Let us substitute into (24.12) the expression of v^2 given by Binet's first formula (this means that we make use of the law of areas):

$$c^2 \left\{ \left[\frac{d\left(\frac{1}{r}\right)}{d\varphi} \right]^2 + \frac{1}{r^2} \right\} = \frac{2\mu}{r} + h$$

We now obtain

$$\begin{aligned} \left[\frac{d\left(\frac{1}{r}\right)}{d\varphi} \right]^2 &= -\frac{1}{r^2} + \frac{2\mu}{c^2}\frac{1}{r} - \frac{\mu^2}{c^4} + \frac{\mu^2}{c^4} + \frac{h}{c^2} \\ &= -\left(\frac{1}{r} - \frac{\mu}{c^2}\right)^2 + \frac{\mu^2}{c^4} + \frac{h}{c^2} \end{aligned} \tag{24.13}$$

Equation (24.13) is the differential equation describing the motion of a particle in field of central force (24.11). In order to integrate this equation let us introduce a new function $u = u(\varphi)$ instead of $r = r(\varphi)$ with the aid of the substitution

$$\frac{1}{r}-\frac{\mu}{c^2}=\sqrt{\frac{\mu^2}{c^4}+\frac{h}{c^2}}\,u \tag{24.14}$$

It follows that

$$\frac{d\left(\frac{1}{r}\right)}{d\varphi}=\sqrt{\frac{\mu^2}{c^4}+\frac{h}{c^2}}\,\frac{du}{d\varphi}$$

and the substitution of this expression into (24.13) results in

$$\left(\frac{\mu^2}{c^4}+\frac{h}{c^2}\right)\left(\frac{du}{d\varphi}\right)^2=-\left(\frac{\mu^2}{c^4}+\frac{h}{c^2}\right)u^2+\frac{\mu^2}{c^4}+\frac{h}{c^2}$$

Cancelling by the binomial and extracting the square root we find

$$\frac{du}{d\varphi}=\pm\sqrt{1-u^2}$$

Now we separate the variables and perform integration:

$$\mp\int\frac{du}{\sqrt{1-u^2}}=\int d\varphi\,,\ \text{whence}\ \arccos u=\varphi-\varphi_0,\ u=\cos(\varphi-\varphi_0)$$

(we have taken the plus sign; the minus sign would yield $u = \sin(\varphi-\varphi_0^*)$, that is an initial phase shift). The substitution of this result into (24.14) yields

$$\frac{1}{r}=\frac{\mu}{c^2}+\sqrt{\frac{\mu^2}{c^4}+\frac{h}{c^2}}\cos(\varphi-\varphi_0) \tag{24.15}$$

The comparison of (24.15) with the standard form of an equation of a conic section (24.9) shows that the trajectory of motion is a conic section with one of its foci at the point O, and

$$\frac{1}{p}=\frac{\mu}{c^2}\,,\ \frac{e}{p}=\sqrt{\frac{\mu^2}{c^4}+\frac{h}{c^2}},\ \text{whence}\ p=\frac{c^2}{\mu}\,,\ e=\sqrt{1+\frac{hc^2}{\mu^2}} \tag{24.16}$$

The type of the trajectory is specified by the initial conditions of motion.

If $h \geqslant 0$ (the case of a comet) then the eccentricity e is greater than or equal to unity ($e \geqslant 1$) and the trajectory is a branch of a hyperbola ($e > 1$) or of a parabola ($e = 1$). If $h < 0$ (the case of a planet) the trajectory is an ellipse ($e < 1$). It should be noted that, by virtue of (24.13), we have $h \geqslant -\mu^2/c^2$; this however can also be proved directly. In addition we note that if the initial velocity is directed towards the centre the trajectory degenerates into a line segment.

2.3. Trajectories of the Earth's Artificial Satellites. Besides the gravitational field of the Earth, a spaceship or an artificial satellite

undergoes the action of gravitational fields of other celestial bodies (the Sun, the Moon, etc.). But when the distance from the Earth is not very large the main role is played by the Earth's gravitational field which, in the first approximation, can be regarded as a spherically symmetric field of central force whose centre coincides with that of the Earth. The trajectory of a spaceship can be divided into two parts: the active leg trajectory (in the motion along this part the engine works) and the inactive leg trajectory (which is travelled when the rocket engine is switched off). The determination of the inactive leg trajectory in the gravitational field of the Earth reduces to solving the Kepler-Newton problem (see Sec. 2.2). If the inactive leg trajectory of a body launched from the Earth into the outer space is an elliptic orbit the body becomes an artificial satellite of the Earth.

The *circular* (or *orbital*) *velocity* is the smallest velocity which should be imparted to a body at a geocentric distance equal to the radius of the Earth for the body to become an artificial satellite of the Earth. An artificial satellite of the Earth rotating about the Earth in a circular orbit moves with that velocity. The eccentricity of the circular orbit being equal to zero ($e = 0$) and its latus rectum being equal to the radius of the orbit ($p = r_0$), formula (24.16) yields

$$h = -\frac{\mu^2}{c^2} \text{ and } p = \frac{c^2}{\mu} = r_0, \text{ whence } h = -\frac{\mu}{r_0}$$

and formula (24.12) is written in the form

$$v_0^2 = \frac{2\mu}{r_0} - \frac{\mu}{r_0} = \frac{\mu}{r_0}$$

From this relation (under the condition that $v_0 \perp r_0$; it can be proved that this condition is necessary for the existence of the circular orbit) we obtain the following expression for the initial velocity v_0:

$$v_0 = \sqrt{\frac{\mu}{r_0}} = \sqrt{\frac{GM}{r_0}} = \sqrt{g(r_0)\, r_0}$$

where $g(r_0) = GM/r_0^2$ is the acceleration of free fall at a distance r_0 from the centre of the Earth (see Example 13.7). Putting $r_0 = R$, where R is the radius of the Earth, we find the circular velocity (the influence of air resistance is neglected):

$$v_{\text{cir}} = \sqrt{gR} = \sqrt{9.81 \cdot 6.37 \cdot 10^6} = 7910 \text{ m/s}$$

The *escape velocity* is the smallest velocity which should be imparted to a body on the Earth's surface for the body to leave the region of the Earth's attraction and to become a satellite of the Sun. For the realization of this condition it is necessary that the trajectory of the body should be parabolic ($e = 1$). By formula (24.16), for

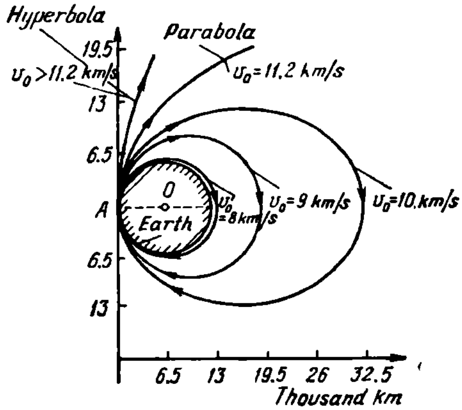

Fig. 24.3

$e = 1$ we have $h = 0$, and then formula (24.12) yields the following expression for the magnitude of the escape velocity:

$$v_0 = \sqrt{\frac{2\mu}{r_0}}$$

Now we compute the value of the escape velocity (neglecting the influence of air resistance):

$$v_{esc} = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} = \sqrt{2 \cdot 9.81 \cdot 6.37 \cdot 10^6} = 11\,170 \text{ m/s}$$

A family of trajectories of satellites and cosmic rockets in the field of the Earth's gravitation is shown in Fig. 24.3 for various values of the initial velocity v_0 ($v_0 \perp r_0$). Here O is the centre of the Earth which is at one of the foci of the trajectory and v_0 is the velocity at the end A of the active leg trajectory.

In the case of elliptic trajectories the body either returns to the Earth (when $v_0 < v_{cir}$; this is the case for ballistic missiles) or becomes an artificial satellite of the Earth (when $v_{cir} \leqslant v_0 < v_{esc}$). For a parabolic trajectory ($v_0 = v_{esc}$) or a hyperbolic trajectory ($v_0 > v_{esc}$) the body becomes a satellite of the Sun or leaves the solar system (when $v_0 > 16.7$ km/s; the velocity of 16.7 km/s is called the *solar escape velocity*).

Chapter 25 Fundamentals of Mechanics of a Thread

In mechanics by a *thread* is meant a thin material line (a wire) whose axis can assume any shape under the action of external forces. If the length of the thread remains constant for any of its motion

and for any action exerted on it the thread is said to be *inextensible*. Finally, if the thread does not resist bending and torsion it is called *ideal*. In the present chapter we shall consider only an ideal inextensible thread.

§ 1. Equilibrium of an Ideal Inextensible Thread in a Stationary Force Field

Let a thread of a finite length l be under the action of *mass forces* distributed along the thread and *tensile forces* applied to its ends.

In what follows the mass forces will be related to the length of the corresponding part of the thread; the mass of the thread will be treated in the same way; we shall also use the usual notation $\boldsymbol{F}$ for the forces and $\mu = dm/ds$ for the density.

Let us consider a thread in equilibrium (Fig. 25.1) and separate out its part Δs. All the remaining parts of the thread not included in that part are (mentally) discarded. For the part Δs to retain its equilibrium it is necessary to apply some tangential forces $\boldsymbol{T}$ and $\boldsymbol{T}_1$ at the ends A and B of that part; these forces are called *tensions*. Consequently, the active external forces applied to the element Δs of the thread are $\boldsymbol{F}\,\Delta m$, $\boldsymbol{T}$ and $\boldsymbol{T}_1$.

At each point of the element there will also be two opposite internal forces $\boldsymbol{T}'$ and $\boldsymbol{T}'' = -\boldsymbol{T}'$ (see Fig. 25.1).

It should be noted that the distance s along the thread is reckoned as the arc length from an arbitrarily chosen point which is taken, depending on the character of the problem, either on the thread itself (in statics) or on the fixed axis of the thread (when the thread is in motion along a contour).

1.1. Vector Form of the Equilibrium Equation of a Free Element of a Thread. We assume that for the element Δs the force $\boldsymbol{T}_1$ is equal to $\boldsymbol{T}_1 = \boldsymbol{T} + \Delta\boldsymbol{T}$; then the equilibrium equation

$$\boldsymbol{T} + \boldsymbol{T}_1 + \boldsymbol{F}\,\Delta m = 0$$

of the element of the thread can be written in the form

$$\frac{\Delta \boldsymbol{T}}{\Delta s} = -\boldsymbol{F}\frac{\Delta m}{\Delta s}$$

Now, passing to the limit for $\Delta s \to 0$, we obtain

$$\frac{d\boldsymbol{T}}{ds} = -\mu\boldsymbol{F}$$

that is

$$\frac{1}{\mu}\frac{d\boldsymbol{T}}{ds} + \boldsymbol{F} = 0 \qquad (25.1)$$

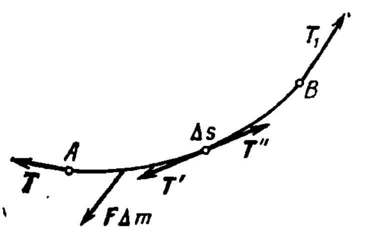

Fig. 25.1

Since the tension T is always along the tangent, that is along the unit vector $\boldsymbol{\tau}$, we can write $\boldsymbol{T} = T\boldsymbol{\tau}$.

Let us take the natural axes with unit vectors $\boldsymbol{\tau}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ (see Fig. 7.9) at the initial point of the element Δs of the thread; then

$$\frac{d\boldsymbol{T}}{ds} = \frac{d(T\boldsymbol{\tau})}{ds} = \frac{dT}{ds}\boldsymbol{\tau} + T\frac{d\boldsymbol{\tau}}{ds}$$

From what was established in particle kinematics (see Sec. 3.3 of Chap. 7) it follows that for the limiting position of the plane passing through three points lying infinitely close to one another the tensions and the velocities at two points of the element of the thread are in the osculating plane and that

$$\frac{d\boldsymbol{\tau}}{ds} = \frac{1}{\rho}\boldsymbol{n}$$

where ρ is the radius of curvature of that element of the thread. Therefore

$$\frac{d\boldsymbol{T}}{ds} = \frac{dT}{ds}\boldsymbol{\tau} + \frac{T}{\rho}\boldsymbol{n}$$

and vector equilibrium equation (25.1) of a free element of a thread takes the form

$$\frac{1}{\mu}\left(\frac{dT}{ds}\boldsymbol{\tau} + \frac{T}{\rho}\boldsymbol{n}\right) + \boldsymbol{F} = 0 \qquad (25.2)$$

1.2. Equilibrium Equations in Terms of the Projections on Natural Axes. If we project equation (25.2) on the natural axes with unit vectors $\boldsymbol{\tau}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ (see Fig. 7.9) then we obviously obtain the equations

$$\frac{1}{\mu}\frac{dT}{ds} + F_\tau = 0, \quad \frac{1}{\mu}\frac{T}{\rho} + F_n = 0, \quad F_b = 0 \qquad (25.3)$$

where F_τ, F_n and F_b are the projections of the mass force $\boldsymbol{F}$ on the tangent, the principal normal and the binormal.

The last equation (25.3) indicates that the force $\boldsymbol{F}$ of the field acting at each point of the free thread entirely lies in the osculating plane.

1.3. Equilibrium Equations of a Thread in Terms of the Projections on Cartesian Coordinate Axes. Analogously, from vector equation (25.2) we can obtain another three scalar equations by projecting the vectors $\boldsymbol{\tau}$ and $\boldsymbol{n}/\rho$ involved in it on three arbitrarily chosen axes of a rectangular Cartesian coordinate system. It can easily be shown that for a point of the thread with coordinates x, y and z these projections are expressed in terms of the quantities

$$\frac{dx}{ds} \text{ and } \frac{d^2x}{ds^2}, \quad \frac{dy}{ds} \text{ and } \frac{d^2y}{ds^2}, \quad \frac{dz}{ds} \text{ and } \frac{d^2z}{ds^2}$$

respectively, and the equilibrium equations take the form

$$\frac{1}{\mu}\left(\frac{dT}{ds}\frac{dx}{ds}+T\frac{d^2x}{ds^2}\right)+F_x=0,\quad \frac{1}{\mu}\left(\frac{dT}{ds}\frac{dy}{ds}+T\frac{d^2y}{ds^2}\right)+F_y=0$$
$$\frac{1}{\mu}\left(\frac{dT}{ds}\frac{dz}{ds}+T\frac{d^2z}{ds^2}\right)+F_z=0 \tag{25.4}$$

These equations can be written in a more compact form:

$$\frac{1}{\mu}\frac{d}{ds}\left(T\frac{dx}{ds}\right)+F_x=0,\quad \frac{1}{\mu}\frac{d}{ds}\left(T\frac{dy}{ds}\right)+F_y=0$$
$$\frac{1}{\mu}\frac{d}{ds}\left(T\frac{dz}{ds}\right)+F_z=0 \tag{25.5}$$

To the system of equations we have written the ordinary condition for the direction cosines of the tangent must be added:

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1$$

Equations (25.5) form a system of six differential equations of the sixth order with respect to x, y, z and T. Its general solution has the form

$$x=x(s,\ c_1,\ c_2,\ \ldots,\ c_6),\quad y=y(s,\ c_1,\ c_2,\ \ldots,\ c_6)$$
$$z=z(s,\ c_1,\ c_2,\ \ldots,\ c_6),\quad T=T(s,\ c_1,\ c_2,\ \ldots,\ c_6)$$

Determining the arbitrary constants c_1, c_2, ..., c_6 from the boundary conditions we can find the complete solution of the problem stated, that is determine the shape of the thread and the tension at any of its points.

EXAMPLE 25.1. *Catenary.* Let us find the equilibrium form of a thread* in the gravity field (Fig. 25.2). In this case we must put $F_x=0$ and $F_y=-g$ in the first two equations (25.5), and the equilibrium equations of the thread take the form

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)=0,\quad \frac{d}{ds}\left(T\frac{dy}{ds}\right)=\gamma$$

where $\gamma=g\mu$ is the weight of unit length of the thread, and since the thread is homogeneous we have $\gamma=\text{const}$. From the first equation it follows that

$$T\frac{dx}{ds}=T_0=\text{const}\qquad (T_0=T|_{x=0})$$

Hence, the projection of the tension T on the x-axis is a constant quantity. The substitution of $T=T_0\,ds/dx$ into the second equilibrium equation results in

$$\frac{dy'}{ds}=k\quad\left(y'=\frac{dy}{dx},\quad k=\frac{\gamma}{T_0}=\text{const}\right)$$

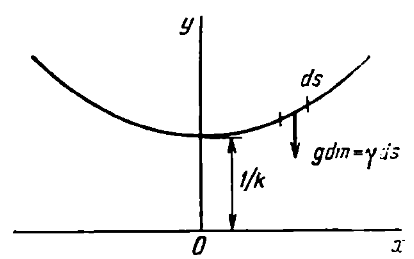

Fig. 25.2

* We remind the reader that here we consider an ideal inextensible thread which is supposed to be homogeneous.

Now we substitute $ds=\sqrt{1+y'^2}\,dx$ (see [1]) into the last equation, separate the variables and integrate taking into account the boundary condition $y'(0)=0$:

$$\int_0^{y'}\frac{dy'}{\sqrt{1+y'^2}}=k\int_0^x dx$$

The left-hand member of the last relation is a tabular integral and we obtain

$$\ln\left(y'+\sqrt{1+y'^2}\right)=kx$$

whence

$$y'+\sqrt{1+y'^2}=e^{kx}$$

Solving the last equation with respect to $y'=dy/dx$ we find

$$\frac{dy}{dx}=\frac{1}{2}\left(e^{kx}-e^{-kx}\right)$$

Finally, we multiply the last equation by dx and integrate repeatedly under the condition $y(0)=1/k$:

$$\int_{1/k}^{y}dy=\frac{1}{2}\int_0^x\left(e^{kx}-e^{-kx}\right)dx$$

From the last relation we obtain the equation describing the equilibrium form of the thread:

$$y=\frac{1}{2k}\left(e^{kx}+e^{-kx}\right)\qquad\left(k=\frac{\gamma}{T_0}\right)$$

This curve is called a *catenary*. The determination of k from the boundary conditions was carried out at the end of Example 17.6.

EXAMPLE 25.2. *Parabolic Thread.* Let us consider the cable AB of a suspension bridge which is under the action of a continuously distributed vertical load (Fig. 25.3). The element ds of the cable is under the action of the force of gravity $\beta\,dx$, where β is the load per unit length. The first two equilibrium equations (25.5) are written in the form

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)=0,\qquad\frac{d}{ds}\left(T\frac{dy}{ds}\right)=\beta\frac{dx}{ds}$$

As in Example 25.1, from the first equation we find

$$T\frac{dx}{ds}=T_0,\ \text{that is}\ T=T_0\frac{ds}{dx}\qquad\left(T_0=T|_{x=0}\right)$$

Substituting this expression of T into the second equation we obtain

$$\frac{d}{dx}\left(T_0\frac{dy}{dx}\right)=\beta$$

whence

$$\frac{d^2y}{dx^2}=\varkappa\qquad\left(\varkappa=\frac{\beta}{T_0}\right)$$

Finally, integrating twice, we derive

$$y=\frac{1}{2}\varkappa x^2+c_1x+c_2$$

where c_1 and c_2 are arbitrary constants of integration.

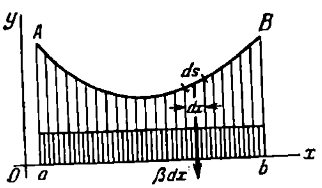

Fig. 25.3

Consequently the equilibrium shape of the cable of the suspension bridge is a parabola with vertical axis. The constants c_1, c_2 and $\varkappa$ are found from the boundary conditions, that is from the coordinates of the points A and B and the length of the cable.

Up till now we considered a free thread; now we proceed to the investigation of the equilibrium of a thread lying on a surface.

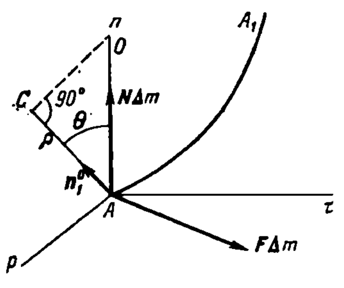

Fig. 25.4

1.4. Equilibrium Equations of a Thread on a Smooth Surface in Terms of the Projections on Local Coordinate Axes. Let the arc AA_1 (Fig. 25.4) represent a part of a thread which is in equilibrium on a smooth surface. Let us draw through the initial point A the axes $A\tau$ and An along the tangent to the thread and the normal to the surface at the point A; the line segment AO along the normal to the surface is taken to be equal to the radius of curvature R of the section of the surface by the plane passing through that normal to the surface and through the tangent to the thread. The radius of curvature ρ of the curve AA_1 itself is represented by the line segment AC of the principal normal n_1^0 of the curve. The angle between the directions of AC and AO is denoted by θ. By Meusnier's* theorem proved in the course of differential geometry,

$$\rho = R\cos\theta$$

As we know, the equilibrium equation for a constrained particle must include the reaction forces; therefore the equilibrium equation for the part of the thread under consideration has the form analogous to equation (25.2) to which the reaction force N along An must be added.

Projecting the forces F and N on the local axes $A\tau$, An and Ap we derive from (25.3) the equations

$$\frac{1}{\mu}\frac{dT}{ds}+F_\tau=0, \quad \frac{1}{\mu}\frac{T}{\rho}\cos\theta+F_n+N=0$$
$$\frac{1}{\mu}\frac{T}{\rho}\sin\theta+F_p=0 \tag{25.6}$$

1.5. Equilibrium Equations of a Thread on a Smooth Surface in Terms of the Projections on the Axes of a Rectangular Cartesian Coordinate System. Let the equation of the surface on which a part of the thread lies be written in the form $f(x, y, z) = 0$; then, as is known from the theory of motion of a particle on a surface (see Sec. 1.1 of Chap. 16, formulas (16.2), (16.3) and (16.5)), the projec-

* Meusnier, J. B. M. (1754-1793), a French engineer.

tions of the normal reaction N of the surface are expressed thus:

$$N_x = \lambda \frac{\partial f}{\partial x}, \quad N_y = \lambda \frac{\partial f}{\partial y}, \quad N_z = \lambda \frac{\partial f}{\partial z}$$

As before, these projections of N must be added to the left-hand side of equation (25.2) and therefore, by analogy with equations (25.5), we obtain the three equations

$$\begin{aligned} &\frac{1}{\mu} \frac{d}{ds}\left(T \frac{dx}{ds}\right) + F_x + \frac{\lambda}{\mu} \frac{\partial f}{\partial x} = 0 \\ &\frac{1}{\mu} \frac{d}{ds}\left(T \frac{dy}{ds}\right) + F_y + \frac{\lambda}{\mu} \frac{\partial f}{\partial y} = 0 \\ &\frac{1}{\mu} \frac{d}{ds}\left(T \frac{dz}{ds}\right) + F_z + \frac{\lambda}{\mu} \frac{\partial f}{\partial z} = 0 \end{aligned} \tag{25.7}$$

expressing the equilibrium condition in Cartesian coordinates; to these equations we must naturally add the relations

$$f(x, y, z) = 0, \quad \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$$

By analogy with what was mentioned in Sec. 1.3, the sought-for quantities x, y, z, T and λ are found as functions of s. As is known from the theory of differential equations, in the case under consideration the number of arbitrary constants reduces to four, namely c_1, c_2, c_3 and c_4, because here we have the additional equation $f(x, y, z) = 0$ of the surface connecting the coordinates x, y and z.

1.6. Equilibrium of a Thread on a Rough Cylindrical Surface. Euler's Formula. Let a thread be passed round the surface of a rough cylinder along its normal section (Fig. 25.5) and let an external active force Q be applied to one of its ends. We shall denote the coefficient of sliding friction by k, the radius of the cylinder (and of its section) by R, the central angle subtended by the part of the arc of the thread lying on the cylinder by α and the arc AB itself by l. The statement of the problem reads: what is the minimum magnitude of the force Q_0 which must be applied to the other end B of the thread for the part AB of the thread to be in equilibrium? Let us separate out an element Δl of the arc AB of the thread with central angle $\Delta\alpha$; the length of this element is $\Delta l = R\,\Delta\alpha$.

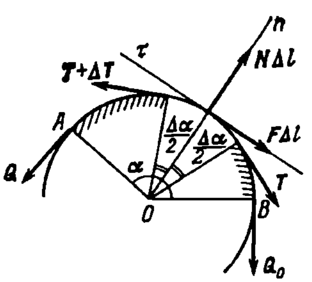

Fig. 25.5

Since the tension at the initial point of the part AB of the thread (we trace this part from the driving end A to the driven end B) is balanced by the tension at the other end B and by the friction force the magnitude of T_1 must exceed that of T and, according to Sec. 1.1,

$$T_1 = T + \Delta T \quad (\Delta T > 0)$$

We shall denote the normal reaction applied to the element Δl by $N\,\Delta l$; then the modulus of the friction force is expressed by

$$kN\,\Delta l$$

The difference ΔT of the tensions is balanced by the friction forces and hence

$$\Delta T = kN\,\Delta l$$

Finally, the quantity $N\,\Delta l$ is found from the equilibrium equation for the element Δl after all the forces acting on that element are projected on the axis On. It is obvious that

$$N\,\Delta l = T\sin\frac{\Delta\alpha}{2} + (T+\Delta T)\sin\frac{\Delta\alpha}{2} = 2T\sin\frac{\Delta\alpha}{2} + \Delta T\sin\frac{\Delta\alpha}{2}$$

In what follows we shall pass to the limit for $\Delta l \to 0$, and therefore the sine of the small angle $\Delta\alpha/2$ can be replaced by the angle $\Delta\alpha/2$ itself, and the second term on the right-hand side can be discarded as an infinitesimal of the second order; we thus obtain

$$N\,\Delta l = 2T\,\frac{\Delta\alpha}{2} = T\,\Delta\alpha$$

As we found earlier, $\Delta T = kN\,\Delta l$ and hence

$$\Delta T = kT\,\Delta\alpha$$

that is in the limit, for $\Delta\alpha \to 0$, we have

$$\frac{dT}{d\alpha} = kT, \text{ whence } \frac{dT}{T} = k\,d\alpha$$

Integrating with respect to the variable α and taking the whole part of the arc AB contacting the surface and corresponding to the increase of the angle α in the direction from A to B indicated above we obtain

$$\int_{Q_0}^{Q} \frac{dT}{T} = k\int_0^{\alpha} d\alpha, \text{ whence } \ln\frac{Q}{Q_0} = k\alpha$$

Finally,

$$\frac{Q}{Q_0} = e^{k\alpha}, \text{ that is } Q_0 = Qe^{-k\alpha} \tag{25.8}$$

We have derived *Euler's formula* representing the solution of the problem stated above.

For equilibrium this formula can also be written in terms of the tension T: in this case the forces Q and Q_0 do in fact play the role of tensions T and T_0 applied to the driving and the driven ends of the thread.

However, it should be stressed that this identification of the external active forces with the tensions is possible only in the case of equilibrium; in this connection it should also be taken into account

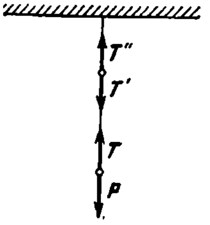

Fig. 25.6

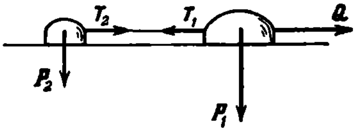

Fig. 25.7

that the tension is interpreted here as an external force P (Fig. 25.6) transferred along the inextensible thread to any of its points or al the reaction T of the thread or, finally, as a system of the internal forces T' and T'' balancing at each point of the thread.

Below we demonstrate by an example that in the case of motion it is incorrect to identify the external active forces with the tensions.

EXAMPLE 25.3. Two particles of weights P_1 and P_2 (Fig. 25.7) connected by an inextensible thread AB lie on a smooth horizontal plane and are in motion under the action of a horizontal driving force Q. Compare the magnitudes of the driving force Q and the tension T.

Solution. Let us consider the motion of the system as a whole; from Newton's second law we find

$$\frac{P_1+P_2}{g}w=Q$$

whence follows that the acceleration is

$$w=\frac{Q}{P_1+P_2}g$$

Now we consider separately the motion of one of the particles, say B; let us determine the tension:

$$T_2=\frac{P_2}{g}w$$

It follows that

$$T_2=T_1=\frac{P_2}{P_1+P_2}Q$$

Since

$$\frac{P_2}{P_1+P_2}<1$$

we see that T is always less than Q, and the greater the weight P_1 in comparison with the weight P_2 the greater is the difference between Q and T.

This example and the considerations in the introduction to the present chapter show that it is also incorrect to assume that the external forces are equal to zero while the tensions remain nonzero.

In any interpretation of forces of tension, both as tensions at the terminal points of a finite part of the thread and as reactions, we must consider these forces in connection with the action of the external active forces.

§ 2. Fundamentals of Dynamics of an Ideal Inextensible Thread

2.1. Vector Equation of Motion of a Thread. Let us consider the motion of an element Δs of an inextensible thread acted upon by forces $\boldsymbol{F}$ of a stationary force field related to unit mass of the thread and tensions $\boldsymbol{T}$ and $\boldsymbol{T}_1$ applied to its ends (Fig. 25.8). As before, $\boldsymbol{T}_1 = -\boldsymbol{T} + \Delta\boldsymbol{T}$. Denoting by $\boldsymbol{w}^*$ the acceleration of the element and by $\Delta m = \mu\,\Delta s$ its mass we obtain, according to Newton's second law, the equation

$$\boldsymbol{w}^*\,\Delta m = \boldsymbol{F}\,\Delta m + \Delta\boldsymbol{T}$$

It can be written as

$$\mu\boldsymbol{w}^*\Delta s = \mu\boldsymbol{F}\,\Delta s + \Delta\boldsymbol{T}, \text{ whence } \frac{\Delta\boldsymbol{T}}{\Delta s} = \mu\,(\boldsymbol{w}^* - \boldsymbol{F})$$

Passing to the limit for $\Delta s \to 0$ we derive the vector equation of motion of the thread:

$$\frac{\partial\boldsymbol{T}}{\partial s} = \mu\,(\boldsymbol{w} - \boldsymbol{F}) \tag{25.9}$$

2.2. The Case of Stationary Motion. Let us consider the motion of a thread characterized by the following properties:

(1) the curve whose shape the thread assumes moves translatory in space without changing its configuration;

(2) the thread itself is in a contour motion, that is it moves along that geometric curve as if it flew along its contour. According to the terminology suggested by A. P. Minakov such a motion of the thread is called *stationary*. For this case let us write equation (25.9) in the form

$$\frac{1}{\mu}\frac{\partial\boldsymbol{T}}{\partial s} = \frac{\partial\boldsymbol{v}}{\partial t} - \boldsymbol{F} \tag{25.10}$$

and resolve the velocity $\boldsymbol{v}$ of the particle of the thread into two components: the relative (contour) velocity $\boldsymbol{u}$ and the transportation velocity $\boldsymbol{v}_{tr}$:

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{v}_{tr}$$

Differentiating this equality with respect to time and then resolving the accelerations $\partial\boldsymbol{u}/\partial t$ and $\partial\boldsymbol{v}_{tr}/\partial t$ along the natural axes τ, $\boldsymbol{n}$ and $\boldsymbol{b}$

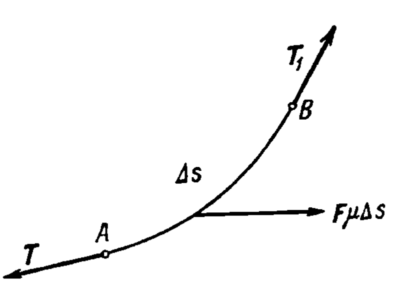

Fig. 25.8

drawn at the given point we obtain

$$w=\left(\frac{\partial u}{\partial \tau}+w_{\tau}^{tr}\right)\tau+\left(\frac{u^2}{\rho}+w_{n}^{tr}\right)n+w_{b}^{tr}b$$

where w_{τ}^{tr}, w_{n}^{tr} and w_{b}^{tr} are the projections of the transportation acceleration w_{tr} of the particle of the thread on the natural axes, and ρ is the radius of curvature of the thread.

Coming back to equation (25.10) and denoting the projections of the force F on the indicated axes as F_{τ}, F_n and F_b respectively we obtain the three equations

$$\frac{1}{\mu}\frac{\partial T}{\partial s}=\frac{\partial u}{\partial \tau}+w_{\tau}^{tr}-F_{\tau},\quad \frac{1}{\mu}\frac{T}{\rho}=\frac{u^2}{\rho}+w_{n}^{tr}-F_n \qquad (25.11)$$
$$0=w_{b}^{tr}-F_b$$

Let us transfer the term u^2/ρ in the second equation to the left:

$$\frac{1}{\mu}\frac{T-\mu u^2}{\rho}=w_{n}^{tr}-F_n$$

Now we denote $T^*=T-\mu u^2$, take into account that $\partial u/\partial s=0$ for an inextensible thread and assume that μ is constant: $\mu=\text{const}$; then

$$\frac{\partial T^*}{\partial s}=\frac{\partial T}{\partial s}$$

In this case equations (25.11) take the form

$$\frac{1}{\mu}\frac{\partial T^*}{\partial s}=\frac{\partial u}{\partial t}+w_{\tau}^{tr}-F_{\tau},\quad \frac{1}{\mu}\frac{T^*}{\rho}=w_{n}^{tr}-F_n,\quad 0=w_{b}^{tr}-F_b \quad (25.12)$$

If, in addition, it is known that the contour motion of the thread is uniform then $\partial u/\partial t=0$; introducing the notation

$$w_{\tau}^{tr}-F_{\tau}=-F_{\tau}^*,\quad w_{n}^{tr}-F_n=-F_n^*,\quad w_{b}^{tr}-F_b=-F_b^*$$

we now bring equations (25.12) to the form

$$\frac{\partial T^*}{\partial s}=-\mu F_{\tau}^*,\quad \frac{T^*}{\rho}=-\mu F_n^*,\quad 0=-\mu F_b^* \qquad (25.13)$$

From the formal point of view these equations have the form of static equations of equilibrium of the thread (see equations (25.3)).

Consequently, *if a homogeneous ideal inextensible thread moves uniformly in its relative contour motion and simultaneously undergoes translatory transportation then both the shape of the thread and its tension satisfy the equilibrium equations of the thread in which, however, besides the acting forces, the additional inertial forces must be present (that is the transportation inertial forces*; see Sec. 1.1 of Chap. 16); *the tension at all the points of the thread exceeds the corresponding static tension by one and the same quantity* μu^2.

If, in addition to the above conditions, the transportation is also uniform then $w_{\tau}^{tr}=w_{n}^{tr}=w_{b}^{tr}=0$ and consequently $F_{\tau}^*=F_{\tau}$,

$F_n^* = F_n$, $F_b^* = F_b$, which means that in the given force field the ideal in xtensible and homogeneous thread preserves the same form as in the state of rest.

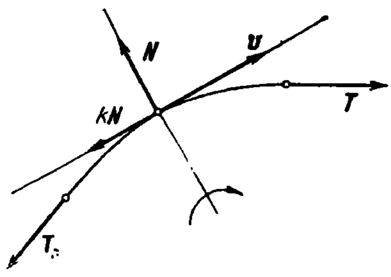

Fig. 25.9

Finally, we mention that when there are no forces F at all the thread which is not in a contour motion may preserve any shape; in this case under the conditions imposed earlier for a thread moving along itself there appears a viscous effect (the "Etkin-Radinger effect").

The phenomenon of this effect is that a thread moving along itself with velocity equal to that of the propagation of transverse waves along that thread preserves its shape (which is given to it initially) in the absence of a force field.

2.3. Motion and Tension of a Thread Sliding along a Fixed Rough Plane Curve. The A. P. Minakov Generalization of Euler's Formula. Let an ideal inextensible thread moving on a rough surface with friction coefficient k slide along an arc on that surface (Fig. 25.9). We assume that the thread is inextensible, that is $\partial v/\partial s = 0$, and that there is no transportation; then taking into account the directions of the friction force kN and the reaction force N we write equations (25.11) in the form

$$\frac{\partial T}{\partial s} = \mu k N + \mu w, \quad \frac{T}{\rho} = \mu N + \frac{\mu v^2}{\rho} \tag{25.14}$$

where v is the modulus of the velocity of the thread (which is assumed to be equal for all the points of the thread) and w is the acceleration of the contour motion, that is the tangential acceleration of the particles of the thread. From the second equation we find N:

$$N = \frac{T - \mu v^2}{\mu \rho} \tag{25.15}$$

Now we state the problem more precisely: let an ideal inextensible thread slide along a rough plane curve of an arbitrary shape. In this case substituting the value of N found from (25.15) into the first equation (25.14) and taking into account that $\rho = \rho(s)$ in the case under consideration we obtain

$$\frac{\partial T}{\partial s} = \frac{1}{\rho} [k (T - \mu v^2) + \mu \rho w]$$

whence

$$\frac{\partial T}{\partial s} - T \frac{k}{\rho} + \mu \left(\frac{k v^2}{\rho} - w \right) = 0 \tag{25.16}$$

This is a linear differential equation of the first order; the well-known substitution

$$T^* = T - \mu v^2$$

transforms this equation into

$$\frac{\partial T^*}{\partial \varphi} - kT^* = \mu w \rho$$

Multiplying both the members of the last equation by $e^{-k\varphi}$ we obtain

$$\frac{\partial}{\partial \varphi}(T^* e^{-k\varphi}) = \mu w \rho e^{-k\varphi}$$

whence, after integration,

$$T = \mu v^2 + \left(\mu w \int \rho e^{-k\varphi}\, d\varphi + C\right) e^{k\varphi} \tag{25.17}$$

The constant C is determined from the condition that $T = T_0$ for $\varphi = 0$. In particular, if $\rho = \text{const} = R$, that is if the thread moves along an arc of a circle of constant radius R, and the central angle subtended by the arc is equal to φ, then, after the constant C is determined in the indicated manner, T is expressed by the formula

$$T = \mu v^2 + (T_0 - \mu v^2)\, e^{k\varphi} + \frac{\mu w R}{k}(e^{k\varphi} - 1) \tag{25.18}$$

Formula (25.18) *generalizes Euler's formula.*

Let us consider the following *special cases*:

(1) *The motion of the thread along its contour is uniform*, that is $v = \text{const} = v_0$, $w = 0$. Then formula (25.18) yields

$$T = \mu v_0^2 + (T_0 - \mu v_0^2)\, e^{k\varphi} \tag{25.19}$$

(2) Let us compute the tension of the thread at the *initial instant of its motion.* This instant is characterized by the condition that $v = 0$, $w \neq 0$. Consequently, for the initial instant we have

$$T = T_0 e^{k\varphi} + \frac{\mu w R}{k}(e^{k\varphi} - 1) \tag{25.20}$$

(3) *The driven end of the thread is free*, that is there is no tension at it ($T_0 = 0$). Then

$$T = \mu \frac{R}{k}\left(w - \frac{kv^2}{R}\right)(e^{k\varphi} - 1) \tag{25.21}$$

Besides the condition indicated above, at the initial instant of motion of such a thread the equality

$$v = 0$$

is fulfilled and hence *for this case* we have

$$T = \frac{\mu R w}{k}(e^{k\varphi} - 1) \tag{25.22}$$

Formulas (25.21) and (25.22) imply an important physical conclusion: even if the driven end of the thread is not fixed and there is no tension at it, there is a nonzero tension at the driving end of the thread both in the case of a nonuniform motion of the thread and at the initial instant of motion.

Finally, it is also interesting to determine the normal reaction N of the surface: the substitution of the value of T found from (25.18) into formula (25.14) results in

$$N=\frac{T_0-\mu v^2}{\mu R}e^{k\varphi}+\frac{w}{k}(e^{k\varphi}-1) \tag{25.23}$$

For a uniform motion we have $w=0$ and

$$N=\frac{T_0-\mu v^2}{\mu R}e^{k\varphi}$$

From this relation for the modulus of the reaction N and consequently for the pressure exerted on the arc of the curve at the initial point, that is for the value $\varphi=0$ of the angle, we obtain

$$N(0)=\frac{T_0-\mu v^2}{\mu R}$$

An interesting case is the one when the thread moves with constant velocity

$$v=\sqrt{\frac{T_0}{\mu}}$$

It is evident that for this velocity there is no pressure at all the points of the circle contacting with the moving thread.

For a more subtle investigation the computations we have presented are insufficiently accurate because in reality the force of sliding friction does not obey Amonton's simple law $F_{fr}=kN$ but is described by more precise Coulomb's formula $F_{fr}=kN+A$ which takes into account the forces of molecular adhesion. Further generalization of the above formulas (obtained by A. P. Minakov) to the case of this more complicated formula for the friction force was carried out by V. S. Shchedrov.

Appendix

Simplified Derivation of Basic Principles of Dynamics of a System of Particles for Absolute Motion

In this Appendix we present a simplified derivation of basic principles of dynamics of a system of particles without using (in contrast to Chap. 19) the general equation of dynamics. It should also be stressed that in the simplified derivation of the first two principles their statements include *constraint reactions* (in contrast to Chap. 19).

(A) Principle of Momentum for a System of Particles and Principle of Motion of the Centre of Mass. We shall consider a system of n particles (see Sec. 1.1 of Chap. 17) with masses $m_1, m_2, \ldots, m_n$. According to the axiom of constraints (see Sec. 1.2 of Chap. 17), let us discard the constraints and replace their action by the corresponding reactions. By $F_\nu^{\text{ext}} = X_\nu^{\text{ext}} i + Y_\nu^{\text{ext}} j + Z_\nu^{\text{ext}} k$ we shall denote the resultant of *all the external forces* (both the given active forces and the passive forces, that is the constraint reactions) acting on the particle m_ν ($\nu = 1, 2, \ldots, n$). Further, by $F_\nu^{(\text{int})}$ we shall denote the resultant of *all the internal forces* acting on that particle (Fig. A.1).

PRINCIPLE OF MOMENTUM FOR A SYSTEM OF PARTICLES (*in differential form*): ***the time derivative of the momentum of a system of particles is equal to the resultant force vector of all the external forces (both active and passive) acting on that system.***

Proof. The motion of each of the particles of the system obeys Newton's second law:

$$m_\nu \frac{dv_\nu}{dt} = F_\nu^{\text{ext}} + F_\nu^{(\text{int})} \quad (\nu = 1, 2, \ldots, n) \tag{1}$$

Adding geometrically these vector equalities we obtain

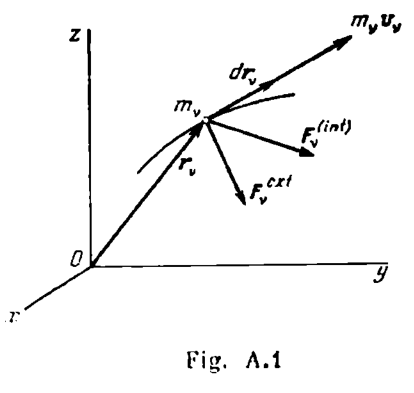

Fig. A.1

$$\sum_{\nu=1}^{n} m_\nu \frac{dv_\nu}{dt} = \sum_{\nu=1}^{n} F_\nu^{\text{ext}} + \sum_{\nu=1}^{n} F_\nu^{(\text{int})} \tag{2}$$

Let us transform the left-hand side of the last equality:

$$\sum_{\nu=1}^{n} m_\nu \frac{dv_\nu}{dt} = \frac{d}{dt} \sum_{\nu=1}^{n} m_\nu v_\nu = \frac{dQ}{dt}$$

Here we have used the fact that, in the first place, the derivative of a vector sum is equal to the vector sum of the derivatives, and the masses of the

particles are assumed to be constant, and, in the second place, the vector $\sum m_\nu v_\nu$ represents the momentum of the system of particles (see Sec. 1.1 of Chap. 19).

Now let us consider the right-hand side of (2). We note that, by virtue of Newton's third law, the resultant force vector of the internal forces of a system of particles is equal to zero at any instant:

$$\sum_{\nu=1}^{n} F_\nu^{(\mathrm{int})}=0$$

Indeed, as is known, the components of the internal forces appearing in the interaction between any two particles of the system are equal in their magnitude and opposite in direction. Let us denote by R^{ext} the *resultant force vector of all the external forces* (including the constraint reactions) acting on the particles of the system:

$$R^{\mathrm{ext}}=\sum_{\nu=1}^{n} F_\nu^{\mathrm{ext}} \tag{3}$$

Finally, for any instant during the motion of the system, we derive from (2) the equality

$$\frac{dQ}{dt}=R^{\mathrm{ext}} \tag{4}$$

which completes the proof of the principle.

PRINCIPLE OF MOMENTUM FOR A SYSTEM OF PARTICLES (*in finite form*): *the change of the projection of the momentum of a system of particles on a fixed or on an inertial axis during the time interval under consideration is equal to the projection of the impulse of the resultant force vector of all the external forces on that axis during the same time.*

Proof. Let us multiply equality (4) by dt:

$$dQ = R^{\mathrm{ext}}\,dt$$

The integration of this equality with respect to time from the initial instant $t=0$ to the terminal instant t yields

$$Q(t)-Q(0)=\int_0^t R^{\mathrm{ext}}\,dt \tag{5}$$

Here

$$Q(t)=\sum_{\nu=1}^{n} m_\nu v_\nu(t), \quad Q(0)=\sum_{\nu=1}^{n} m_\nu v_\nu(0)$$

The integral on the right-hand side of (5) is called the impulse (see Sec. 1.1 of Chap. 15) of the resultant force vector of the external forces acting on the system. Projecting vector equality (5) on fixed (or inertial; see Sec. 1.2 of Chap. 13) axes Ox, Oy and Oz we obtain

$$Q_x(t)-Q_x(0)=\int_0^t R_x^{\mathrm{ext}}\,dt$$

$$Q_y(t)-Q_y(0)=\int_0^t R_y^{\mathrm{ext}}\,dt \tag{6}$$

$$Q_z(t) - Q_z(0) = \int_0^t R_z^{\text{ext}}\, dt$$

where

$$Q_x(t) = \sum_{\nu=1}^{n} m_\nu v_x^\nu(t),\ Q_x(0) = \sum_{\nu=1}^{n} m_\nu v_x^\nu(0),\ R_x^{\text{ext}} = \sum_{\nu=1}^{n} X_\nu^{\text{ext}}$$

(for the projections on the axes Oy and Oz we have analogous expressions). The principle is proved.

Here we also present

PRINCIPLE OF MOTION OF THE CENTRE OF MASS OF A SYSTEM: *the centre of mass of a system of particles moves as if it were a particle possessing the whole mass of the system to which were applied a force equal to the resultant vector of all the external forces (including the constraint reactions) acting on the given system.*

Proof. By the lemma proved in Sec. 1.1 of Chap. 19, the momentum of the system is equal to the product of the total mass of the system by the velocity $\boldsymbol{v}_C$ of its centre of mass:

$$\boldsymbol{Q} = M\boldsymbol{v}_C \quad (M = m_1 + m_2 + \ldots + m_n)$$

The substitution of this expression into formula (4) results in

$$M\frac{d\boldsymbol{v}_C}{dt} = \boldsymbol{R}^{\text{ext}}, \text{ that is } M\boldsymbol{w}_C = \boldsymbol{R}^{\text{ext}} \tag{7}$$

Projecting the last equality on fixed (or inertial) axes Ox, Oy and Oz we obtain

$$M\frac{d^2x_C}{dt^2} = R_x^{\text{ext}} = \sum_{\nu=1}^{n} X_\nu^{\text{ext}},\ M\frac{d^2y_C}{dt^2} = \sum_{\nu=1}^{n} Y_\nu^{\text{ext}},\ M\frac{d^2z_C}{dt^2} = \sum_{\nu=1}^{n} Z_\nu^{\text{ext}} \tag{8}$$

where x_C, y_C and z_C are the coordinates of the centre of mass C of the system with respect to the coordinate system $Oxyz$. The principle is proved.

The three principles we have presented may lead to first integrals and, in particular, to the laws of conservation of the momentum of a system or of its projection on the given axis provided that some special conditions hold (see Sec. 1.4 of Chap. 19).

(B) Principle of Angular Momentum for a System of Particles. Here we present a simplified proof of

PRINCIPLE OF ANGULAR MOMENTUM FOR A SYSTEM OF PARTICLES: *the time derivative of the angular momentum of a system of particles about a fixed centre (about a fixed axis) is equal to the resultant moment of all the external forces acting on that system about that centre (about that axis).*

Proof. Let us perform vector multiplication on the left of equalities (1) by the radius vector $\boldsymbol{r}_\nu$ (see Fig. A.1):

$$\left[\boldsymbol{r}_\nu,\ m_\nu\frac{d\boldsymbol{v}_\nu}{dt}\right] = [\boldsymbol{r}_\nu,\ \boldsymbol{F}_\nu^{\text{ext}}] + [\boldsymbol{r}_\nu,\ \boldsymbol{F}_\nu^{(\text{int})}] \quad (\nu = 1, 2, \ldots, n) \tag{9}$$

By analogy with Sec. 2.2 of Chap. 15, according to formula (3), page 154, we can write the equation

$$\frac{d}{dt}[\boldsymbol{r}_\nu,\ m_\nu\boldsymbol{v}_\nu] = \left[\frac{d\boldsymbol{r}_\nu}{dt},\ m_\nu\boldsymbol{v}_\nu\right] + \left[\boldsymbol{r}_\nu,\ m_\nu\frac{d\boldsymbol{v}_\nu}{dt}\right] \quad (\nu = 1, 2, \ldots, n)$$

The first summand on the right-hand side is equal to zero because it is a vector product of two collinear vectors (that is of two vectors parallel to one straight

line): $dr_\nu/dt = v_\nu$ and $m_\nu v_\nu$. Therefore relation (9) can be written as

$$\frac{d}{dt}[r_\nu, m_\nu v_\nu] = [r_\nu, F_\nu^{\text{ext}}] + [r_\nu, F_\nu^{(\text{int})}] \quad (\nu = 1, 2, \ldots, n)$$

Adding together geometrically these vector equalities for $\nu = 1, 2, \ldots, n$ we obtain

$$\frac{d}{dt}\sum_{\nu=1}^{n}[r_\nu, m_\nu v_\nu] = \sum_{\nu=1}^{n}[r_\nu, F_\nu^{\text{ext}}] + \sum_{\nu=1}^{n}[r_\nu, F_\nu^{(\text{int})}] \qquad (10)$$

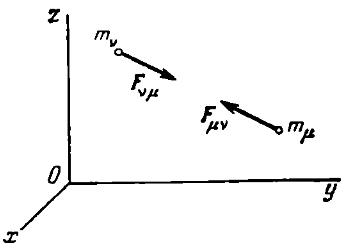

Fig. A.2

Let us elucidate the mechanical meaning of each of the expressions involved in the last relation. On the left-hand side under the sign of differentiation there stands the vector K_O representing the angular momentum of the system about the centre O (see Sec. 2.1 of Chap. 19); it is equal to the vector sum of the momenta of the particles of the system about that centre:

$$K_O = \sum_{\nu=1}^{n} \text{Mom}\,(m_\nu v_\nu) = \sum_{\nu=1}^{n}[r_\nu, m_\nu v_\nu]$$

The first sum on the right-hand side of (10) is the *resultant moment* M_O^{ext} *of all the external forces acting on the system* (including the reaction forces) about the centre O:

$$M_O^{\text{ext}} = \sum_{\nu=1}^{n} \text{Mom}_O\, F_\nu^{\text{ext}} = \sum_{\nu=1}^{n}[r_\nu, F_\nu^{\text{ext}}]$$

The second sum on the right-hand side of (10) is the *resultant moment* $M_O^{(\text{int})}$ *of all the internal forces of the system of particles* about the centre O. For the internal forces within a given system not only their resultant vector but also their resultant moment about any fixed centre is equal to zero at any instant. Indeed, the components of the internal forces appearing in the interaction between any two particles of the system obey Newton's third law, that is they are equal in their magnitude and opposite in direction along one line, and consequently

$$F_{\nu\mu} + F_{\mu\nu} = 0 \text{ and } \text{Mom}_O\, F_{\nu\mu} + \text{Mom}_O\, F_{\mu\nu} = 0 \quad (\nu, \mu = 1, 2, \ldots, n;\ \nu \neq \mu)$$

(Fig. A.2). Hence, we not only have

$$R^{(\text{int})} = \sum_{\nu=1}^{n} F_\nu^{(\text{int})} = \sum_{\substack{\nu,\,\mu=1\\(\nu>\mu)}}^{n} (F_{\nu\mu} + F_{\mu\nu}) = 0$$

but also

$$M_O^{(\text{int})} = \sum_{\nu=1}^{n} \text{Mom}_O\, F_\nu^{(\text{int})} = \sum_{\substack{\nu,\,\mu=1\\(\nu>\mu)}}^{n} (\text{Mom}_O\, F_{\nu\mu} + \text{Mom}_O\, F_{\mu\nu}) = 0$$

Finally, equality (10) takes the form

$$\frac{dK_O}{dt} = M_O^{\text{ext}} \tag{11}$$

and expresses in vector form the principle we had to prove. Projecting vector equality (11) on fixed (or inertial) axes Ox, Oy and Oz and taking into account formulas (5.6) and (19.17) we obtain

$$\frac{dK_x}{dt} = M_x^{\text{ext}}, \quad \frac{dK_y}{dt} = M_y^{\text{ext}}, \quad \frac{dK_z}{dt} = M_z^{\text{ext}} \tag{12}$$

Here K_z is the angular momentum of the system of particles about the fixed axis Oz:

$$K_z = \sum_{\nu=1}^{n} \text{mom}_z\,(m_\nu \boldsymbol{v}_\nu)$$

The quantity M_z^{ext} is the resultant moment of all the external forces acting on the system (including the constraint reactions) about the same axis:

$$M_z^{\text{ext}} = \sum_{\nu=1}^{n} \text{mom}_z\, \boldsymbol{F}_\nu^{\text{ext}}$$

The other quantities involved in (12) have analogous meaning. Formulas (12) express the principle of angular momentum for a system in scalar form. The principle is proved.

The two principles proved in Sec. (B) may lead to first integrals; in particular, if some special conditions hold they lead to the law of conservation of the angular momentum of a system about a fixed centre (or about a fixed axis); see Sec. 2.3 of Chap. 19.

(C) Principle of Energy for a System of Particles. We begin with proving PRINCIPLE OF ENERGY FOR A SYSTEM OF PARTICLES (*in differential form*): *the differential of the kinetic energy of a system of particles is equal to the total elementary work of all the forces acting on the system (both the external forces including the constraint reactions and the internal forces) along the real displacement of that system.*

Proof. Let us multiply scalarly vector equations (1) by the real elementary displacements $d\boldsymbol{r}_\nu = \boldsymbol{v}_\nu\, dt$ of the particles of the system (see Fig. A.1):

$$m_\nu (\boldsymbol{v}_\nu,\ d\boldsymbol{v}_\nu) = (\boldsymbol{F}_\nu^{\text{ext}},\ d\boldsymbol{r}_\nu) + (\boldsymbol{F}_\nu^{(\text{int})},\ d\boldsymbol{r}_\nu) \quad (\nu = 1, 2, \ldots, n) \tag{13}$$

As is known, the scalar square of a vector is equal to the square of its modulus:

$$(\boldsymbol{v}_\nu,\ \boldsymbol{v}_\nu) = \boldsymbol{v}_\nu^2 = v_\nu^2$$

The differentiation of the last equality yields

$$2\,(\boldsymbol{v}_\nu,\ d\boldsymbol{v}_\nu) = 2v_\nu\, dv_\nu \quad (\nu = 1, 2, \ldots, n)$$

Therefore the left-hand side of (13) is brought to the form

$$m_\nu (\boldsymbol{v}_\nu,\ d\boldsymbol{v}_\nu) = m_\nu v_\nu\, dv_\nu = d\left(\frac{m_\nu v_\nu^2}{2}\right) \quad (\nu = 1, 2, \ldots, n)$$

Adding together equalities (13) for $\nu = 1, 2, \ldots, n$ we conclude that at any instant during the motion of the system the relation

$$d \sum_{\nu=1}^{n} \frac{m_\nu v_\nu^2}{2} = \sum_{\nu=1}^{n} (\boldsymbol{F}_\nu^{\text{ext}},\ d\boldsymbol{r}_\nu) + \sum_{\nu=1}^{n} (\boldsymbol{F}_\nu^{(\text{int})},\ d\boldsymbol{r}_\nu) \tag{14}$$

holds. The expression under the sign of differential in (14) is the kinetic energy of the system:

$$T = \frac{1}{2} \sum_{\nu=1}^{n} m_\nu v_\nu^2.$$

The expressions on the right-hand side of (14) are, respectively, the total elementary work of the external and of the internal forces of the system along the real displacement. The principle is proved.

PRINCIPLE OF ENERGY FOR A SYSTEM OF PARTICLES (*in finite form*): *the change of the kinetic energy of a system of particles is equal to the total work of all the forces acting on the system (both the external forces including the constraint reactions and the internal forces) along the given displacement of that system.*

Proof. The integration of differential equality (14) results in

$$T - T_0 = A^{\text{ext}} + A^{(\text{int})} \tag{15}$$

Here the quantities A^{ext} and $A^{(\text{int})}$ are the line integrals

$$A^{\text{ext}} = \int_{\Gamma_0}^{\Gamma} \sum_{\nu=1}^{n} (F_\nu^{\text{ext}}, dr_\nu) = \int_{\Gamma_0}^{\Gamma} \sum_{\nu=1}^{n} (X_\nu^{\text{ext}}\, dx_\nu + Y_\nu^{\text{ext}}\, dy_\nu + Z_\nu^{\text{ext}}\, dz_\nu)$$

and

$$A^{(\text{int})} = \int_{\Gamma_0}^{\Gamma} \sum_{\nu=1}^{n} (F_\nu^{(\text{int})}, dr_\nu) = \int_{\Gamma_0}^{\Gamma} \sum_{\nu=1}^{n} (X_\nu^{(\text{int})}\, dx_\nu + Y_\nu^{(\text{int})}\, dy_\nu + Z_\nu^{(\text{int})}\, dz_\nu)$$

They express, respectively, the work of the external and of the internal forces corresponding to the displacement of the system from its initial configuration Γ_0 to its terminal configuration Γ. The principle is proved.

REMARK ON THE WORK OF INTERNAL FORCES. For a system of particles with invariable configuration (in particular, for a rigid body) the work of the internal forces is equal to zero for any real displacement of the system (see Sec. 3.3 of Chap. 19). For such a system formulas (14) and (15) take the form

$$dT = \sum_{\nu=1}^{n} (F_\nu^{\text{ext}}, dr_\nu) \tag{14a}$$

and

$$T - T_0 = A^{\text{ext}} = \int_{\Gamma_0}^{\Gamma} \sum_{\nu=1}^{n} (X_\nu^{\text{ext}}\, dx_\nu + Y_\nu^{\text{ext}}\, dy_\nu + Z_\nu^{\text{ext}}\, dz_\nu) \tag{15a}$$

REMARK ON THE WORK OF EXTERNAL FORCES. If the constraints imposed on a system of particles are stationary, that is if they do not depend on time explicitly and are ideal (see Secs. 1.1 and 1.2 of Chap. 17) then the real displacements belong to the class of virtual displacements and, by virtue

of the axiom of ideal constraints, the total work of the constraint reactions is equal to zero for any real displacement. In such a case the work of the external forces is equal to the work of the external active forces applied to the system:

$$A^{\text{ext}} = A^{(\text{ext})} = \int_{\Gamma_0}^{\Gamma} \sum_{\nu=1}^{n} (X_\nu^{(\text{ext})} dx_\nu + Y_\nu^{(\text{ext})} dy_\nu + Z_\nu^{(\text{ext})} dz_\nu)$$

here $\boldsymbol{F}_\nu^{(\text{ext})} = X_\nu^{(\text{ext})} \boldsymbol{i} + Y_\nu^{(\text{ext})} \boldsymbol{j} + Z_\nu^{(\text{ext})} \boldsymbol{k}$ is the resultant of the external active forces applied to the νth particle of the system ($\nu = 1, 2, \ldots, n$).

Examples demonstrating the application of the basic principles of dynamics of a system of particles are given in Chap. 19.

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